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Remark on fundamental solution for vorticity equation of two dimensional Navier – Stokes flows

(Dedicated to Professor Kôji Kubota on his sixtieth birthday)

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Abstract. In this paper we treat a perturbed heat equation related to the vorticity equation for the Navier–Stokes flow in $\boldsymbol{R}^2$. We get estimate for the fundamental solution of this equation. We note that estimate like ours played the essential role in the paper by Giga, Miyakawa and Osada [4] where they discussed existence of solution for Navier–Stokes equation in $\boldsymbol{R}^2$ with measure as initial vorticity.

Key words: the incompressible Navier–Stokes equations, vorticity equation, fundamental solution, 2 dimensional flow.

1. Introduction and Results

Consider the incompressible Navier–Stokes equations in two dimensional Euclidean space $\boldsymbol{R}^2$:

$$\text{(NS)} \quad \begin{cases} u_t - \nu\Delta u + (u,\nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0 & \text{in } (0,\infty)\times\boldsymbol{R}^2, \\ \qquad u|_{t=0} = u_0 \quad \text{in } \boldsymbol{R}^2, \end{cases}$$

where $u = u(t,x) = (u_1(t,x), u_2(t,x))$ is the velocity vector field, $p = p(t,x)$ is the pressure, $\nu > 0$ is the kinematic viscosity, $u_t = \partial u/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$ and $\operatorname{div} u = \partial u_1/\partial x_1 + \partial u_2/\partial x_2$.

For the vorticity $\omega(t,x) = \operatorname{rot} u(t,x) = \partial u_1/\partial x_2 - \partial u_2/\partial x_1$, we reduce (NS) to the following equations by the well known Biot–Savart law:

$$\text{(NSR)} \quad \begin{cases} \omega_t - \nu\Delta\omega + (u,\nabla)\omega = 0, \quad u(t,x) = \mathbf{K} * \omega(t,x) \\ \qquad\qquad \text{in } (0,\infty)\times\boldsymbol{R}^2, \\ \omega|_{t=0} = \omega_0 \equiv \operatorname{rot} u_0 \quad \text{in } \boldsymbol{R}^2, \end{cases}$$

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where the kernel $\mathbf{K}(x) = x^\perp/(2\pi|x|^2)$ with $x^\perp = (-x_2, x_1)$. The symbol $*$ means a convolution with respect to space variable x. That is, for two functions $f = f(t,x)$ and $g = g(t,x)$ which may be independent of time variable t, we define

$$f(t) * g(t) = \int_{\boldsymbol{R}^2} f(t, x-y) \cdot g(t,y)\, dy.$$

Here we note that $\operatorname{rot}(\mathbf{K} * f) = f$ and $\operatorname{div}(\mathbf{K} * f) = 0$ formally.

Here we summarize notations which we need throughout this paper. A function or a vector field $f = f(t,x)$ is denoted by $f(t)$ for simplicity. If $f = f(x)$, we denote only f. The Banach space $L^p(\boldsymbol{R}^2)$ represents scalar or $\boldsymbol{R}^2$ valued Lebesgue's space with exponent p and we use $||\cdot||_p$ for its norm. We say that a vector field $f(t,x) = (f_1, f_2)$ is in $B^{1,1}_\sigma((0,T] \times \boldsymbol{R}^2)$, if $f(t)$ and all its derivatives are bounded and continuous in $(0,T] \times \boldsymbol{R}^2$ and $f(t)$ satisfies $\operatorname{div} f = 0$ in $(0,T] \times \boldsymbol{R}^2$.

In [4] Giga, Miyakawa and Osada constructed a global solution to (NS) when initial vorticity ω_0 is a integrable function (i.e. $\omega_0 \in L^1(\boldsymbol{R}^2)$) or more generally a finite Radon measure by solving (NSR). Note that no smallness assumption on ω_0 was imposed there and that u_0 may not be square integrable even locally. They also proved that their solution is unique when $\omega_0 \in L^2(\boldsymbol{R}^2)$. However for general finite Radon measure ω_0 the uniqueness of solution seems to be a still open problem. Later their proof was simplified by Kato [5]. A difference proof was given by Ben–Artzi [1] when $\omega_0 \in L^1(\boldsymbol{R}^2)$. An extention to bounded domain with zero boundary vorticity was given in Miyakawa and Yamada [6].

