Hokkaido Mathematical Journal Vol. 26 (1997) p. 669–684

Weak convergence on the first exit time of randomly perturbed dynamical systems with a repulsive equilibrium point

Toshio MIKAMI

(Received August 27, 1996)

Abstract. We show that the first exit times of small random perturbations of dynamical systems from a bounded domain D ($\subset \mathbf{R}^d$) weakly converge to the life time of an explosive diffusion process and that the mean first exit times converge to the mean explosion time, as random perturbations disappear, when they are appropriately scaled. We consider the case when D contains only one equilibrium point o of dynamical systems and when o is polynomially unstabe and is repulsive.

Key words: small random perturbations of dynamical systems, first exit time, weak convergence, polynomially unstabe, repulsive.

1. Introduction

Let us consider the following stochastic differential equation: for $t > 0$, $x \in \mathbf{R}^d$ and $\varepsilon > 0$,

$$\begin{aligned} dX^\varepsilon(t,x) &= b(X^\varepsilon(t,x))dt + \varepsilon^{1/2}\sigma(X^\varepsilon(t,x))dW(t), \\ X^\varepsilon(0,x) &= x, \end{aligned} \tag{1.1}$$

where $b(\cdot) = (b^i(\cdot))_{i=1}^d : \mathbf{R}^d \mapsto \mathbf{R}^d$ is bounded and globally Lipschitz continuous, where $\sigma(\cdot) = (\sigma^{ij}(\cdot))_{i,j=1}^d : \mathbf{R}^d \mapsto M_d(\mathbf{R})$ is bounded, globally Lipschitz continuous, and uniformly nondegenerate, and where $W(\cdot)$ is a d-dimensional Wiener process (see [9]). $X^\varepsilon(t,x)$ can be considered as the small random perturbations of $X^0(t,x)$ for small ε (see [4, 5, 15]).

Let D ($\subset \mathbf{R}^d$) be a bounded domain which contains the origin o and suppose that $b(x) = o$ iff $x = o$. Then the asymptotic behavior, as $\varepsilon \to 0$, of the first exit time $\tau_D^\varepsilon(x)$ of $X^\varepsilon(t,x)$ from D defined by

$$\tau_D^\varepsilon(x) \equiv \inf\{t > 0; X^\varepsilon(t,x) \notin D\} \tag{1.2}$$

has been studied by many authors.

When $X^0(t,x) \to o$ as $t \to \infty$ for all $x \in \overline{D}$, it is studied by Freidlin

1991 *Mathematics Subject Classification* : 60F10.

and Wentzell [15], Day [1], and Fleming and James [3].

When o is a hyperbolic equilibrium point (see [7]) of $X^0(t,x)$ and when it is unstable, it is considered by Kifer [10], Mikami [13], and Day [2].

Mikami [14] considered the following case.

(A.0) D ($\subset \mathbf{R}^d$) is a bounded domain which contains o. For any $x \in \overline{D}\backslash\{o\}$ there exists $s = s(x) \geq 0$ such that $X^0(t,x) \notin \overline{D}$ for $t > s$ and such that $X^0(t,x) \in D$ for $t < s$, and $X^0(t,x) \to o$ as $t \to -\infty$.

(A.1) There exist positive constants ℓ, C_0 and δ_0 such that $U_{\delta_0}(o) \equiv \{y \in \mathbf{R}^d; |y| < \delta_0\} \subset D$ and that

$$\begin{aligned} |b(x)| \leq C_0|x|^{\ell+1} \quad & \text{for all} \quad x \in D, \\ < x, b(x) > \geq |x|^{\ell+2}/C_0 \quad & \text{for all } x \text{ for which } |x| < \delta_0. \end{aligned} \tag{1.3}$$

Remark 1. Kifer [10], Mikami [13], and Day [2] considered the case when $|b(x)| \sim |x|$ as $x \to o$ and when $b(x)$ is differentiable at o. If $d = 1$, $D = (\alpha, \beta)$ ($\alpha < 0 < \beta$) and

$$b(x) = \begin{cases} x(2 + \sin(1/x)) \ ; & \text{if} \quad x \neq o, \\ o \ ; & \text{if} \quad x = o, \end{cases}$$

then (A.0)–(A.1) hold with $\ell = 0$, $C_0 = 3$, and $|b(x)| \sim |x|$ as $x \to o$, but $b(x)$ is not differentiable at o. The case when $b(x)$ is not differentiable at o has not been studied yet. Mikami [14] and this paper consider the case $|b(x)| \sim |x|^{\ell+1}$ as $x \to o$ for $\ell > 0$ (see Example 1 in section 2). In this case $Db(o)$ is a zero matrix, and henceforth o is not a hyperbolic equilibrium point of $X^0(t,x)$.

In Kifer's case, $\tau_D^\varepsilon(o) \sim -\log\varepsilon$ as $\varepsilon \to 0$.

Under (A.0)–(A.1), the following is known.

Theorem 1 (I) *Suppose that* (A.0)–(A.1) *hold. Then for any* $\delta > 0$,

$$\lim_{\varepsilon \to 0} P(\varepsilon^{-(1-\delta)\ell/(\ell+2)} \leq \tau_D^\varepsilon(o) \leq \varepsilon^{-(1+\delta)\ell/(\ell+2)}) = 1 \tag{1.4}$$

(*see Corollary* 1.2, [14]).

