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On strongly separable Frobenius extensions

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Abstract. Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, h^*\}$, and define a map h by $h(d) = \sum x_k d y_k$ for each d in D $(= V_A(B))$. A is a strongly separable extension of B if and only if D is a Frobenius C-algebra with h a Frobenius homomorphism, where C is the center of A. In this case A is an H-separable Frobenius extension of $B^* = V_A(V_A(B))$.

Key words: strongly separable extension, Frobenius extension. H-separable extension.

Introduction

In their paper [4] A. Mewborn and E. McMahon defined the strongly separable extension of a non commutative ring, which is a generalization of the separable algebra over a commutative ring. Strong separability is weaker than H-separability, but stronger than separability. In [4] Mewborn and McMahon gave a few necessary and sufficient conditions for ring extensions to be strongly separable (See Theorem 3.5 [4]). Afterwords K. Yamashiro gave some new characterizations of strong separability which are the complete improvements of Proposition 3.10 [4] (See Theorem 1.4 [11]). These characterizations of strong separability are generalizations of the characterizations of H-separability. In the first section of this paper we will give a new proof of Mewborn-McMahon and Yamashiro's theorem concerning with the characterizations of strong separability. We will give also a new characterization of strong separability, which is the improvement of Proposition 3.9 [4] (Theorem 1.2).

H-separable extensions have a close relationship with Frobenius extensions. For example each projective H-separable extension of a (not necessarily artinian) simple ring is a Frobenius extension, and every H-separable extension of full liniar ring is a free Frobenius extension (See Theorem 4 [9]). Recently in [10] the author gave some necessary and sufficient conditions for the Frobenius extension to be H-separable. The aim of this paper is to generalize the results obtained in [10] to the case of strongly separable

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extension.

Let A be a ring with its center C, B a subring of A and D the centralizer of B in A. In the case where D is a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$, A is a strongly separable extension of B if and only if there exists $\sum x_k \otimes y_k$ in $(A \otimes_B A)^A$ such that $d = \sum x_k d d_i y_k e_i = \sum d_i x_k e_i d y_k$ holds for each d in D. Furthermore if these conditions are satisfied, A is an H-separable Frobenius extension of $B^* = V_A(V_A(B))$ (Theorem 2.1). Since the strongly separable extension and the H-separable extension are generalizations of the separable algebra and the Azumaya algebra, respectively, the following conjecture naturally comes out:

"In the case where A is a strongly separable extension of B, is A an H-separable extension of B^* and B^* separable extension of B."

Theorem 2.1 of this paper gives a partial answer to this problem. In the second theorem we will show that, in the case where A is a Frobenius extension of B with a Frobenius system $\{x_k, y_k, h^*\}$, A is a strongly separable extension of B if and only if D is a Frobenius C-algebra with a Frobenius homomorphism h, where h is defined by $h(d) = \sum x_k d y_k$ for each d in D.

1. Throughout this paper we will use the same notation as the author's previous paper [10]. In particular A will always be a ring with identity 1, B a subring of A containing 1, C the center of A and $D = V_A(B)$, the centralizer of B in A. For an A-A-module M we write $M^A = \{m \mid m \in M, am = ma \text{ for each } a \in M\}$. We have an isomorphism of $\mathrm{Hom}({}_AA_A, {}_AM_A)$ to M^A, via $f \to f(1)$ for each f in $\mathrm{Hom}({}_AA_A, {}_AM_A)$.

Lemma 1.1 *For an A-A-module M the following conditions are equivalent*;

(i) *M is isomorphic to a direct summand of a finite direct sum of copies of A as A-A-module.*

(ii) *There exist finite f_i in $\mathrm{Hom}({}_AM_A, {}_AA_A)$ and m_i in M^A such that $m = \sum f_i(m) m_i$ holds for each m in M*

(iii) *M^A is C-finitely generated projective, and the map μ of $A \otimes_C M^A$ to M such that $\mu(a \otimes m) = am$ for a in A and m in M^A is an isomorphism.*

Proof. (i) $\Rightarrow$ (iii) is shown in Proposition 5.2 [3]. (i) $\Rightarrow$ (ii) can be shown by the same method as the existance of the dual basis of a finitely generated projective module, using the isomorphism $M^A \cong \mathrm{Hom}({}_AM_A, {}_AM_A)$. So we will omit it. $\square$

Following [3] we will give a definition.