The method in [4] is based on the delicate estimates from above for the fundamental solution of the equation:

$$\text{(RE)} \qquad \begin{cases} \omega_t - \nu\Delta\omega + (u,\nabla)\omega = 0 & \text{in } (0,T] \times \boldsymbol{R}^2, \\ \omega|_{t=0} = \omega_0 & \text{in } \boldsymbol{R}^2 \end{cases}$$

with a given coefficient $u(t)$. To obtain their estimates, they assumed that

$$\operatorname{div} u = 0 \quad \text{and} \quad ||v(t)||_1 \leq M_0$$

with $M_0 > 0$ independent of t, where $v(t) = \operatorname{rot} u(t)$ so that $u(t) = \mathbf{K} * v(t)$. They used the special structure of $\mathbf{K}$ in $u(t) = \mathbf{K} * v(t)$ to obtain their estimate. However, it is not clear in what may their constant depend on M_0. The purpose of our paper is to establish similar estimate under the assumptions that $\operatorname{div} u = 0$ and

$$\sup_{0 \leq t \leq T} \sqrt{t} \cdot ||u(t)||_\infty \leq M \tag{1.1}$$

for some positive constant M (instead of $||v(t)||_1 \leq M_0$) with explicit dependence of constants in M. Our main result is

Theorem 1 *Assume that the coefficient $u \in B^{1,1}_\sigma((0,T] \times \boldsymbol{R}^2)$ satisfies* (1.1). *Then the fundamental solution $\Gamma_u(t,x;s,y)$ for* (RE) *satisfies:*

$$\Gamma_u(t,x;s,y) \leq \frac{Ce^{K_1 M^2}}{\nu\delta(t-s)} \cdot \exp\left(-\frac{K_2|x-y|^2}{\nu(t-s)}\right)$$

for $0 \leq s < t \leq T$ and $x, y \in \boldsymbol{R}^2$ with a numerical constant C, where the constants K_1 and K_2 are obtained as

$$K_1 = \frac{2(1+\delta)}{\nu(\sqrt{N}-2)} \quad \text{and} \quad K_2 = \frac{1}{N(1+\delta)}$$

for any $0 < \nu$, $\delta \leq 1$ and any $N > 4$.

Note that one can take $K_2 < 1/4$ as close as $1/4$ which is the constant appeared in exponent of the standard Gauss kernel. Here and hereafter we denote by C or C_j numerical positive constants ($j = 0, 1, \cdots$). Their value may differ from one occasion to another.

Similar estimate was given in [4] with assumption $||v(t)||_1 \leq M_0$. However, so just mentioned before K_1 and K_2 may depend on M_0 in [4]. To show Theorem 1, we essentially use the methods developed by Nash [7] and prove it along the way in Fabes and Strook [3] (see also [2]) with some simplification. Although our result applies to the general dimension with standard modification, we restrict ourselves into two dimensional case.

In [5] Kato obtained the unique global solution $\omega(t)$ of (NSR) which is smooth for $t > 0$, $\omega(0) = \omega_0$ and satisfeis

$$||\omega(t)||_p \leq C_1 \cdot t^{1-1/p} ||\omega_0||_1$$

for $1 \leq p \leq \infty$. By the Calderón–Zygmund inequality $||\nabla u||_r \leq C_2 ||\operatorname{rot} u||_r$ for $1 < r < \infty$ and the Gagliardo–Nierenberg inequality this estimates implies (1.1) with $u(t) = K * \omega(t)$, $M = C_0 ||\omega_0||_1$ and $T = \infty$. Our Theorem 1 yields