(II) *Suppose that* (A.0) *holds. Then for any* $x \in D\backslash\{o\}$ *and* $\delta > 0$,

$$\lim_{\varepsilon \to 0} P(|\tau_D^\varepsilon(x) - \tau_D^0(x)| < \delta) = 1. \tag{1.5}$$

(*see* [5]).

Theorem 2 (Theorem 1.3 and Corollary 1.4, [14]). *Suppose that* (A.0)–(A.1) *hold. Then*

$$\lim_{\varepsilon \to 0} \{\log E[\tau_D^\varepsilon(o)]\} / \log(\varepsilon^{-\ell/(\ell+2)}) = 1, \tag{1.6}$$

and for any $x \in D \backslash \{o\}$,

$$\lim_{\varepsilon \to 0} E[\tau_D^\varepsilon(x)] = \tau_D^0(x). \tag{1.7}$$

In this paper we show that $\varepsilon^{\ell/(\ell+2)} \tau_D^\varepsilon(o)$ weakly converge, as $\varepsilon \to 0$, to the life time $\tau(o)$ of an explosive diffusion process and that $E[\varepsilon^{\ell/(\ell+2)} \tau_D^\varepsilon(o)]$ converge, as $\varepsilon \to 0$, to $E[\tau(o)]$, under the stronger assumption than (A.0)–(A.1).

In section 2 we state our result and prove it in section 4, using lemmas which are stated and proved in section 3.

2. Main result

In this section we state our result. Let us first introduce the assumption.
(H.0)=(A.0).

(H.1) There exists $\ell > 0$ such that $b(x)$ can be written as follows; $b(x) = B(x) + R(x)$ for $x \in D$. Here $B(x)$ and $R(x)$ are locally Lipschitz continuous functions which satisfy the following;

$$B(tx) = t^{\ell+1} B(x) \quad \text{for all} \quad t > 0 \quad \text{and} \quad x \in \mathbf{R}^d, \tag{2.1}$$

and there exist C_1 and $\gamma_1 \in (0, 1]$ such that

$$R(x) \le C_1 |x|^{\ell+1+\gamma_1} \quad \text{for all} \quad x \in \mathbf{R}^d. \tag{2.2}$$

(H.2)

$$\inf_{|x|=1} < B(x), x >> 0. \tag{2.3}$$

Remark 2. (H.1) holds with $\gamma_1 = 1$ when $\ell \in \mathbf{N}$, $b \in C_b^{\ell+2}(\mathbf{R}^d; \mathbf{R}^d)$, and

$$\partial^k b^i(o) / \partial x_1^{j_1} \cdots \partial x_d^{j_d} = 0$$

for all $i = 1, \ldots, d$, $k = 0, \ldots, \ell$ and $(j_1, \ldots, j_d)$ for which $j_1 + \cdots + j_d = k$ and $j_m \ge 0$ $(m = 1, \ldots, d)$, by Taylor's expansion. It also holds if $b(x) = |x|^\ell x (\notin C_b^{\ell+1}(\mathbf{R}^d; \mathbf{R}^d))$ for $\ell \in \mathbf{N}$.

Let $Y^{\varepsilon}(t,x)$ $(t \geq 0, x \in \mathbf{R}^d, \varepsilon > 0)$ be the solution of the following stochastic differential equation, up to the life time (see [9]):

$$\begin{aligned} dY^{\varepsilon}(t,x) &= B(Y^{\varepsilon}(t,x))dt + \varepsilon^{1/2}\sigma(o)dW(t), \\ Y^{\varepsilon}(0,x) &= x, \end{aligned} \tag{2.4}$$

and denote by $\tau(x)$ the life time of $\{Y^1(t,x)\}_{0\leq t}$:

$$\tau(x) \equiv \inf\Big\{t > 0;\ \sup_{0\leq s\leq t} |Y^1(s,x)| = \infty\Big\} \tag{2.5}$$

Then we can prove the following.

Theorem 3 *Suppose that* (H.0)–(H.2) *hold. Then* $\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(o)$ *weakly converges to* $\tau(o)$ *as* $\varepsilon \to 0$, *and*

$$\lim_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^{\varepsilon}(o)] = E[\tau(o)] < \infty. \tag{2.6}$$

As a corollary to Theorem 3, we easily get the following whose proof is omitted.

Corollary 1 *Suppose that* (H.0)–(H.2) *hold. Then* $\tau_D^{\varepsilon}(o)/E[\tau_D^{\varepsilon}(o)]$ *weakly converges to* $\tau(o)/E[\tau(o)]$ *as* $\varepsilon \to 0$.

Remark 3. (H.2) implies that $P(\tau(x) < \infty) = 1$ for all $x \in \mathbf{R}^d$ by Has'minskii's test (see [8, 12]). It is easy to see that (H.1)–(H.2) imply (A.1) in section 0, and hence that (1.4)–(1.7) holds under (H.0)–(H.2).

Let us give an example which shows that (H.1)–(H.2) is stronger than (A.1).

Example 1. Let $d = 1$ and $D = (\alpha, \beta)$ $(\alpha < 0 < \beta)$ and put for $\ell > 0$

$$b(x) = \begin{cases} |x|^{\ell}x(2 + \sin(1/x))\ ; & \text{if } x \neq o \text{ and } \alpha < x < \beta, \\ o & ;\ \text{if } x = o. \end{cases} \tag{2.7}$$

Then this $b(x)$ satisfies (A.1) with $C_0 = 3$, but does not satisfy (H.1). In fact, $b(x)/x^{\ell+1}$ does not converge as $x \downarrow 0$, though

$$\lim_{x\downarrow 0} b(x)/x^{\ell+1} = B(1) \tag{2.8}$$

if (H.1) holds.