Definition A two-sided A-module M is centrally projective over A, if M satisfies the condition of Lemma 1.1.

Let R be an arbitrary ring, and M and N any left R-modules. Write $S = \mathrm{Hom}({}_RM, {}_RM)$ and $T = \mathrm{Hom}({}_RN, {}_RN)$. We regard M and N as left R-S and R-T-modules, respectively. By writing $f(x)$ for each f in S (or T) and x in M (or N). Now we have the following natural R-S-homomorphism

$$\tau : M \to \mathrm{Hom}({}_T\mathrm{Hom}({}_RM, {}_RN), {}_TN)$$

such that $\tau(m)(f) = f(m)$ for each f in $\mathrm{Hom}({}_RM, {}_RN)$ and m in M. Write $H = \mathrm{Hom}({}_RM, {}_RN)$. Let $m \in M$. Then we have $f(m) = 0$ for each f in H, if and only if $\tau(m)(f) = 0$ for each f in H, that is, $\tau(m) = 0$. This means that $\mathrm{Ker}\,\tau = \bigcap_{f \in H} \mathrm{Ker}\, f (= \mathrm{Rej}({}_MN))$ (See page 109 [1]).

On the other hand in Theorem 1.2 [2] K. Hirata showed that, if M is isomorphic to a direct summand of a finite direct sum of copies of N as R-module, τ is an isomorphism.

Now let M be an A-A-module, and put $N = A$ and $R = A \otimes_C A^\circ$, where A° is the opposite ring of A. Then M and N are left R-modules. Under this situation let S and T be as above. Then T is isomorphic to C, and the above map τ becomes the following A-A-homomorphism

$$\tau : M \to \mathrm{Hom}(\mathrm{Hom}({}_AM_A, {}_AA_A)_C, A_C)$$

Lemma 1.2 *If an A-A-module M is centrally projective over A, the map τ given above is an isomorphism.*

Proof. Under the above situation, M is isomorphic to a direct summand of a finite direct sum of copies of N as R-module if and only if M is centrally projective over A. Therefore the lemma is an immediate consequence of Theorem 1.2 [2].

There always exists an A-A-homomorphism

$$\eta : A \otimes_B A \to \mathrm{Hom}({}_CD, {}_CA)$$

such that $\eta(a \otimes b)(d) = adb$ for any a, b in A and d in D. Throughout this paper η will always be the map defined as above, and K will stand for $\mathrm{Ker}\,\eta$.

Also there exists the well known isomorphism ν of $\mathrm{Hom}({}_AA \otimes_B A_A,$

${}_AA_A)$ to D such that $\nu(f) = f(1\otimes 1)$ for each f in $\mathrm{Hom}({}_AA\otimes_B A_A, {}_AA_A)$. ν induces the isomorphism

$$\nu^* : \mathrm{Hom}({}_CD, {}_CA) \to \mathrm{Hom}({}_C\mathrm{Hom}({}_AA\otimes_B A_A, {}_AA_A), {}_CA)$$

such that $\nu^*(f) = f\nu$ for each f in $\mathrm{Hom}({}_CD, {}_CA)$, and direct calculation shows that the composition $\nu^*\eta$ is exactly equal to the map τ of M to $\mathrm{Hom}(\mathrm{Hom}({}_AM_A, {}_AA_A)_C, A_C)$ introduced above.

In [4] Mewborn and McMahon defined that A is a strongly separable extension of B in the case where η is an A-A-split epimorphism and D is C-finitely generated projective.

Now we will give a new proof of Theorem 3.5 [4] and Theorem 1.4 [11].