Theorem 2 *Let $\omega(t)$ be the unique global solution for* (NSR) *and $u(t) =$*

$\mathbf{K} * \omega(t)$. *Then we obtain*

$$\Gamma_u(t, x; s, y) \leq \frac{Ce^{CK_1\|\omega_0\|_1^2}}{\nu\delta(t-s)} \cdot \exp\left(-\frac{K_2|x-y|^2}{\nu(t-s)}\right)$$

for $0 \leq s < t < \infty$ *and* $x, y \in \boldsymbol{R}^2$, *where* K_1, K_2, ν *and* δ *are in Theorem* 1.

2. Proof of Theorem 1

Here we prove Theorem 1 along the way in [3]. Let $A = \nu\Delta - (u(t), \nabla)$ and $A_\psi = e^{-\psi}Ae^{\psi}$ for $\psi(x) = \alpha \cdot x$ (the inner product of vectors $\alpha, x \in \boldsymbol{R}^2$). Then we have

Lemma 2.1 *Let* f *be a non negative rapidly decreasing function in* $\boldsymbol{R}^2$, p *be a natural number and* $0 \leq t \leq T$. *Then we obtain*

$$\int_{\boldsymbol{R}^2} A_\psi f \cdot f^{2p-1}\, dx \leq -\frac{C\nu}{p} \cdot \frac{\|f\|_{2p}^{4p}}{\|f\|_p^{2p}} + q_p(t) \cdot \|f\|_{2p}^{2p},$$

here $q_p(t) = p\nu|\alpha|^2 + M|\alpha|/\sqrt{t}$.

Proof. By simple calculus we have

$$\begin{aligned} \int_{\boldsymbol{R}^2} A_\psi f \cdot f^{2p-1}\, dx &= \nu \int_{\boldsymbol{R}^2} e^{-\psi}\Delta\left(e^{\psi} f\right) \cdot f^{2p-1}\, dx \\ &\quad - \int_{\boldsymbol{R}^2} (u, \nabla)\left(e^{\psi} f\right) \cdot e^{-\psi} f^{2p-1}\, dx \\ &\equiv \nu \cdot I_1 - I_2. \end{aligned}$$

In I_1 we use the integral by parts, then we have

$$\begin{aligned} I_1 &= -\int_{\boldsymbol{R}^2} \nabla\left(e^{\psi} f\right) \cdot \nabla\left(e^{-\psi} f^{2p-1}\right) dx \\ &= -(2p-1)\int_{\boldsymbol{R}^2} f^{2p-2}\left|\nabla f\right|^2 dx \\ &\quad - 2(p-1)\int_{\boldsymbol{R}^2} f^{2p-1}\alpha \cdot \nabla f\, dx + |\alpha|^2 \int_{\boldsymbol{R}^2} f^{2p}\, dx. \end{aligned}$$

If $p \geq 2$, we get $f^{2p-2}\left|\nabla f\right|^2 = \left|\nabla\left(f^p\right)\right|^2/p^2$ and

$$2(p-1)\left|f^{2p-1}\alpha \cdot \nabla f\right| \leq 2(p-1)\left\{|\alpha| f^p \cdot f^{p-1}|\nabla f|\right\}$$

$$\le (p-1)\cdot|\alpha|^2 f^{2p} + (p-1)\cdot f^{2p-2}|\nabla f|^2$$
$$= (p-1)\cdot|\alpha|^2 f^{2p} + \frac{p-1}{p^2}\cdot|\nabla(f^p)|^2.$$

Thus I_1 satisfies

$$I_1 \le -\frac{1}{p}\int_{\boldsymbol{R}^2}|\nabla(f^p)|^2\,dx + p|\alpha|^2\int_{\boldsymbol{R}^2} f^{2p}\,dx.$$

Since the case $p=1$ is trivial, this estimate is valid for $p \ge 1$.

