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A characterization of the standard Reeb flow

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Abstract. Among the topological conjugacy classes of the continuous flows $\{\phi^t\}$ whose orbit foliations are the planar Reeb foliation, there is one special class called the standard Reeb flow. We show that $\{\phi^t\}$ is conjugate to the standard Reeb flow if and only if $\{\phi^t\}$ is conjugate to $\{\phi^{\lambda t}\}$ for any $\lambda > 0$.

Key words: Reeb foliations, flows, topological conjugacy.

1. Introduction

Let

$$P = \{(\xi, \eta) | \xi \geq 0, \eta \geq 0\} - \{(0,0)\}.$$

A nonsingular flow $\{\Phi^t\}$ on P defined by

$$\Phi^t(\xi, \eta) = (e^t \xi, e^{-t} \eta)$$

is called the *standard Reeb flow*. In this note the oriented foliation $\mathcal{R}$ whose leaves are the orbits of $\{\Phi^t\}$ with the orientation given by the time direction is called the *Reeb foliation*. A continuous flow on P with orbit foliation $\mathcal{R}$ is called an $\mathcal{R}$-flow. The topological conjugacy classes of $\mathcal{R}$-flows $\{\phi^t\}$ are classified in [L] in the following way. Let $\gamma_1 : [0, \infty) \to P$ (resp. $\gamma_2 : [0, \infty) \to P$) be a continuous path such that $\gamma_1(0) \in \{\xi = 0\}$ (resp. $\gamma_2(0) \in \{\eta = 0\}$) which intersects every interior leaf of $\mathcal{R}$ at exactly one point. Then one can define a continuous function

$$f_{\{\phi^t\},\gamma_1,\gamma_2} : (0, \infty) \to \mathbb{R}$$

by setting that $f_{\{\phi^t\},\gamma_1,\gamma_2}(x)$ is the time needed for the flow $\{\phi^t\}$ to move

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from the point $\gamma_1(x)$ until it reaches a point on the curve γ_2. Then $f_{\{\phi^t\},\gamma_1,\gamma_2}$ belongs to the following space

$$E = \Big\{ f : (0,\infty) \to \mathbb{R} | f \text{ is continuous and } \lim_{x\to 0} f(x) = \infty \Big\}.$$

Of course $f_{\{\phi^t\},\gamma_1,\gamma_2}$ depends upon the choices of γ_1 and γ_2. There are two umbiguities, one coming from the parametrization of γ_1, and the other coming from the positions of γ_1 and γ_2. Let H be the space of homemorphisms of $[0,\infty)$ and C the space of continuous functions on $[0,\infty)$. Define an equivalence relation $\sim$ on E by

$$f \sim f' \iff f' = f \circ h + k, \quad \exists h \in H, \quad \exists k \in C.$$

Then clearly the equivalence class of $f_{\{\phi^t\},\gamma_1,\gamma_2}$ does not depend on the choice of γ_1 and γ_2. Moreover it is an invariant of the topological conjugacy classes of $\mathcal{R}$-flows. Thus if we denote by $\mathcal{E}$ the set of the topological conjugacy classes of the $\mathcal{R}$-flows, then there is a well defined map

$$\iota : \mathcal{E} \to E/\sim .$$

The main result of [L] states that ι is a bijection. In particular any $f \in E$ is obtained as $f = f_{\{\phi^t\},\gamma_1,\gamma_2}$ for some $\mathcal{R}$-flow $\{\phi^t\}$ and paths γ_i.

Clearly any strictly monotone function of E belongs to a single equivalence class, and this corresponds to the standard Reeb flow $\{\Phi^t\}$. The purpose of this note is to show the following characterization of the standard Reeb flow.

Theorem 1 *An $\mathcal{R}$-flow $\{\phi^t\}$ is topologically conjugate to the standard Reeb flow $\{\Phi^t\}$ if and only if $\{\phi^{\lambda t}\}$ is topologically conjugate to $\{\phi^t\}$ for any $\lambda > 0$.*

Of course the only if part is immediate. We shall show the if part in the next section.