3. Lemmas

In this section we state and prove lemmas.

Let us give some notation.

Put, for $t \geq 0$, $x \in \mathbf{R}^d$ and $\varepsilon > 0$,

$$
\begin{aligned}
Z_1^\varepsilon(t,x) &\equiv \varepsilon^{-1/(\ell+2)} X^\varepsilon(\varepsilon^{-\ell/(\ell+2)}t, \varepsilon^{1/(\ell+2)}x), \\
Z_2^\varepsilon(t,x) &\equiv \varepsilon^{-1/(\ell+2)} Y^\varepsilon(\varepsilon^{-\ell/(\ell+2)}t, \varepsilon^{1/(\ell+2)}x)
\end{aligned}
\tag{3.1}
$$

(see Eqs (1.1) and (2.4)).

The following lemma plays a crucial role in the proof of Theorem 3.

Lemma 1 *Suppose that* (H.0)–(H.1) *hold. Then there exist positive constants* $C(\sigma)$ *and* $C(r)$ $(r > 0)$, *and a one-dimensional Wiener process* $\tilde{W}(\cdot)$ *such that for* $t > 0$, $x \in \mathbf{R}^d$, $r > 0$ *and* ε (< 1) *for which* $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$,

$$
\begin{aligned}
&\sup_{0 \leq s \leq t} |Z_1^\varepsilon(s,x) - Z_2^\varepsilon(s,x)| \\
&\qquad \leq \exp(C(r)t)\varepsilon^{\gamma_1/(\ell+2)}(1 + tC_1 r^{\ell+1+\gamma_1} + C(\sigma)^2 r^2 t/2 \\
&\qquad\quad + C(\sigma) r \sup_{0 \leq u \leq t} |\tilde{W}(u)|),
\end{aligned}
\tag{3.2}
$$

as far as $\max(|Z_1^\varepsilon(s,x)|, |Z_2^\varepsilon(s,x)|) \leq r$ *for all* $s \in [0,t]$.

Proof. Suppose that $\max(|Z_1^\varepsilon(s,x)|, |Z_2^\varepsilon(s,x)|) \leq r$ for all $s \in [0,t]$. Then there exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that for $s \in [0,t]$,

$$
\begin{aligned}
&|Z_1^\varepsilon(s,x) - Z_2^\varepsilon(s,x)| \\
&\quad \leq \varepsilon^{1/(\ell+2)} + tC_1 r^{\ell+1+\gamma_1}\varepsilon^{\gamma_1/(\ell+2)} + C(\sigma)^2 r^2 t \varepsilon^{1/(\ell+2)}/2 \\
&\quad\quad + C(\sigma) r \varepsilon^{1/(\ell+2)} \sup_{0 \leq u \leq s} |\tilde{W}(u)| + \int_0^s C(r)|Z^\varepsilon(u,x) - Z(u,x)|du
\end{aligned}
\tag{3.3}
$$

from (H.1), which will be proved later. Here we put

$$
\begin{aligned}
C(r) &\equiv \sup\{|B(x') - B(x)|/|x' - x|; x' \neq x, |x'|, |x| \leq r\}, \\
C(\sigma) &\equiv \sup\left\{\left[\sum_{i,j=1}^d |\sigma^{ij}(x) - \sigma^{ij}(o)|^2\right]^{1/2} /|x|; x \neq o, x \in \mathbf{R}^d\right\}.
\end{aligned}
$$

By Gronwall's inequality (see [7]), the proof is over from (3.3) since $\varepsilon < 1$ and $\gamma_1 \in (0,1]$ from (H.1).

Let us prove (3.3). For any $s \in [0, \varepsilon^{-\ell/(\ell+2)}t]$, putting $y = \varepsilon^{1/(\ell+2)}x$

$$
\begin{aligned}
&|X^\varepsilon(s,y) - Y^\varepsilon(s,y)| \\
&\le \varepsilon^{2/(\ell+2)} + \int_0^s |B(X^\varepsilon(u,y)) - B(Y^\varepsilon(u,y))| du \\
&\qquad + \int_0^s |R(X^\varepsilon(u,y))| du \\
&\quad + \varepsilon^{1/2} \Big| \int_0^s < (X^\varepsilon(u,y) - Y^\varepsilon(u,y))(|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 \\
&\qquad + \varepsilon^{4/(\ell+2)})^{-1/2}, [\sigma(X^\varepsilon(u,y)) - \sigma(o)] dW(u) > \Big| \\
&\quad + (\varepsilon^{\ell/(\ell+2)}/2) \int_0^s \sum_{i,j=1}^d |\sigma^{ij}(X^\varepsilon(u,y)) - \sigma^{ij}(o)|^2 du, \qquad (3.4)
\end{aligned}
$$