For I_2 we obtain

$$\begin{aligned} I_2 &= \int_{\boldsymbol{R}^2} e^{\psi} f\cdot\{-(u\cdot\alpha)e^{-\psi}f^{2p-1} + (2p-1)f^{2p-2}e^{-\psi}(u,\nabla)f\}\,dx \\ &= -\int_{\boldsymbol{R}^2}(u\cdot\alpha)f^{2p}\,dx + (2p-1)\int_{\boldsymbol{R}^2} f^{2p-1}(u,\nabla)f\,dx \\ &= -\int_{\boldsymbol{R}^2}(u\cdot\alpha)f^{2p}\,dx, \end{aligned}$$

here we use $\operatorname{div} u = 0$. So by the assumption (1.1) we get

$$|I_2| \le \left|\int_{\boldsymbol{R}^2}(u\cdot\alpha)f^{2p}\,dx\right| \le \frac{M|\alpha|}{\sqrt{t}}\int_{\boldsymbol{R}^2} f^{2p}\,dx$$

Combining these estimates, we arrive at

$$\int_{\boldsymbol{R}^2} A_\psi f\cdot f^{2p-1}\,dx \le -\frac{\nu}{p}\int_{\boldsymbol{R}^2}|\nabla(f^p)|^2\,dx + q_p(t)\int_{\boldsymbol{R}^2} f^{2p}\,dx. \tag{2.1}$$

Furthermore by Gagriado–Nirenberg inequality

$$||f||_2^2 \le C||f||_1\cdot||\nabla f||_2$$

holds. Replacing f by f^p, we get

$$||f||_{2p}^{2p} \le C||f||_p^p\cdot||\nabla(f^p)||_2.$$

Hence we obtain (see, [7])

$$||\nabla(f^p)||_2^2 = \int_{\boldsymbol{R}^2}|\nabla(f^p)|^2\,dx \ge \frac{1}{C}\cdot\frac{||f||_{2p}^{4p}}{||f||_p^{2p}}$$

This and (2.1) prove our lemma. □

For a non negative rapidly decreasing function $f = f(x)$ we put

$$F(t) = F(t,x) = e^{-\psi(x)} \int_{\boldsymbol{R}^2} \Gamma_u(t,x;0,y) e^{\psi(y)} f(y)\, dy.$$

Since Γ_u is the fundamental solution of (RE), $F(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}||F(t)||_{2p}^{2p} &= 2p \int_{\boldsymbol{R}^2} \frac{dF}{dt}(t) \cdot (F(t))^{2p-1}\, dx \\ &= 2p \int_{\boldsymbol{R}^2} A_\psi F(t) \cdot (F(t))^{2p-1}\, dx\end{aligned}$$

for any natural number p. On the other hand, we have

$$\frac{d}{dt}||F(t)||_{2p}^{2p} = 2p||F(t)||_{2p}^{2p-1} \cdot \frac{d}{dt}||F(t)||_{2p}.$$

Thus, by Lemma 2.1, we obtain

$$\frac{d}{dt}||F(t)||_{2p} \leq -\frac{C\nu}{p} \cdot \frac{||F(t)||_{2p}^{2p+1}}{||F(t)||_p^{2p}} + q_p(t) \cdot ||F(t)||_{2p} \tag{2.2}$$

If $p = 1$, neglecting the first term in the right hand side of (2.2) and applying the Gronwall's inequality, (2.2) implies

$$||F(t)||_2 \leq \exp\left(\int_0^t q_1(s)\, ds\right) \cdot ||f||_2 = e^{Q(t)} \cdot ||f||_2, \tag{2.3}$$

where we use $F(0) = f$ and we define a new function $Q(t) = \nu|\alpha|^2 t + 2M|\alpha|\sqrt{t}$.

In the case of $p \geq 2$, we apply the following lemma on differential inequality to (2.2).