Remark 1.1 A single λ is not enough for Theorem 1. In fact there is an $\mathcal{R}$-flow $\{\phi^t\}$ not topologically conjugate to $\{\Phi^t\}$ such that $\{\phi^{2t}\}$ is topologically conjugate to $\{\phi^t\}$. This will be given in Example 2.4 below.

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2. Proof of the if part

The equivalence class of $f \in E$ is determined by how $f(x)$ oscilates while it tends to ∞ as $x \to 0$. So to measure the degree of oscilation of $f \in E$, define a nonnegative valued continuous function f^* defined on $(0,1]$ by

$$f^*(x) = \max(f|_{[x,1]}) - f(x).$$

Then we have the following lemma.

Lemma 2.1 (1) *If $\lambda > 0$, then $(\lambda f)^* = \lambda f^*$.*
(2) *If c is a constant, then $(f + c)^* = f^*$.*
(3) *If $h \in H$, then there is $0 < a < 1$ such that $(f \circ h)^* = f^* \circ h$ on $(0, a)$.*
(4) *If $k \in C$ and $x \to 0$, then $(f + k)^*(x) - f^*(x) \to 0$.*
(5) *There is a sequence $\{x_n\}$ tending to 0 such that $f^*(x_n) = 0$.*

Proof. Points (1) and (2) are immediate. To show (3) notice that

$$(f \circ h)^*(x) = \max(f|_{[h(x),h(1)]}) - f(h(x)) \quad \text{and}$$
$$f^* \circ h(x) = \max(f|_{[h(x),1]}) - f(h(x)).$$

Since $f(x) \to \infty$ $(x \to 0)$, both maxima coincide for small x.

Let us show (4). By (2) we only need to show (4) assuming that $k(0) = 0$. Now given $\epsilon > 0$, there is $\delta > 0$ such that if $0 < x < \delta$, then $|k(x)| < \epsilon$. Choose $\eta > 0$ small enough so that if $0 < x < \eta$, then we have

$$f(x) \geq \max(f|_{[\delta,1]}) \quad \text{and} \quad (f + k)(x) \geq \max((f + k)|_{[\delta,1]}).$$

This implies that for $x \in (0, \eta)$,

$$|f^*(x) - (f + k)^*(x)|$$
$$\leq |f(x) - (f + k)(x)| + |\max((f + k)|_{[x,\delta]}) - \max(f|_{[x,\delta]})| < 2\epsilon.$$

This shows (4). Finally (5) follows from the assumption $f(x) \to \infty$ as $x \to 0$. $\square$

For $f \in E$ define an invariant $\sigma(f) = \limsup_{x\to 0} f^*(x)$ which takes value in $[0,\infty]$. In fact $\sigma(f)$ coincides with the invariant $\mathcal{A}(f)$ defined in [L] and used to show that $\mathcal{E}$ is uncountable.

Lemma 2.2 *Assume $f, f' \in E$ and $\lambda > 0$.*

(1) *We have $\sigma(\lambda f) = \lambda\sigma(f)$.*
(2) *If $f \sim f'$, then $\sigma(f) = \sigma(f')$. In particular f corresponds to the standard Reeb flow if and only if $\sigma(f) = 0$.*

Proof. Clearly (1) follows from Lemma 2.1 (1), while the first statement of (2) is an easy consequence of Lemma 2.1 (3) and (4). To show the last statement, assume $\sigma(f) = 0$. Extend the function f^* defined on $(0,1]$ to $[0,\infty)$ by letting

$$f^* = 0 \quad \text{on} \quad \{0\} \cup (1,\infty).$$

Since $\sigma(f) = 0$, f^* is continuous, i.e. $f^* \in C$. Thus $f \sim f + f^*$, and the latter is (weakly) monotone near 0. Still adding a suitable function, one can show that f is equivalent to a function g which is strictly monotone on the whole $(0,\infty)$ such that $g(x) \to 0$ $(x \to \infty)$. Clearly such functions are mutually equivalent by a pre-composition of some $h \in H$, and correspond to the standard Reeb flow $\{\Phi^t\}$. □

Now since

$$f_{\{\phi^{\lambda t}\},\gamma_1,\gamma_2} = \lambda^{-1} f_{\{\phi^t\},\gamma_1,\gamma_2}, \tag{2.1}$$

for $\lambda > 0$, Theorem 1 reduces to the following proposition.