since by the Ito formula (see [9]),

$$
\begin{aligned}
&(|X^\varepsilon(s,y) - Y^\varepsilon(s,y)|^2 + \varepsilon^{4/(\ell+2)})^{1/2} \\
&= \varepsilon^{2/(\ell+2)} + \int_0^s < (X^\varepsilon(u,y) - Y^\varepsilon(u,y))(|X^\varepsilon(u,y) \\
&\qquad - Y^\varepsilon(u,y)|^2 + \varepsilon^{4/(\ell+2)})^{-1/2}, B(X^\varepsilon(u,y)) \\
&\qquad - B(Y^\varepsilon(u,y)) + R(X^\varepsilon(u,y)) > du \\
&\quad + \varepsilon^{1/2} \int_0^s < (X^\varepsilon(u,y) - Y^\varepsilon(u,y))(|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 \\
&\qquad + \varepsilon^{4/(\ell+2)})^{-1/2}, [\sigma(X^\varepsilon(u,y)) - \sigma(o)] dW(u) > \\
&\quad + (\varepsilon/2) \int_0^s \sum_{i,j=1}^d |\sigma^{ij}(X^\varepsilon(u,y)) - \sigma^{ij}(o)|^2 \\
&\qquad \times (|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 + \varepsilon^{4/(\ell+2)})^{-1/2} du \\
&\quad - (\varepsilon/2) \int_0^s |(\sigma(X^\varepsilon(u,y)) - \sigma(o))^*(X^\varepsilon(u,y) - Y^\varepsilon(u,y))|^2 \\
&\qquad \times (|X^\varepsilon(u,y) - Y^\varepsilon(u,y)|^2 + \varepsilon^{4/(\ell+2)})^{-3/2} du.
\end{aligned}
$$

Here we used that $X^\varepsilon(u,y) \subset D$ $(0 \le u \le s)$ from the assumption since $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$, and denote by $\sigma(x)^*$ the transposed matrix of $\sigma(x)$.

From (3.4) and (3.5)–(3.7) below, we get (3.3).

For $s \in [0,t]$ and $y = \varepsilon^{1/(\ell+2)}x$

$$
\varepsilon^{-1/(\ell+2)} \int_0^{\varepsilon^{-\ell/(\ell+2)}s} |B(X^\varepsilon(u,y)) - B(Y^\varepsilon(u,y))| du
$$

$$= \int_0^s |B(Z_1^\varepsilon(u,x)) - B(Z_2^\varepsilon(u,x))|du \quad \text{(from (2.1))}$$
$$\le C(r) \int_0^s |Z_1^\varepsilon(u,x) - Z_2^\varepsilon(u,x)|du \tag{3.5}$$

from the assumption; and

$$\varepsilon^{-1/(\ell+2)} \int_0^{\varepsilon^{-\ell/(\ell+2)}s} |R(X^\varepsilon(u,y))|du$$
$$\le \varepsilon^{-(1+\ell)/(\ell+2)} sC_1(r\varepsilon^{1/(\ell+2)})^{\ell+1+\gamma_1} = C_1 sr^{\ell+1+\gamma_1}\varepsilon^{\gamma_1/(\ell+2)} \tag{3.6}$$

from (2.2) and the assumption; and

$$\varepsilon \int_0^{\varepsilon^{-\ell/(\ell+2)}s} \sum_{i,j=1}^d |\sigma^{ij}(X^\varepsilon(u,y)) - \sigma^{ij}(o)|^2 du$$
$$\le \varepsilon\varepsilon^{-\ell/(\ell+2)} sC(\sigma)^2 (r\varepsilon^{1/(\ell+2)})^2 = \varepsilon^{4/(\ell+2)} sC(\sigma)^2 r^2 \tag{3.7}$$

from the assumption. □

Put for $x \in D$, and $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$ (see (A.1)),

$$T_{r,\varepsilon}(x) \equiv \{t > 0; |X^\varepsilon(t,x)| \ge r\varepsilon^{1/(\ell+2)}\}, \tag{3.8}$$
$$T_\varepsilon(x) \equiv \{t > 0; |X^\varepsilon(t,x)| \ge \delta_0\}. \tag{3.9}$$

Then the following lemma can be proved by the strong Markov property (see [9]) of $X^\varepsilon(t,x)$.

Lemma 2 *Suppose that* (H.0) *holds. Then for $x \in D$, $T > 0$, $\delta \in (0,T)$, $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$,*

$$P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x) \le T)$$
$$\ge \inf_{|y|\le r} P\Big(\sup_{0\le s\le T-\delta} |Z_1^\varepsilon(s,y)| \ge r\Big)$$
$$\times \inf_{|y|\ge r\varepsilon^{1/(\ell+2)}} P\Big(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| \ge \delta_0\Big)$$
$$\times \inf_{|y|\ge\delta_0} P(\tau_D^\varepsilon(y) \le \varepsilon^{-\ell/(\ell+2)}\delta/2). \tag{3.10}$$

In particular,

$$\begin{aligned}
&P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) \\
&\quad \ge P\Big(\sup_{0\le s\le T-\delta} |Z_1^\varepsilon(s,o)| \ge r\Big) \\
&\qquad \times \inf_{|y|\ge r\varepsilon^{1/(\ell+2)}} P\Big(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| \ge \delta_0\Big) \\
&\qquad \times \inf_{|y|\ge \delta_0} P(\tau_D^\varepsilon(y) \le \varepsilon^{-\ell/(\ell+2)}\delta/2).
\end{aligned} \tag{3.11}$$

Proof. For $x \in D$, $T > 0$, $\delta \in (0,T)$, $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$,

$$\begin{aligned}
&P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x) \le T) \\
&\quad \ge P\Big(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}(T-\delta)} |X^\varepsilon(s,x)| \ge r\varepsilon^{1/(\ell+2)}, \\
&\qquad \sup_{T_{r,\varepsilon}(x)\le s\le T_{r,\varepsilon}(x)+\varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,x)| \ge \delta_0, \\
&\qquad \tau_D^\varepsilon(X^\varepsilon(T_\varepsilon(X^\varepsilon(T_{r,\varepsilon}(x),x)), X^\varepsilon(T_{r,\varepsilon}(x),x))) \le \varepsilon^{-\ell/(\ell+2)}\delta/2\Big).
\end{aligned} \tag{3.12}$$

By the strong Markov property of $X^\varepsilon(t,x)$, the proof is over. □

The following lemma can be proved in the same way as in Lemma 2.4 in [14] and omit the proof.