Lemma 2.2 *Assume that $g(t) \in L^1(0,T)$ and $h(t)$ on $[0,T]$ hold*

$$\int_0^t h(s) \cdot \exp\left(2p \int_0^s g(\theta)\, d\theta\right) ds > 0$$

for any $t \in [0,T]$ and a natural number p. If a function $u \in C^1([0,T])$ satisfies

$$\frac{d}{dt}u(t) \leq -h(t) \cdot u^{1+2p}(t) + g(t) \cdot u(t)$$

for any $t \in [0,T]$*, then we obtain*

$$(u(t))^{2p} \le \frac{\exp\left(2p\int_0^t g(s)\,ds\right)}{2p\int_0^t h(s)\cdot\exp\left(2p\int_0^s g(\theta)\;d\theta\right)\,ds}$$

for any $t \in [0,T]$.

Proof. Putting $v(t) = u(t)\cdot e^{-\int_0^s g(s)\,ds}$, then the differential inequality for $u(t)$ implies

$$\begin{aligned}\frac{d}{dt}(v^{-2p}) &= -2pv^{-2p-1}\left(u' - g\right)e^{-\int_0^t g(s)\,ds} \\ &\ge 2pe^{(2p+1)\int_0^t g(s)\,ds}u^{-2p-1}\cdot hu^{2p+1}e^{-\int_0^t g(s)\,ds} \\ &= 2ph(t)e^{2p\int_0^t g(s)\,ds}.\end{aligned}$$

Thus integrating in $[0,t]$ and neglecting $1/u^{2p}(0)$, we have

$$\frac{e^{2p\int_0^t g(s)\,ds}}{u^{2p}(t)} \ge 2p\int_0^t \;h(s)e^{2p\int_0^s g(\theta)\;d\theta}\,ds.$$

Hence we get our assertion. □

Applying Lemma 2.2 to (2.2) with $p \ge 2$ as $u(t) = ||F(t)||_p$ and $q(t) = q_p(t)$, we obtain

$$||F(t)||_{2p}^{2p} \le \frac{e^{2pQ_p(t)}}{\int_0^t C\nu||F(s)||_p^{-2p}\cdot e^{2pQ_p(s)}\,ds} \tag{2.4}$$

for $t \in [0,T]$, where $Q_p(t) \equiv \int_0^t q_p(s)\,ds = p\nu|\alpha|^2\cdot t + 2M|\alpha|\cdot\sqrt{t}$. Now we set

$$w_p(t) \equiv \;\sup\{s^{(p-2)/(2p)}\cdot||F(s)||_p;\; 0 \le s \le t\}$$

and obtain

$$\begin{aligned}&\int_0^t C\nu||F(s)||_p^{-2p}\cdot e^{2pQ_p(s)}\,ds \\ &\qquad\qquad \ge C\nu\left(w_p(t)\right)^{-2p}\cdot\int_0^t s^{p-2}\cdot e^{2pQ_p(s)}\,ds.\end{aligned}$$

Moreover for $\kappa = 1 - \delta/(p^2)$ with $0 < \delta \leq p^2$, we have

$$\int_0^t s^{p-2} \cdot e^{2pQ_p(s)}\, ds \geq \int_{\kappa t}^t s^{p-2} \cdot e^{2pQ_p(s)}\, ds \geq e^{2pQ_p(\kappa t)} \cdot \int_{\kappa t}^t s^{p-2}\, ds$$
$$= e^{2pQ_p(\kappa t)} \cdot \frac{(1-\kappa^{p-1})t^{p-1}}{p-1}.$$

Hence from (2.4) it follows that

$$(t^{((2p)-2)/(2\cdot(2p))} \cdot \|F(t)\|_{2p})^{2p} \leq \frac{(p-1)\cdot(w_p(t))^{2p}}{C\nu\cdot(1-\kappa^{p-1})} \cdot e^{2p(Q_p(t)-Q_p(\kappa t))}.$$