Among five conditions in the next theorem, the equivalence of (i), (ii) and (v) was first proved by Mewborn-McMahon, and the equivalence of (i), (iii) and (iv) was proved by Yamashiro. □

Theorem 1.1 *The following conditions are equivalent*

(i) *A is a strongly separable extension of B.*

(ii) *There exist finite d_i in D and $\sum x_{ij}\otimes y_{ij}$ in $(A\otimes_B A)^A$ such that $d = \sum d_i x_{ij} d y_{ij}$ holds for each d in D.*

(iii) *For each A-A-module M there exists a C-submodule X such that $M^B = DM^A \oplus X$ and $X \subseteq \mathrm{Rej}_M(A)$.*

(iv) *We have $(A\otimes_B A)^B = D(A\otimes_B A)^A \oplus X$ as C-module with $X \subseteq \mathrm{Ker}\,\eta$.*

(v) *$A\otimes_B A = N\oplus L$ as A-A-module, where N is A-centrally projective, and $\mathrm{Hom}({}_AL_A, {}_AA_A) = 0$.*

Proof. Assume (i). Then there exists an A-A-map ψ of $\mathrm{Hom}({}_CD, {}_CA)$ to $A\otimes_B A$ such that $\eta\psi$ is equal to the identity map on $\mathrm{Hom}({}_CD, {}_CA)$. Then we have $A\otimes_B A = K\oplus N$, where $K = \mathrm{Ker}\,\eta$ and $N = \mathrm{Im}\,\psi = \mathrm{Hom}({}_CD, {}_CA)$. The latter is A-centrally projective, since D is C-finitely generated projective. Let f be any A-A-map of K to A. Then since K is an A-A-direct summand of $A\otimes_B A$, f is extended to an A-A-map f^* of $A\otimes_B A$ to A. But as is remarked above K is equal to $\mathrm{Rej}_{A\otimes_B A}(A)$. Therefore we have $f(K) = f^*(K) = 0$, and $\mathrm{Hom}({}_AK_A, {}_AA_A) = 0$. Thus we have (v). Next let $\{g_i, d_i\}$ be a dual basis of D over C, and $\psi(g_i) = \sum x_{ij}\otimes y_{ij}$ for each i. Then since $g_i \in \mathrm{Hom}({}_CD, {}_CC) = [\mathrm{Hom}({}_CD, {}_CA)]^A$, and ψ is an A-A-map, we have $\sum x_{ij}\otimes y_{ij} \in (A\otimes_B A)^A$ for each i. Let ι be the inclusion map of D to A. Then $\iota(d) = d = \sum d_i g_i(d)$ for each d in D, which implies that $\iota = \sum d_i g_i$. Then we have $\iota = \eta\psi(\iota) = \eta(\sum d_i\psi(g_i)) = \eta(\sum d_i x_{ij}\otimes y_{ij})$, and $d = \iota(d) = \eta(\sum d_i x_{ij} \otimes y_{ij})(d) = \sum d_i x_{ij} d y_{ij}$ for each d in D. Thus we have also (ii). Next assume (ii). Let M be an A-A-module, and $X = \{m - \sum d_i x_{ij} m y_{ij} \mid m \in M^B\}$. Then X is a C-submodule of M, and we have $M^B = X + \sum d_i x_{ij} M^B y_{ij}$. For each m in M^B and each A-A-map f of M to A we have $f(m) \in D$ and

$$\begin{aligned} f(m - \sum d_i x_{ij} m y_{ij}) &= f(m) - \sum d_i x_{ij} f(m) y_{ij} \\ &= f(m) - f(m) = 0. \end{aligned}$$

