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# A new MacWilliams type identity for linear codes

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**Abstract.** We obtain a new MacWilliams type identity which generarizes MacWilliams identity for linear codes into another direction.

*Key words*: code, support, weight, weight emumerater, MacWilliams identity.

## 1. Introduction

Let $F := \boldsymbol{F}_q$ be a field of $q$-elements and let $V := F^n$ be the row vector space of dimensional $n$. Put $N := \{1, 2, \ldots, n\}$. The *inner product* and the *distance* of vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ of $V$ are defined by

$$\begin{aligned} \langle u, v \rangle &:= \sum_{i=1}^{n} u_i v_i, \\ d(u, v) &:= \sharp\{i \in N \mid u_i \neq v_i\} \end{aligned}$$

The *support* and the *weight* of a vector $v = (v_1, \ldots, v_n)$ of $V$ are defined by

$$\begin{aligned} \mathrm{supp}(v) &:= \{i \in N \mid v_i \neq 0\}, \\ |v| &:= \mathrm{wt}(v) := |\mathrm{supp}(v)| = \sharp\{i \in N \mid v_i \neq 0\}. \end{aligned}$$

A *code* $C$ of length $n$ is a subspace of $V$ and its *minimum distance* $d(C)$ of $C$ is defined by

$$d(C) := \min\{|v| \mid 0 \neq v \in C\}$$

When a code $C$ of length $n$ is of dimensional $k$ as a subspace and of minimum distance $d$, the code is called an $[n, k]$-*code* or an $[n, k, d]$-*code*. The *dual code* of $C$ is defined by

$$C^{\perp} := \{v \in V \mid \langle u, v \rangle = 0 \quad (\forall u \in C)\}.$$

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The *weight enumerator* of $C$ is defined by

$$W_C(x,y) := \sum_{u\in C} x^{n-|u|}y^{|u|} = \sum_{r=0}^{n} a_r x^{n-r}y^r.$$

where $a_r = \sharp\{u \in C \mid |u| = r\}$.

*Example* 1. Let $G_{24}$ be the binary Golay code of length 24, then $a_0 = a_{24} = 1$, $a_8 = a_{16} = 759$, $a_{12} = 2576$, and so

$$W_{G_{24}}(x,y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}.$$

There is a well known theorem by MacWilliams [2] about the weight enumerator of the dual code.

**Theorem 1** (F.J. MacWilliams (1963))

$$W_{C^\perp}(x,y) = \frac{1}{|C|}W_C(x+(q-1)y, x-y).$$

For any pair of row vectors $u, v \in V$, we now define

$$\begin{aligned} i(u,v) &:= \sharp\{i \in N \mid u_i = 0, v_i = 0\},\\ j(u,v) &:= \sharp\{i \in N \mid u_i = 0, v_i \neq 0\},\\ k(u,v) &:= \sharp\{i \in N \mid u_i \neq 0, v_i = 0\},\\ l(u,v) &:= \sharp\{i \in N \mid u_i \neq 0, v_i \neq 0\}, \end{aligned}$$

so that clearly

$$\begin{aligned} n &= i(u,v) + j(u,v) + k(u,v) + l(u,v),\\ |v| &= j(u,v) + l(u,v),\\ |u| &= k(u,v) + l(u,v). \end{aligned}$$

Let $C$ and $D$ be codes of length $n$. Define the *joint weight enumerator* of $C$ and $D$ by

$$\begin{aligned} W_{C,D}(a,b,c,d) &:= \sum_{u\in C}\sum_{v\in D} a^{i(u,v)}b^{j(u,v)}c^{k(u,v)}d^{l(u,v)}\\ &= \sum_{i,j,k,l} A_{i,j,k,l}a^ib^jc^kd^l, \end{aligned}$$

where $a$, $b$, $c$, $d$ are indeterminates and $A_{i,j,k,l}$ is the number of pairs of

$u \in C$ and $v \in D$ such that

$$i(u,v) = i,\ j(u,v) = j,\ k(u,v) = k,\ l(u,v) = l.$$

Then the following *generalized MacWilliams identity* holds:

**Theorem 2** ([2], [1])

$$\begin{aligned} W_{C^\perp,D}(a,b,c,d) &= \frac{1}{|C|} W_{C,D}(a+\gamma c, b+\gamma d, a-c, b-d), \\ W_{C,D^\perp}(a,b,c,d) &= \frac{1}{|D|} W_{C,D}(a+\gamma b, a-b, c+\gamma d, c-d), \\ W_{C^\perp,D^\perp}(a,b,c,d) &= \frac{1}{|C||D|} W_{C,D}(a+\gamma(b+c)+\gamma^2 d, a-b \\ &\quad + \gamma(c-d), a-c+\gamma(b-d), a-b-c+d), \end{aligned}$$

*where* $\gamma = q-1$.

The purpose of this note is to generarize Theorem 1 into another direction:

**Theorem 3** (Main theorem) *For an* $[n,k]$*-code* $C$ *and a positive integer* $\lambda$*, define the* $\lambda$-ply weight enumerator $W_C^{(\lambda)}(x,y)$ *by*

$$W_C^{(\lambda)}(x,y) := \sum_{u^{(1)},u^{(2)},\ldots,u^{(\lambda)} \in C} x^{n-s(u^{(1)},\ldots,u^{(\lambda)})} y^{s(u^{(1)},\ldots,u^{(\lambda)})}$$

*where*

$$\begin{aligned} s(u^{(1)},\ldots,u^{(\lambda)}) &:= \sharp\{i \in N \mid u_i^{(j)} \neq 0, \mathit{for}\ \exists j \in \{1,2,\ldots,\lambda\}\} \\ &= |\mathrm{supp}(u^{(1)}) \cup \cdots \cup \mathrm{supp}(u^{(\lambda)})|. \end{aligned}$$

*Then*

$$W_{C^\perp}^{(\lambda)}(x,y) = \frac{1}{|C|^\lambda} W_C^{(\lambda)}(x+(q^\lambda-1)y, x-y).$$

Of course, $W_C^{(1)}(x,y) = W_C(x,y)$ is the usual weight enumerator and the main theorem is just the ordinary MacWilliams identity. Note that even in this classical case the proof of the main theorem in this paper gives a new and short proof (cf. [5]). Furthermore, the main theorem for $\lambda = 2$ gives the MacWilliams identity for the *bi-weight enumerator*:

$$\begin{aligned} BW_C(x,y) &:= W_{C,C}(x,y,y,y) = W_C^{(2)}(x,y) \\ &= \sum_{u,v\in C} x^{n-|\mathrm{supp}(u)\cup\mathrm{supp}(v)|}y^{\mathrm{supp}(u)\cup\mathrm{supp}(v)|}, \end{aligned}$$

that is,

$$BW_{C^\perp}(x,y) = \frac{1}{|C|^2}BW_C(x+(q^2-1)y, x-y)$$

Furthermore, we can prove various kinds of MacWilliams type identities (e.g. Theorem 2) for linear codes by the way that we used to prove the main theorem.

## 2. Proof of the main theorem

For a subspace $D$ of $V$ and for a subset $R \subseteq N = \{1,2,\dots,n\}$, let

$$\begin{aligned} D(R) &:= \{u \in D \mid \mathrm{supp}(u) \subseteq R\}, \\ D^* &:= \mathrm{Hom}_F(D,F) \cong V/D^\perp. \end{aligned}$$

Clearly $D(R) = D \cap V(R)$ is a subspace of $V$ and $|V(R)| = q^{|R|}$. Then there is an exact sequence of $F$-vector spaces ([5]):

$$0 \longrightarrow C^\perp(R) \xrightarrow{\mathrm{inc}} V(R) \xrightarrow{f} C^* \xrightarrow{\mathrm{res}} C(N-R)^* \longrightarrow 0,$$

where the map $f$ is defined by

$$f : v \longmapsto (\hat{v} : u \mapsto \langle u, v\rangle).$$

Thus

$$|C| \cdot |C^\perp(R)| = |V(R)| \cdot |C(N-R)|. \tag{1}$$

(This identity can be proved directly, too.)