Proposition 2.3 *If $f \in E$ and $f \sim \lambda f$ for any $\lambda > 0$, then $\sigma(f) = 0$.*

The rest of the paper is devoted to the proof of Proposition 2.3. But before starting, let us mention an example for Remark 1.1.

Example 2.4 By (2.1) and the main result of [L], it suffices to construct a function $f \in E$ such that $f(x/2) = 2f(x)$ and that $\sigma(f) = \infty$. Set for example

$$f(x) = \frac{1}{x} 2^{\sin(2\pi \log_2 x)}.$$

The following lemma, roughly the same thing as the linearization in one dimensional local dynamics, plays a crucial role in what follows.

Lemma 2.5 *Assume $f \in E$ satisfies $\lambda f = f \circ h + k$ for some $h \in H$, $k \in C$ and $\lambda > 1$. Then 0 is an attracting fixed point of h and there exists $f_\infty \in E$ such that $f_\infty - f \in C$, $\lambda f_\infty = f_\infty \circ h$ and $f_\infty(x) \to 0$ $(x \to \infty)$.*

Proof. Any equivalence class of E has a representative f such that

$$f|_{[1,\infty)} \text{ is bounded.} \tag{2.2}$$

So it is no loss of generality to assume that the function f in the lemma satisfies (2.2). We can also assume that $k(0) = 0$, by adding a suitable constant to f if necessary. Choose $a' \in (0,1)$ so that if $a \in (0, a')$,

$$f(a) > \frac{2}{\lambda - 1} \max(|k||_{[0,1]}).$$

Then we have

$$f \circ h(a) > \frac{\lambda + 1}{2} f(a), \quad \forall a \in (0, a'). \tag{2.3}$$

If a is sufficiently near 0, we have

$$f(a) > \sup(f|_{[1,\infty)}).$$

If furthermore $f^*(a) = 0$, then

$$\{x | f(x) > f(a)\} \subset (0, a).$$

Thus (2.3) implies $h(a) < a$ for such a. But this allows us to use (2.3) repeatedly for $h^n(a)$ $(n = 1, 2, \dots)$ instead of a, showing that $f \circ h^n(a) \to \infty$ as $n \to \infty$. Clearly this implies that $[0, a]$ is contained in the attracting domain of an attractor 0 of the homeomorphism h, showing the first point of Lemma 2.5.

For the rest of the proof, let us divide the argument into two cases according to the dynamics of h. First assume that the whole line $[0, \infty)$ is the attracting domain of 0. Let

$$f_n(x) = \lambda^{-n} f(h^n(x)).$$

Then we have

$$f_{n+1}(x) - f_n(x) = -\lambda^{-n-1} k(h^n(x)),$$

showing that $f_n \to f_\infty$ uniformly on compact subsets of $(0,\infty)$ for some continuous function f_∞. Now since

$$\lambda f_{n+1}(x) = f_n(h(x)),$$

we have

$$\lambda f_\infty = f_\infty \circ h.$$

We also have

$$|f(x) - f_\infty(x)| \leq \sum_{n=0}^{\infty} \lambda^{-n-1} |k(h^n(x))|.$$

The continuity of k, together with the assumption $k(0) = 0$, implies that

$$\lim_{x\to 0} |f(x) - f_\infty(x)| = 0,$$

showing that $f_\infty - f \in C$.

Finally since $h^{-n}(x) \to \infty$ $(n \to \infty)$ and

$$f_\infty \circ h^{-n}(x) = \lambda^{-n} f_\infty(x), \quad \forall x \in (0,\infty),$$

we have $f_\infty(x) \to 0$ $(x \to \infty)$.

Next assume there is a fixed point b of h such that $(0,b)$ is an attracting domain of 0. Thus we have $h^{-n}(x) \to b$ $(n \to \infty)$ for any $x \in (0,b)$.