Thus we have $f(X) = 0$, and $X \subseteq \mathrm{Rej}_M(A)$. Since $\sum x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A$, it is obvious that $\sum x_{ij} M^B y_{ij} \subseteq M^A$. Hence we have $M^B = DM^A + K$. We will show that the above sum is a direct sum. Let $\sum e_i n_i = m - \sum d_i x_{ij} m y_{ij}$ for some $e_i \in D$, $n_i \in M^A$ and $m \in M^B$. Then we have

$$\begin{aligned} \sum d_k x_{kl} \sum e_i n_i y_{kl} &= \sum d_k x_{kl} m y_{kl} - \sum d_k x_{kl} d_i x_{ij} m y_{ij} y_{kl} \\ &= \sum d_k x_{kl} m y_{kl} - \sum d_k x_{kl} d_i y_{kl} x_{ij} m y_{ij} \\ &= \sum d_k x_{kl} m y_{kl} - \sum d_i x_{ij} m y_{ij} = 0 \end{aligned}$$

But $\sum d_k x_{kl} e_i n_i y_{kl} = \sum d_k x_{kl} e_i y_{kl} n_i = \sum e_i n_i$. Hence $\sum e_i n_i = 0$, and we have $X \cap DM^A = 0$. Thus we have (iii). It is obvious that (iii) implies (iv). As for the proofs of (iv) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) see the proofs of (3) $\Rightarrow$ (1) of Theorem 1.4 [11] and (2) $\Rightarrow$ (1) of Theorem 3.5 [4], respectively. Lastly assume (v). Let ι be the A-A-split monomorphism of N to $A \otimes_B A$. Since $\mathrm{Hom}({}_A L_A, {}_A A_A) = 0$, ι induces the isomorphism ι^* of $\mathrm{Hom}({}_A A \otimes_B A_A, {}_A A_A)$ ($\cong D$) to $\mathrm{Hom}({}_A N_A, {}_A A_A)$. Then since N is A-centrally projective, $\mathrm{Hom}({}_A N_A, {}_A A_A)$ and D are A-centrally projective. On the other hand we have the A-A-isomorphism τ_N of N to $\mathrm{Hom}({}_C\mathrm{Hom}({}_A N_A, {}_A A_A), {}_C A)$ defined in the same way as Lemma 1.2. Then we have the following commutative diagram of A-A-maps

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[d, "\tau_N"'] & A \otimes_B A \arrow[d, "\tau"'] \\ \mathrm{Hom}({}_C\mathrm{Hom}({}_A N_A, {}_A A_A), {}_C A) \arrow[r, "\iota^{**}"'] & \mathrm{Hom}({}_C\mathrm{Hom}({}_A A \otimes_B A_A, {}_A A_A), {}_C A) \arrow[d, "\nu^*"'] \\ & \mathrm{Hom}({}_C D, {}_C A) \end{tikzcd}$$

where τ and ν are the same as defined above, and ι^{**} and ν^* are the maps induced canonically by ι and ν, respectively. Then since ι splits as A-A-

map, and ι^{**}, τ_N and ν^* are A-A-isomorphisms, we can see that τ and η $(= \nu^*\tau)$ splits as A-A-map.

We have also the following A-A-homomorphism

$$\eta_l : D \otimes_C A \to \mathrm{Hom}({}_BA, {}_BA)$$

such that $\eta_l(d \otimes a)(x) = dxa$ for x, $a \in A$ and $d \in D$. The map η_r of $A \otimes_C D$ to $\mathrm{Hom}(A_B, A_B)$ is similarly defined. □

Now we will give a new characterization of the strongly separable extension which is a generalization of Lemma 2 [10].

Theorem 1.2 *Let A be left B-finitely generated projective. Then the following three conditions are equivalent;*

(i) *A is a strongly separable extension of B.*

(ii) $\mathrm{Hom}({}_BA, {}_BA) = N \oplus X$ *as A-A-module with N A-centrally projective and $X^A = 0$.*

(iii) *η_l is an A-A-split monomorphism, and D is C-finitely generated projective.*