Lemma 3 *Suppose that* (A.0)–(A.1) *hold. Then for any* $\delta > 0$, $r \ge (2^{\ell+1}C_0/[\ell\delta])^{1/\ell}$ *and* y *for which* $|y| \ge r\varepsilon^{1/(\ell+2)}$,

$$\begin{aligned}
&P\Big(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| < \delta_0\Big) \\
&\quad \le P\Big(\sup\Big\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} < X^\varepsilon(s,y), \\
&\qquad \sigma(X^\varepsilon(s,y))dW(s) > |; 0 \le t \le \varepsilon^{-\ell/(\ell+2)}\delta/2, n \ge 1\Big\} \ge |y|/2\Big).
\end{aligned} \tag{3.13}$$

4. Proof of main result

In this section, we prove Theorem 3 in section 2. We devide the proof into the four steps; for any $T > 0$,

$$\limsup_{\varepsilon\to 0} P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) \le P(\tau(o) \le T), \tag{4.1}$$

$$\liminf_{\varepsilon\to 0} P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) \ \ge\ P(\tau(o) \le T), \tag{4.2}$$

$$\liminf_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] \ \ge\ E[\tau(o)], \tag{4.3}$$

$$\limsup_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] \ \le\ E[\tau(o)] < \infty. \tag{4.4}$$

Let us first prove (4.1).

Proof of (4.1). For $r > 0$ and $\varepsilon(< 1)$ for which $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$,

$$\begin{aligned}
&P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T)\\
&\le P\Big(\sup_{0\le s\le \varepsilon^{-\ell/(\ell+2)}T} |X^\varepsilon(s,o)| \ge r\varepsilon^{1/(\ell+2)}\Big)\\
&\le P\Big(\sup_{0\le s\le T} |Z_2^\varepsilon(s,o)| \ge r-1\Big)\\
&\qquad + P\Big(\sup_{0\le s\le T} |Z_1^\varepsilon(s,o)| \ge r,\ \sup_{0\le s\le T} |Z_2^\varepsilon(s,o)| \le r-1\Big)
\end{aligned} \tag{4.5}$$

(see Eq (3.1)).

The first probability in the last part of (4.5) is shown to converge to $P(\tau \le T)$ as $r \to \infty$;

$$\begin{aligned}
&P\Big(\sup_{0\le s\le T} |Z_2^\varepsilon(s,o)| \ge r-1\Big)\\
&\quad = P\Big(\sup_{0\le s\le T} |Y^1(s,o)| \ge r-1\Big) \to P(\tau(o) \le T),
\end{aligned} \tag{4.6}$$

as $r \to \infty$ since the probability law of $Z_2^\varepsilon(s,o)$ $(0 \le s)$ is the same as that of $Y^1(s,o)$ $(0 \le s)$ from (H.1) (see Eqs (2.4) and (3.1)).

The second probability in the last part of (4.5) converges to 0 as $\varepsilon \to 0$ for any r and $T > 0$. Let us prove this. Put for r, $\varepsilon > 0$, $y \in \mathbf{R}^d$ and $i = 1, 2$

$$\tau_r^{i,\varepsilon}(y) \equiv \{t > 0; |Z_i^\varepsilon(t,y)| \ge r\} \tag{4.7}$$

(see Eq (3.1)). Suppose that $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$. Then from Lemma 1, there exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that

$$\begin{aligned}
&P\Big(\sup_{0\le s\le T} |Z_1^\varepsilon(s,o)| \ge r,\ \sup_{0\le s\le T} |Z_2^\varepsilon(s,o)| \le r-1\Big)\\
&\le P(\tau_r^{1,\varepsilon}(o) \le T < \tau_{r-1}^{2,\varepsilon}(o), 1 \le |Z_1^\varepsilon(\tau_r^{1,\varepsilon}(o),o) - Z_2^\varepsilon(\tau_r^{1,\varepsilon}(o),o)|)\\
&\le P\Big(1 \le \exp(C(r)T)\varepsilon^{\gamma_1/(\ell+2)}(1 + TC_1 r^{\ell+1+\gamma_1} + C(\sigma)^2 r^2 T/2
\end{aligned}$$

$$+ C(\sigma) r \sup_{0 \le u \le T} |\tilde{W}(u)|)\Big) \to 0 \quad \text{as} \ \ \varepsilon \to 0. \tag{4.8}$$

□

Next we prove (4.2).