The same argument as above shows the existence of a continuous function f_∞ on $(0,b)$. Since

$$f_\infty \circ h^{-n}(x) = \lambda^{-n} f_\infty(x), \quad \forall x \in (0,b),$$

we have

$$\lim_{x \uparrow b} f_\infty(x) = 0.$$

Now extend f_∞ by setting $f_\infty = 0$ on $[b, \infty)$. □

Let us start the proof of Proposition 2.3. Assume $f \in E$ satisfies $f \sim 2^{1/N} f$ for any $N \in \mathbb{N}$. Applying Lemma 2.5, f can be changed within the equivalence class to one which satisfies the condition of f_∞ for $\lambda = 2$. We also assume for contradiction that $\sigma(f) > 0$. Then by Lemma 2.2 (1) it follows that $\sigma(f) = \infty$.

Thus the proof of Proposition 2.3 reduces to showing that there is no $f \in E$ which satisfies the following assumption.

Assumption 2.6 A function $f \in E$ satisfies

$$2f = f \circ h, \quad \exists h \in H, \quad f(x) \to 0 \ (x \to \infty), \tag{2.4}$$

$$2^{1/N} f - f \circ h_N \in C \quad \exists h_N \in H, \ \forall N \geq 2 \ \text{ and} \tag{2.5}$$

$$\sigma(f) = \infty. \tag{2.6}$$

Define

$$E_0 = \{f \in E | f(x) \to 0 \ \ (x \to \infty)\}.$$

Henceforth all the functions dealt with will be in E_0, and the following definition is more convenient. For $f \in E_0$ define

$$f^\sharp(x) = \max(f|_{[x,\infty)}) - f(x).$$

Clearly $f^\sharp$ and f^* are the same near 0 and Lemma 2.1 (1), (4) and (5) hold also for $f^\sharp$, while (3) becomes stronger. In summary we have:

Lemma 2.7 *Assume $f, f' \in E_0$.*

(1) *If $\lambda > 0$, then $(\lambda f)^\sharp = \lambda f^\sharp$.*
(3) *If $h \in H$, then $(f \circ h)^\sharp = f^\sharp \circ h$.*
(4) *If $f' - f \in C$ and $x \to 0$, then $f^\sharp(x) - (f')^\sharp(x) \to 0$.*
(5) *There is a sequence $\{x_n\}$ tending to 0 such that $f^\sharp(x_n) = 0$.*

Hereafter f is always to be a function satisfying Assumption 2.6. Thus we have

$$2f^\sharp = f^\sharp \circ h. \tag{2.7}$$

Fix N for a while and let $h_1 = h_N^N$. Notice that by Lemma 2.5 both h and h_1 have 0 as their attractors and that

$$\begin{aligned} f \circ h - f \circ h_1 &= 2f - f \circ h_1 \\ &= \sum_{\nu=0}^{N-1} 2^{(N-\nu-1)/N}\big(2^{1/N} f \circ h_N^\nu - f \circ h_N^{\nu+1}\big) \in C. \end{aligned}$$

The following is an easy corollary of Lemma 2.7.

Corollary 2.8 *We have*

$$\lim_{x\to 0} |f^\sharp \circ h(x) - f^\sharp \circ h_1(x)| = 0.$$

Our overall strategy is to show that $f^\sharp$ is too much oscilating in a fundamental domain of h, thanks to condition (2.5). For that purpose first of all we have to compare the dynamics of h and h_1 near the common attractor 0 and to show that they have more or less the same fundamental domains.

Lemma 2.9 *Either there exists a sequence $\{a_n\}$ such that $a_n \to 0$ and that $h^2(a_n) \le h_1(a_n) \le h(a_n)$ or there exists a sequence $\{a_n\}$ such that $a_n \to 0$ and that $h_1^2(a_n) \le h(a_n) \le h_1(a_n)$.*

Proof. If there is a sequence $\{a_n\}$ such that $a_n \to 0$ and that $h(a_n) = h_1(a_n)$, there is nothing to prove. So there are two cases to consider. One is when $h_1(x) < h(x)$ for any small x, and the other $h_1(x) > h(x)$.