Proof. By Proposition 3.9 [4] and the definition of the strongly separable extension (i) implies (iii). Now assume (iii). Then we have $\mathrm{Hom}({}_BA, {}_BA) \cong D \otimes_C A \oplus X$ as A-A-module. Since D is C-finitely generated projective. $D \otimes_C A$ is A-centrally projective, and we have $(D \otimes_C A)^A = D \otimes_C C \cong D$ by Lemma 3 [7]. Then we have

$$\begin{aligned} D &\cong \mathrm{Hom}({}_BA_A, {}_BA_A) \\ &= [\mathrm{Hom}({}_BA, {}_BA)]^A \cong (D \otimes_C A)^A \oplus X^A \cong D \oplus X^A \end{aligned}$$

which implies that $X^A = 0$. Thus we have (ii). Next assume (ii). Since $X^A = 0$, we have $D \cong [\mathrm{Hom}({}_BA, {}_BA)]^A = X^A \oplus N^A = N^A$. We will denote this isomorphism by μ. For each $d \in D$ $\mu(d)$ is a B-A-endomorphism of A given by $\mu(d)(a) = da$ for $a \in A$. We have also a natural A-A-isomorphism

$$\sigma : N \to \mathrm{Hom}({}_A\mathrm{Hom}(N_A, A_A), {}_AA)$$

such that $\sigma(x)(f) = f(x)$ for $x \in N$ and $f \in \mathrm{Hom}(N_A, A_A)$, since N is right A-finitely generated projective. Then since σ is an A-A-isomorphism, σ induces a C-isomorphism

$$\sigma : N^A \to \mathrm{Hom}({}_A\mathrm{Hom}(N_A, A_A)_A, {}_AA_A)$$

Furthermore since A is left B-finitely generated projective, we have a natural isomorphism

$$\rho : A \otimes_B A \to \mathrm{Hom}(\mathrm{Hom}({}_BA, {}_BA)_A, A_A)$$

such that $\rho(a \otimes b)(f) = af(b)$ for $a, b \in A$ and $f \in \mathrm{Hom}({}_BA, {}_BA)$. Now by assumption we have $\mathrm{Hom}(\mathrm{Hom}({}_BA, {}_BA)_A, A_A) \cong \mathrm{Hom}(X_A, A_A) \oplus \mathrm{Hom}(N_A, A_A)$ as A-A-module. Thus we have $A \otimes_B A \cong X^* \oplus N^*$, where $X^* = \mathrm{Hom}(X_A, A_A)$ and $N^* = \mathrm{Hom}(N_A, A_A)$ as A-A-module. Since N is A-centrally projective, it is obvious that N^* is also A-centrally projective. Therefore we need only to show that $\mathrm{Hom}({}_AX^*{}_A, {}_AA_A) = 0$. Let ι be the A-A-split monomorphism of N to $\mathrm{Hom}({}_BA, {}_BA)$, and write $\iota^* = \mathrm{Hom}(\iota, A)$, $\iota^{**} = \mathrm{Hom}(\iota^*, A)$ and $\rho^* = \mathrm{Hom}(\rho, A)$. Now consider the composition of all those maps

$$\begin{aligned} D &\to {}_\mu N^A \to_\sigma \mathrm{Hom}({}_AN^*{}_A, {}_AA_A) \\ &\to {}_{\iota^{**}}\mathrm{Hom}({}_A\mathrm{Hom}(\mathrm{Hom}({}_BA, {}_BA)_A, A_A)_A, {}_AA_A) \\ &\to {}_{\rho^*}\mathrm{Hom}({}_AA \otimes_B A_A, {}_AA_A) \to D \end{aligned}$$

Then we have

$$\begin{aligned} &(\rho^*\iota^{**}\sigma\mu(d))(1 \otimes 1) = (\iota^{**}\sigma\mu(d))(\rho(1 \otimes 1)) \\ &= (\sigma\mu(d))(\iota^*(\rho(1 \otimes 1))) = (\sigma\mu(d))(\rho(1 \otimes 1)\iota) = \rho(1 \otimes 1)(\iota\mu(d)) \\ &= \rho(1 \otimes 1)(\mu(d)) = 1\mu(d)(1) = d \end{aligned}$$

for each d in D. Hence the composition of all of the above maps is the identity map on D. This means that $\mathrm{Hom}({}_AX^*{}_A, {}_AA_A) = 0$. □

2. Strongly separable Frobenius extensions

In this section we will give the generalization of the results obtained in [10]. The next theorem is a generalization of Theorem 2 [10].