Proof of (4.2). From Lemma 2, for $\delta \in (0, T)$, $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$,

$$\begin{aligned} &P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le T) \\ &\quad \ge P\Big(\sup_{0 \le s \le T-\delta} |Z_1^\varepsilon(s, o)| \ge r\Big) \\ &\qquad \times \inf_{|y| \ge r\varepsilon^{1/(\ell+2)}} P\Big(\sup_{0 \le s \le \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s, y)| \ge \delta_0\Big) \\ &\qquad \times \inf_{|y| \ge \delta_0} P(\tau_D^\varepsilon(y) \le \varepsilon^{-\ell/(\ell+2)}\delta/2). \end{aligned} \tag{4.9}$$

The following (4.10)–(4.12) which are proved later complete the proof.

$$\liminf_{\delta \to 0} \liminf_{r \to \infty} \liminf_{\varepsilon \to 0} P\Big(\sup_{0 \le s \le T-\delta} |Z_1^\varepsilon(s, o)| \ge r) \ge P(\tau(o) \le T\Big) \tag{4.10}$$

; and for any $\delta \in (0, T)$,

$$\lim_{r \to \infty} \liminf_{\varepsilon \to 0} \inf_{|y| \ge r\varepsilon^{1/(\ell+2)}} P\Big(\sup_{0 \le s \le \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s, y)| \ge \delta_0\Big) = 1 \tag{4.11}$$

; and for any $\delta \in (0, T)$,

$$\lim_{\varepsilon \to 0} \inf_{|y| \ge \delta_0} P(\tau_D^\varepsilon(y) \le \varepsilon^{-\ell/(\ell+2)}\delta/2) = 1. \tag{4.12}$$

Let us first prove Eq (4.12). From (H.0), there exist $\delta_1 > 0$ and $T_1 > 0$ such that

$$\inf_{|x| \ge \delta_0} \sup\{dist(X^0(t, x), D); 0 \le t \le T_1\} \ge 2\delta_1. \tag{4.13}$$

Therefore

$$\begin{aligned} &P(\tau_D^\varepsilon(y) \le \varepsilon^{-\ell/(\ell+2)}\delta/2) \\ &\qquad \ge P\Big(\sup_{0 \le t \le T_1} |X^\varepsilon(s, y) - X^0(s, y)| < \delta_1\Big) \to 1 \quad \text{as} \ \ \varepsilon \to 0, \end{aligned} \tag{4.14}$$

uniformly in y for which $|y| \geq \delta_0$ (see [5]).

Next we prove (4.10).

For any r and $\delta \in (0, T)$

$$
\begin{aligned}
&P\Big(\sup_{0\leq s\leq T-\delta} |Z_1^\varepsilon(s,o)| \geq r\Big) \\
&\geq P\Big(\sup_{0\leq s\leq T-\delta} |Z_2^\varepsilon(s,o)| \geq r+1\Big) \qquad (4.15)\\
&\quad - P\Big(\sup_{0\leq s\leq T-\delta} |Z_2^\varepsilon(s,o)| \geq r+1, \sup_{0\leq s\leq T-\delta} |Z_1^\varepsilon(s,o)| < r\Big).
\end{aligned}
$$

The first probability on the right hand side of (4.15) converges to $P(\tau(o) \leq T)$ as $r \to \infty$, and then $\delta \to 0$, in the same way as in (4.6).

Let us prove that the second probability on the right hand side of (4.15) converges to 0 as $\varepsilon \to 0$ for any $r > 0$ and $\delta \in (0, T)$, which can be done in the same way as in (4.8). Suppose that $U_{r\varepsilon^{1/(\ell+2)}}(o) \subset D$. Then from Lemma 1, there exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that

$$
\begin{aligned}
&P\Big(\sup_{0\leq s\leq T-\delta} |Z_2^\varepsilon(s,o)| \geq r+1, \sup_{0\leq s\leq T-\delta} |Z_1^\varepsilon(s,o)| < r\Big) \\
&\leq P(\tau_{r+1}^{2,\varepsilon}(o) \leq T-\delta < \tau_r^{1,\varepsilon}(o), 1 \\
&\leq |Z_2^\varepsilon(\tau_{r+1}^{2,\varepsilon}(o), o) - Z_1^\varepsilon(\tau_{r+1}^{2,\varepsilon}(o), o)|) \\
&\leq P\Big(1 \leq \exp(C(r+1)T)\varepsilon^{\gamma_1/(\ell+2)}(1 + TC_1[r+1]^{\ell+1+\gamma_1} \\
&\qquad + C(\sigma)^2(r+1)^2T/2 + C(\sigma)(r+1)\sup_{0\leq u\leq T}|\tilde{W}(u)|)\Big) \\
&\to 0 \quad \text{as } \varepsilon \to 0. \qquad (4.16)
\end{aligned}
$$

Finally we prove Eq (4.11) from Lemma 3. Since (H.1)–(H.2) is stronger than (A.1) (see Remark 3 in section 2), we can suppose that (A.1) holds.

For $C_0 > 0$ in (A.1), $r \geq (2^{\ell+1}C_0/[\ell\delta])^{1/\ell}$ and y for which $|y| \geq r\varepsilon^{1/(\ell+2)}$,

$$
\begin{aligned}
&P\Big(\sup_{0\leq s\leq \varepsilon^{-\ell/(\ell+2)}\delta/2} |X^\varepsilon(s,y)| < \delta_0\Big) \\
&\leq P\Big(\sup\Big\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} \\
&\qquad < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > |; \\
&\qquad\qquad 0 \leq t \leq \varepsilon^{-\ell/(\ell+2)}\delta/2, n \geq 1\Big\} \geq |y|/2\Big) \qquad (4.17)
\end{aligned}
$$

from Lemma 3; and by the time change (see [9]), there exists a one dimensional Wiener process $\tilde{W}$ such that

$$\begin{aligned}
&P\Big(\sup\Big\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y),\\
&\quad \sigma(X^\varepsilon(s,y))dW(s) > |; 0 \le t \le \varepsilon^{-\ell/(\ell+2)}\delta/2, n \ge 1\Big\} \ge |y|/2\Big)\\
&\le P\Big(C_1(\sigma)(2\delta)^{1/2}/r \sup_{0\le s\le 1}|\tilde{W}(s)| \ge 1\Big)\\
&\to 0 \quad \text{as} \quad r \to \infty. \qquad (4.18)
\end{aligned}$$

Here we put $C_1(\sigma) = \sup_{x\in\mathbf{R}^d}(\sum_{i,j=1}^d \sigma^{ij}(x)^2)^{1/2}$. □

Let us prove (4.3).