For the moment assume the former. In way of contradiction assume the contrary of the assertion of the lemma. This is equivalent to saying that $h_1(x) < h^2(x)$ for *any small* x. For small x, let $y = y(x) \in [h_1(x), x]$ be any point which gives $\max(f^\sharp|_{[h_1(x),x]})$. Notice that $f^\sharp(y)$ can be as large as we wish by choosing x even smaller. Then since $f^\sharp(h^2(y)) = 4f^\sharp(y) > f^\sharp(y)$, the point $h^2(y)$ is contained in

$$[h^2 \circ h_1(x), h^2(x)] - (h_1(x), x] = [h^2 \circ h_1(x), h_1(x)] \subset [h_1^2(x), h_1(x)].$$

The last inclusion follows from the assumption for a contradiction.

Put $h^2(y) = h_1(z)$ for some $z = z(x) \in [h_1(x), x]$. Then we have

$$f^\sharp \circ h_1(z) = 4f^\sharp(y) \geq 4f^\sharp(z) \quad \text{and} \quad f^\sharp \circ h(z) = 2f^\sharp(z). \tag{2.8}$$

If we choose x near enough to 0, then the associated $z = z(x)$ is also near, and thus

$$|2f^\sharp(z) - f^\sharp \circ h_1(z)| = |f^\sharp \circ h(z) - f^\sharp \circ h_1(z)|$$

can be arbitrarily small by Corollary 2.8. Then we have

$$f^\sharp(z) \approx \frac{1}{2} f^\sharp \circ h_1(z) = 2f^\sharp(y) \gg 1$$

for any such $z = z(x)$. On the other hand $z(x)$ can be arbitrarily near to 0, and thus (2.8) contradicts Corollary 2.8.

The opposite case where $h(x) < h_1(x)$ for any small x can be dealt with similarly by considering $f' \in E_0$, equivalent to f, such that $2f' = f' \circ h_1$, instead of f. □

Now fix a large number N and choose $f_1 \in E_0$ such that

$$f_1 - f \in C, \quad 2^{1/N} f_1 = f_1 \circ h_N.$$

The existence of such f_1 is guaranteed by Lemma 2.5 applied to $\lambda = 2^{1/N}$. We have then

$$2^{1/N} f_1^\sharp = f_1^\sharp \circ h_N. \tag{2.9}$$

Together with Lemma 2.9 which asserts that the fundamental domain of h_N^N is more or less comparable with that of h, this implies that $f_1^\sharp$ is oscilating in an extremely high frequency for N big. We are going to get a contradiction from this.

We still assume (2.4) for f. According to Lemma 2.9, there are two cases to consider. One is when there is a sequence $a_n \to 0$ such that $h^2(a_n) \leq h_N^N(a_n) \leq h(a_n)$, the other being $h_N^{2N}(a_n) \leq h(a_n) \leq h_N^N(a_n)$.

Assume for the moment that the former holds for infinitely many N. Let x_n^1 be the largest point such that $x_n^1 \leq a_n$ and $f_1^\sharp(x_n^1) = 0$. Notice that by Lemma 2.7 (5) and the equation (2.9), we have

$$x_n^1 \in (h_N(a_n), a_n]. \tag{2.10}$$

Then again by (2.9) $f_1^\sharp$ vanishes at the points $x_n^\nu = h_N^{\nu-1}(x_n^1)$ for any $1 \leq \nu \leq N$. Let y_n^1 be any point in $[x_n^2, x_n^1]$ at which $f_1^\sharp$ takes the maximal value and let $y_n^\nu = h_N^{\nu-1}(y_n^1)$ for $1 \leq \nu \leq N-1$. By (2.10) the order of these points are as follows.

$$h^2(a_n) < h_N^N(a_n) \leq x_n^N < y_n^{N-1} < \cdots < y_n^\nu < x_n^\nu < \cdots < y_n^1 < x_n^1 \leq a_n.$$

Notice that y_n^ν is a point in $[x_n^{\nu+1}, x_n^\nu]$ at which $f_1^\sharp$ takes the maximal value, and

$$f_1^\sharp(y_n^\nu) = 2^{(\nu-1)/N} f_1^\sharp(y_n^1).$$

We also have

$$f_1^\sharp(y_n^\nu) \geq \frac{1}{2} \max\big(f_1^\sharp|_{[h_N^N(a_n), a_n]}\big). \tag{2.11}$$