Theorem 2.1 *Suppose that D is a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$. Then A is a strongly separable extension of B if and only if there exists $\sum x_k \otimes y_k$ in $(A \otimes_B A)^A$ such that $d = \sum x_k d d_i y_k e_i = \sum d_i x_k e_i d y_k$ hold for each d in D.*

Added in proof. Section 1 of this paper, except for the proof of the equivalence of (ii), (iii) and (iv), have been introduced by the same author at the summer seminar which was held at Zhongshan University, Guangzhou China, in July 1993.

If these conditions are satisfied, then A is an H-separable Frobenius extension of $B^* = V_A(V_A(B))$ with a Frobenius system $\{x_k, y_k, h^*\}$, where h^* is defined by $h^*(x) = \sum d_i x e_i$ for each x in A.

Proof. Suppose that A is a strongly separable extension of B. Then there exists an A-A-map ψ of $\mathrm{Hom}({}_C D, {}_C A)$ to $A \otimes_B A$ such that $\eta\psi$ is the identity map on $\mathrm{Hom}({}_C D, {}_C A)$, and we have $A \otimes_B A = K \oplus N$, where $K = \mathrm{Ker}\,\eta$ and $N = \mathrm{Im}\,\psi$. Since the restriction of η on N is an A-A-isomorphism, η induces a C-isomorphism of N^A to $[\mathrm{Hom}({}_C D, {}_C A)]^A = \mathrm{Hom}({}_C D, {}_C C)$. Hence there exists $\sum x_k \otimes y_k$ in $N^A (\subseteq A \otimes_B A)^A)$ such that $\eta(\sum x_k \otimes y_k) = h$. Then we have

$$\begin{aligned} d &= \sum d_i h(e_i d) = \sum d_i x_k e_i d y_k \quad \text{and} \\ d &= \sum h(d d_i) e_i = \sum x_k d d_i y_k e_i \end{aligned}$$

for each d in D. Conversely assume that there exists $\sum x_k \otimes y_k$ in $(A \otimes_B A)^A$ which satisfies the above condition. Then also $\sum x_k e_i \otimes y_k$ belongs to $(A \otimes_B A)^A$ for each i, and we have $d = \sum d_i x_k e_i d y_k$ for each d in D. Then A is a strongly separable extension of B by Theorem 3.5 [4]. Thus we have proved the equivalence. Next define h^* as in the theorem. Since $\sum x_k D y_k \subseteq C$, we have $\sum x_k h^*(y_k x) = \sum x_k d_i y_k x e_i = x \sum x_k d_i y_k e_i = x1 = x$ for each x in A. Similarly we have $\sum h^*(x x_k) y_k = x$ for each x in A. Furthermore since $\sum d_i \otimes e_i \in (D \otimes_C D)^D$, we have $h^*(A) = \sum d_i A e_i \subseteq V_A(D) = B^*$. Hence h^* is a B^*-B^*-map of A to B^*. Then $\{x_k, y_k, h^*\}$ is a Frobenius system of A over B^*. On the other hand since A is a strongly separable extension of B, η is an A-A-split epimorphism. Then since $B^* \supseteq B$ and $V_A(B^*) = D$, η induces an epimorphism η^* of $A \otimes_{B^*} A$ to $\mathrm{Hom}({}_C D, {}_C A)$. Now let $\sum a_j \otimes b_j \in \mathrm{Ker}\,\eta^*$. Then since $\sum a_j d_i b_j = 0$ for each i, we have in $A \otimes_{B^*} A$

$$\begin{aligned} \sum a_i \otimes b_j &= \sum a_j \otimes h^*(b_j x_k) y_k = \sum a_j h^*(b_j x_k) \otimes y_k \\ &= \sum a_j d_i b_j x_k e_i \otimes y_k = \sum 0 \otimes y_k = 0 \end{aligned}$$

Thus $\mathrm{Ker}\,\eta^* = 0$, and we see that η^* is an isomorphism. Then since $V_A(B^*)$ $(= D)$ is C-finitely generated projective, A is an H-separable extension of B^*. □

The next theorem is a generalization of Theorem 1 [10].