We now introduce another weight enumerator as follows:

$$\widetilde{W}_C^{(\lambda)}(x,y) := \sum_{R\subseteq N} |C(R)|^\lambda x^{n-|R|}y^{|R|}. \tag{2}$$

Then by (1),

$$\widetilde{W}_{C^\perp}^{(\lambda)}(x,y) = \sum_{R\subseteq N} |C^\perp(R)|^\lambda x^{n-|R|}y^{|R|}$$

$$\begin{aligned}
&= \frac{1}{|C|^\lambda} \sum_{R \subseteq N} |V(R)|^\lambda \cdot |C(N-R)|^\lambda x^{n-|R|} y^{|R|} \\
&= \frac{1}{|C|^\lambda} \sum_{R \subseteq N} |C(N-R)|^\lambda x^{n-|R|} (q^\lambda y)^{|R|} \\
&= \frac{1}{|C|^\lambda} \sum_{R \subseteq N} |C(R)|^\lambda x^{|R|} (q^\lambda y)^{n-|R|} \\
&= \widetilde{W}_C^{(\lambda)}(q^\lambda y, x). \qquad (3)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\widetilde{W}_C^{(\lambda)}(x,y) &= \sum_{R \subseteq N} |C(R)|^\lambda x^{n-|R|} y^{|R|} \\
&= \sum_{R \subseteq N} \left( \sum_{S \subseteq R} \sharp\{u \in C \mid \mathrm{supp}(u) = S\} \right)^\lambda x^{n-|R|} y^{|R|} \\
&= \sum_{R \subseteq N} \sum_{S_1, \ldots, S_\lambda \subseteq R} \sharp\{(u^{(1)}, \ldots, u^{(\lambda)}) \mid \\
&\qquad u^{(j)} \in C, \mathrm{supp}(u^{(j)}) = S_j\} x^{n-|R|} y^{|R|} \\
&= \sum_{S_1, \ldots, S_\lambda \subseteq N} \Big\{ \sharp\{(u^{(1)}, \ldots, u^{(\lambda)}) \mid u^{(j)} \in C, \mathrm{supp}(u^{(j)}) = S_j\} \\
&\qquad \times \sum_{S_1 \cup \cdots \cup S_\lambda \subseteq R \subseteq N} x^{n-|R|} y^{|R|} \Big\} \\
&= \sum_{S_1, \ldots, S_\lambda \subseteq N} \Big\{ \sharp\{(u^{(1)}, \ldots, u^{(\lambda)}) \mid u^{(j)} \in C, \mathrm{supp}(u^{(j)}) = S_j\} \\
&\qquad \times y^{|S_1 \cup \cdots \cup S_\lambda|} (x+y)^{n-|S_1 \cup \cdots \cup S_\lambda|} \Big\} \\
&= W_C^{(\lambda)}(x+y, y),
\end{aligned}$$

where we used the binomial identity

$$\begin{aligned}
\sum_{T \subseteq R \subseteq N} x^{n-|R|} y^{|R|} &= \sum_{R' \subseteq N-T} x^{n-|T|-|R'|} y^{|T|+|R'|} \\
&= y^{|T|} (x+y)^{n-|T|}. \qquad (4)
\end{aligned}$$

By (3) and (4), we have

$$W_{C^\perp}^{(\lambda)}(x,y) = \widetilde{W}_{C^\perp}^{(\lambda)}(x-y, y)$$

$$= \frac{1}{|C|^\lambda}\widetilde{W}_C^{(\lambda)}(q^\lambda y, x-y)$$

$$= \frac{1}{|C|^\lambda}W_C^{(\lambda)}(x+(q^\lambda-1)y, x-y).$$

The theorem is proved.

*Remark.* We do not need to assume that $\lambda$ is a positive integer in the definition (1) of the weight enumerator $\widetilde{W}_C^\lambda(x,y)$ contrary to the definition of $W_C^\lambda(x,y)$. The MacWilliams type identity (3) for $\widetilde{W}_C^\lambda(x,y)$ is true even for any complex number $\lambda$.

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## References

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