In fact on one hand

$$\max\big(f_1^\sharp|_{[x_n^N, a_n]}\big) = f_1^\sharp\big(y_n^{N-1}\big) = 2^{(N-2)/N} f_1^\sharp(y_n^1) \leq 2 f_1^\sharp(y_n^1).$$

On the other hand

$$\max\big(f_1^\sharp|_{[h_N^N(a_n), x_n^N]}\big) \leq 2^{(N-1)/N} \max\big(f_1^\sharp|_{[x_n^2, x_n^1]}\big) \leq 2 f_1^\sharp(y_n^1),$$

because

$$h_N^{-N+1}[h_N^N(a_n)), x_n^N] = [h_N(a_n), x_n^1] \subset [x_n^2, x_n^1].$$

Henceforth we focus our attention to the other homeomorphism $h \in H$. There is a sequence $\{m_n\}$ of integers such that the points $h^{-m_n}(a_n)$ belong to a fixed fundamental domain in the basin of 0 for h. Notice that $m_n \to \infty$ since $a_n \to 0$. Passing to a subsequence if necessary, we may assume that

$$h^{-m_n}(a_n) \to a, \quad h^{-m_n}(x_n^\nu) \to x^\nu \text{ and } h^{-m_n}(y_n^\nu) \to y^\nu,$$

for some points a, x^ν and y^ν. There is an ordering

$$h^2(a) \leq x^N \leq y^{N-1} \leq \cdots \leq y^\nu \leq x^\nu \leq \cdots \leq y^1 \leq x^1 \leq a.$$

We shall show that $f^\sharp(x^\nu) = 0$ and that $f^\sharp(y^\nu)$ is bounded away from 0 with a bound *independent of* N. Since these points can be taken in the same compact interval $[h^2(a), a]$, this will contradict the continuity of $f^\sharp$.

By Lemma 2.7 (4), $f_1^\sharp(x_n^\nu) = 0$ implies $f^\sharp(x_n^\nu) \le 1$ for any large n. Therefore by(2.7)

$$f^\sharp(h^{-m_n}(x_n^\nu)) \le 2^{-m_n},$$

showing that $f^\sharp(x^\nu) = 0$.

On the other hand since $h_N^N(a_n) \le h(a_n)$, we have by (2.11)

$$f_1^\sharp(y_n^\nu) \ge \frac{1}{2}\max\big(f_1^\sharp|_{[h_N^N(a_n),a_n]}\big) \ge \frac{1}{2}\max\big(f_1^\sharp|_{[h(a_n),a_n]}\big),$$

and therefore again by Lemma 2.7 (4), for any large n,

$$f^\sharp(y_n^\nu) \ge \frac{1}{2}\max\big(f^\sharp|_{[h(a_n),a_n]}\big) - 1.$$

Let $M = \max(f^\sharp|_{[h(a),a]})$ and notice that $M > 0$ since $\sigma(f) > 0$ (2.6) and by (2.7).

For any large n, the interval $h^{-m_n}[h(a_n), a_n]$ is near $[h(a), a]$, and is composed of a subinterval of $[h(a), a]$ and the iterate by $h^{\pm 1}$ of the complementary subinterval, and therefore

$$\max\big(f^\sharp|_{h^{-m_n}[h(a_n),a_n]}\big) \ge M/2.$$

This implies by (2.7)

$$\max\big(f^\sharp|_{[h(a_n),a_n]}\big) \ge \frac{1}{2}M2^{m_n},$$

showing that for any large n

$$f^\sharp(y_n^\nu) \ge \frac{1}{4}M2^{m_n} - 1.$$

This concludes that

$$f^{\sharp}(y^{\nu}) \geq \frac{1}{4}M,$$

as is desired.

The opposite case where $h_N^{2N}(a_n) \leq h(a_n) \leq h_N^N(a_n)$ ($\exists a_n \to 0$) holds for infinitely many N can be dealt with in a similar way, although the argument is not completely symmetric.

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