Theorem 2.2 *Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, h^*\}$, then the following conditions are equivalent*

(i) *A is a strongly separable extension of B*

(ii) *There exist finite d_i, e_i in D such that $\sum d_i x_k e_i d y_k = d$ holds for each d in D*

(iii) *D is a Frobenius C-algebra with a Frobenius homomorphism h which is defined by $h(d) = \sum x_k d y_k$ for each d in D*

Proof. Since $(A \otimes_B A)^A = \sum x_k D \otimes y_k$, the equivalence of (i) and (ii) is an immediate consequence of Theorem 3.5 [4]. Now assume (ii), and define h as in (iii). Then since $\sum x_k \otimes y_k \in (A \otimes_B A)^A$ and $\sum x_k e_i \otimes y_k \in (A \otimes_B A)^A$ for each i, if we define $f_i(d) = \sum x_k e_i d y_k$ $(= h(e_i d))$ for each d in D, h and each f_i belong to $\mathrm{Hom}({}_C D, {}_C C)$, and we have $d = \sum d_i x_k e_i d y_k = \sum d_i f_i(d)$ by assumption. Therefore $\{f_i, d_i\}$ forms a dual basis for D over C. Now write $D^* = \mathrm{Hom}({}_C D, {}_C C)$, and let θ be the map of D to D^* defined by $\theta(d) = hd$ for each d in D. Then since $d = \sum d_i h(e_i d)$ we have

$$\begin{aligned} f(d) &= f(\textstyle\sum d_i h(e_i d)) = \sum f(d_i) h(e_i d) = h(\sum f(d_i) e_i d) \\ &= (h \textstyle\sum f(d_i) e_i)(d) \end{aligned}$$

and consequently $f = h \sum f(d_i) e_i$ for each f in D^*. Thus θ is an epimorphism. Moreover since D is C-finitely generated projective, D^* is also C-finitely generated projective. Hence θ splits as C-map, and we can write $D = D^* \oplus X$, where $X = \mathrm{Ker}\,\theta$. Furthermore for each maximal ideal $\mathfrak{m}$ of C we have $D^* \otimes_C C/\mathfrak{m} \cong \mathrm{Hom}_{C/\mathfrak{m}}(D \otimes_C C/\mathfrak{m}, C/\mathfrak{m}) \cong D \otimes_C C/\mathfrak{m}$ as vector space over $C/\mathfrak{m}$ (See e.g., Corollary 2.5 [1]). Then we have $X/X\mathfrak{m} = 0$ for each maximal ideal $\mathfrak{m}$ of C, which implies that $X = 0$ by generalized Nakayama's Lemma. Therefore θ is a C-isomorphism. It is obvious that θ is a right D-map. Thus we have (iii). Next assume (iii). Then there exist a_j, b_j in D such that $\{a_j, b_j, h\}$ forms a Frobenius system of D over C, and we have $d = \sum a_j h(b_j d) = \sum a_j x_k b_j d y_k$ for each d in D. But we have also $\sum x_k b_j \otimes y_k \in (A \otimes_B A)^A$ for each j, since $\sum x_k \otimes y_k \in (A \otimes_B A)^A$. Then we have (i). □

Corollary 2.1 *If A is a strongly separable Frobenius extension of B, A is an H-separable Frobeius extension of B^* $(= V_A(V_A(B)))$.*

Proof. This is an immediate consequence of Theorems 2.1 and 2.2.

□

Furthermore by the above theorem we can obtain a new characterization of projective separable algebras.

Corollary 2.2 *Let A be a Frobenius algebra over a commutative ring R with $\{x_k, y_k, h\}$ a Frobenius system, and define a map h^* by $h^*(x) = \sum x_k x y_k$ for each x in A. Then A is a separable R-algebra if and only if h^* is also a Frobenius homomorphism of A over R.*

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