Proof of (4.3). For any $T > 0$,

$$\liminf_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] \ge \int_0^T P(\tau(o) > t)dt \qquad (4.19)$$

from (4.1) by Fatou's lemma, since

$$\begin{aligned}
E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] &= \int_0^\infty P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt\\
&\ge \int_0^T P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt. \qquad (4.20)
\end{aligned}$$

Let $T \to \infty$ in (4.19). Then we get (4.3). □

Finally we prove (4.4).

Proof of (4.4). We only have to show the following to complete the proof;

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n] = 0 \qquad (4.21)$$

since for any $n \in \mathbf{N}$,

$$\begin{aligned}
&E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)]\\
&= E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o); \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n] - \int_0^n t dP(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)\\
&= E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o); \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n]\\
&\qquad - nP(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n) + \int_0^n P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt\\
&= E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n]
\end{aligned}$$

$$+\int_0^n P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt \tag{4.22}$$

; and

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_0^n P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > t)dt \\
&\quad\le \int_0^n P(\tau(o) > t)dt \quad \text{(from (4.2) by Fatou's lemma)} \\
&\quad\to E[\tau(o)] \quad \text{as} \quad n\to\infty
\end{aligned} \tag{4.23}$$

; and for $n \in \mathbf{N}$, from (4.3) and (4.22)–(4.23),

$$\begin{aligned}
E[\tau(o)] &\le \liminf_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)] \\
&\le \limsup_{\varepsilon\to 0} E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n] \\
&\qquad + \int_0^n P(\tau(o) > t)dt.
\end{aligned} \tag{4.24}$$

To prove (4.21), we only have to prove

$$\limsup_{\varepsilon\to 0}\left[\sup_{x\in D} P(1 < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x))\right] < 1 \tag{4.25}$$

since

$$\begin{aligned}
&E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) > n] \\
&\quad= \sum_{k=n}^\infty E[\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) - n; k < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le k+1] \\
&\quad\le \sum_{k=n}^\infty (k+1-n)P(k < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o) \le k+1) \\
&\quad\le \sum_{k=n}^\infty P(k < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(o)) \\
&\quad\le \sum_{k=n}^\infty \left\{\sup_{x\in D} P(1 < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x))\right\}^k \\
&\quad= \left\{\sup_{x\in D} P(1 < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x))\right\}^n \Big(1 - \sup_{x\in D} P(1 < \varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x))\Big)^{-1}
\end{aligned}$$

by the strong Markov property of $X^\varepsilon(t,x)$.

Though (4.25) can be shown to be true in the same way as in the proof of (4.2), we prove it for the sake of completeness.

From Lemma 2, for $r > 0$ and $\varepsilon > 0$ for which $r\varepsilon^{1/(\ell+2)} < \delta_0$, and $x \in D$,

$$\begin{aligned} &P(\varepsilon^{\ell/(\ell+2)}\tau_D^\varepsilon(x) \leq 1) \\ &\quad \geq \inf_{|y|\leq r} P\Big(\sup_{0\leq s\leq 1/2} |Z_1^\varepsilon(s,y)| \geq r\Big) \\ &\qquad \times \inf_{|y|\geq r\varepsilon^{1/(\ell+2)}} P\Big(\sup_{0\leq s\leq \varepsilon^{-\ell/(\ell+2)}/4} |X^\varepsilon(s,y)| \geq \delta_0\Big) \\ &\qquad \times \inf_{|y|\geq\delta_0} P(\tau_D^\varepsilon(y) \leq \varepsilon^{-\ell/(\ell+2)}/4). \end{aligned} \tag{4.26}$$

Take sufficiently large $r_0 > 0$ so that

$$\liminf_{\varepsilon\to 0} \inf_{|y|\geq r_0\varepsilon^{1/(\ell+2)}} P\Big(\sup_{0\leq s\leq \varepsilon^{-\ell/(\ell+2)}/4} |X^\varepsilon(s,y)| \geq \delta_0\Big) \geq 1/2, \tag{4.27}$$

which is possible from (4.11).

From (4.27) and (4.12), we only have to show the following;

$$\liminf_{\varepsilon\to 0} \inf_{|y|\leq r_0} P\Big(\sup_{0\leq s\leq 1/2} |Z_1^\varepsilon(s,y)| \geq r_0\Big) > 0, \tag{4.28}$$

which can be proved in the same way as in (4.15)–(4.16).

In fact, for any y for which $|y| \leq r_0$

$$\begin{aligned} &P\Big(\sup_{0\leq s\leq 1/2} |Z_1^\varepsilon(s,y)| \geq r_0\Big) \\ &\quad \geq P\Big(\sup_{0\leq s\leq 1/2} |Z_2^\varepsilon(s,y)| \geq r_0 + 1\Big) \\ &\qquad - P\Big(\sup_{0\leq s\leq 1/2} |Z_2^\varepsilon(s,y)| \geq r_0 + 1, \sup_{0\leq s\leq 1/2} |Z_1^\varepsilon(s,y)| < r_0\Big). \end{aligned} \tag{4.29}$$

The first probability on the right hand side of (4.29) can be shown to be bounded from below by a positive constant, uniformly in $\varepsilon > 0$ and y for which $|y| \leq r_0$;

$$\begin{aligned} &P\Big(\sup_{0\leq s\leq 1/2} |Z_2^\varepsilon(s,y)| \geq r_0 + 1\Big) \\ &\quad = P\Big(\sup_{0\leq s\leq 1/2} |Y^1(s,y)| \geq r_0 + 1\Big) \\ &\quad \geq \inf_{|z|\leq r_0} P\Big(\sup_{0\leq s\leq 1/2} |Y^1(s,z)| \geq r_0 + 1\Big) > 0. \end{aligned}$$

This is true since $u(t,x) \equiv P(\sup_{0\le s\le t}|Y^1(s,x)| \ge r_0+1)$ is a classical solution of the following P.D.E. from (H.1) (see p. 383, [6], problems 1 and 2);

$$\begin{aligned}
\partial u(t,x)/\partial t &= \left[\sum_{i,j=1}^{d} a^{ij}(o)\partial^2 u(t,x)/\partial x_i \partial x_j\right]/2 \\
&\quad + \sum_{i=1}^{d} B^i(x)\partial u(t,x)/\partial x_i \quad \text{for } t>0,\ |x|<r_0+1, \\
u(0,x) &= 0 \quad \text{for } |x|<r_0+1, \\
u(t,x) &= 1 \quad \text{for } t\ge 0,\ |x|=r_0+1.
\end{aligned}$$

Let us prove that the second probability on the right hand side of (4.29) converges to 0 as $\varepsilon \to 0$, uniformly in y for which $|y| \le r_0$. From Lemma 1, there exists a one-dimensional Wiener process $\tilde{W}(\cdot)$ such that for y for which $|y| \le r_0$

$$\begin{aligned}
&P\Big(\sup_{0\le s\le 1/2}|Z_2^\varepsilon(s,y)| \ge r_0+1,\ \sup_{0\le s\le 1/2}|Z_1^\varepsilon(s,y)| < r_0\Big) \\
&\le P(\tau_{r_0+1}^{2,\varepsilon}(y) \le 1/2 < \tau_{r_0}^{1,\varepsilon}(y), 1 \le |Z_2^\varepsilon(\tau_{r_0+1}^{2,\varepsilon}(y),y) \\
&\quad - Z_1^\varepsilon(\tau_{r_0+1}^{2,\varepsilon}(y),y)|) \\
&\le P\Big(1 \le \exp(C(r_0+1)/2)\varepsilon^{\gamma_1/(\ell+2)}(1+C_1[r_0+1]^{\ell+1+\gamma_1}/2 \\
&\quad + C(\sigma)^2(r_0+1)^2/4 + C(\sigma)(r_0+1)\sup_{0\le u\le 1/2}|\tilde{W}(u)|)\Big) \\
&\to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

□

Acknowledgment The author would like to thank Prof. A. A. Novikov for suggesting him to consider the problem discussed in this paper.

References

[1] Day M.V., *On the exponential exit law in the small parameter exit problem.* Stochastics **8** (1983), 297–323.

[2] Day M.V., *On the exit law from saddle points.* Stoc. Proc. and Appl. **60** (1995), 287–311.

[3] Fleming W.H. and James M.R., *Asymptotic series and exit time probabilities.* Ann. Probab. **20** (1992), 1369–1384.

[4] Freidlin M.I., *Functional integration and partial differential equations.* Princeton University Press, Princeton, 1985.

[5] Freidlin M.I. and Wentzell A.D., *Random perturbations of dynamical systems.* Springer, Berlin, Heidelberg, New York, Tokyo, 1984.

[6] Friedman A., *Stochastic differential equations and applications*, **2**. Academic press, New York, 1976.

[7] Hartman P., *Ordinary differential equations.* John Wiley & Sons, New York, London, Sydney, 1964.

[8] Has'minskii R.Z., *Ergodic properties of Recurrent Diffusion Processes and Stabilization of the Solution of the Cauchy Problem for Parabolic Equations.* Theory Probab. Appl. **5** (1960), 179–196.

[9] Ikeda N. and Watanabe S., *Stochastic differential equations and diffusion processes.* 2nd ed. North-Holland/Kodansha, Amsterdam, New York, Oxford, Tokyo, 1989.

[10] Kifer Y., *The exit problem for small random perturbations of dynamical systems with a hyperbolic fixed point.* Israel J. Math. **40** (1981), 74–96.

[11] Kifer Y., *Random perturbations of dynamical systems.* Birkhäuser, Boston, Basel, 1988.

[12] McKean Jr H.P., *Stochastic Integral.* Academic Press, New York, London, 1969.

[13] Mikami T., *Large deviations for the first exit time on small random perturbations of dynamical systems with a hyperbolic equilibrium point.* Hokkaido Math. J. **24** (1995), 491–525.

[14] Mikami T., *Asymptotic behavior of the first exit time of randomly perturbed dynamical systems with a repulsive equilibrium point.* To appear in J. Math. Soc. Japan.

[15] Wentzell A.D. and Freidlin M.I., *On small random perturbations of dynamical systems.* Russian Math. Survey **25** (1970), 1–55.

Department of Mathematics
Hokkaido University
8 Nishi, 10 Kita, Kita-ku,
Sapporo 060, Japan
E-mail: mikami@math.sci.hokudai.ac.jp