Since $1-\kappa^{p-1} \geq 1-\kappa = \delta/(p^2)$, we have $2p(Q_p(t) - Q_p(\kappa t)) \leq 2\delta Q(t)$. Thus we get

$$(t^{((2p)-2)/(2\cdot(2p))} \cdot \|F(t)\|_{2p})^{2p} \leq \frac{p^2(p-1)\cdot(w_p(t))^{2p}}{C\nu\delta} \cdot e^{2\delta Q(t)}$$
$$\leq \frac{p^3\cdot(w_p(t))^{2p}}{C\nu\delta} \cdot e^{2\delta Q(t)}.$$

This arrives at

$$\frac{w_{2p}(t)}{w_p(t)} \leq \left(\frac{p^3}{C\nu\delta}\right)^{1/(2p)} \cdot e^{(\delta/p)Q(t)} \tag{2.5}$$

for $0 \leq t \leq T$. Here for $p = 2^k$ we put $v_k(t) = w_p(t)$ provided that k is a natural number. Now we use (2.5) inductively to get

$$\sup_{k\geq 1} v_k(t) \leq \sup_{k\geq 1} 8^{A_k} \cdot (C\nu\delta)^{B_k} \cdot e^{\delta Q(t)C_k} \cdot v_1(t)$$
$$\leq \frac{C_1}{\sqrt{\nu\delta}} e^{\delta Q(t)} \cdot v_1(t),$$

where $A_k = \sum_{j=1}^k j2^{-(j-1)}$, $B_k = \sum_{j=1}^k 2^{-(j+1)}$, and $C_k = \sum_{j=1}^k 2^{-j}$.

Since $v_k(t) = \sup\ s^{(p-2)/(2p)} \cdot \|F(t)\|_p$ with $p = 2^k$, this estimate and (2.3) imply that

$$\|F(t)\|_\infty \leq \frac{C}{\sqrt{\nu\delta t}} e^{(1+\delta)Q(t)} \cdot \|f\|_2 \tag{2.6}$$

for $F(t,x) = e^{-\alpha\cdot x} \int_{\boldsymbol{R}^2} \Gamma_u(t,x;0,y) e^{\alpha\cdot y} f(y)\, dy$.

Now we prove the estimate in Theorem 1. We define a operator $\mathcal{F}_u(t) : L^2(\boldsymbol{R}^2) \to L^\infty(\boldsymbol{R}^2)$ by $\mathcal{F}_u(t)f = F(t,x)$. From (2.6) we have

$$||\mathcal{F}_u(t)f||_\infty \leq \frac{C}{\sqrt{\nu\delta t}} e^{(1+\delta)Q(t)} \cdot ||f||_2.$$

At the same time, since the fundamental solution which define the adjoint operator $(\mathcal{F}_u(t))^* : L^1(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}^2)$ equals to $\Gamma_{(-u)}$, then we can see that $(\mathcal{F}_u(t))^*$ is operator from $L^2(\boldsymbol{R}^2)$ to $L^\infty(\boldsymbol{R}^2)$. So we also obtain

$$||(\mathcal{F}_u(t))^* f||_\infty \leq \frac{C}{\sqrt{\nu\delta t}} e^{(1+\delta)Q(t)} \cdot ||f||_2.$$

Thus by duality

$$||\mathcal{F}_u(t)f||_2 \leq \frac{C}{\sqrt{\nu\delta t}} e^{(1+\delta)Q(t)} \cdot ||f||_1.$$

Here, we put $v(\cdot) = u(\cdot + t)$. Then we have $\mathcal{F}_u(2t) = \mathcal{F}_v(t) \circ \mathcal{F}_u(t)$. Hence we obtain

$$||\mathcal{F}_u(2t)f||_\infty \leq \frac{C}{\sqrt{\nu\delta t}} e^{(1+\delta)Q(t)} \cdot ||\mathcal{F}_v(t)f||_2 \leq \frac{C^2}{\nu\delta t} e^{2(1+\delta)Q(t)} \cdot ||f||_1.$$

In this we put $f(y) = \rho_\varepsilon(y-z)$ for Friedrichs' mollifier ρ_ε and let $\varepsilon \to 0$, then we get

$$\Gamma_u(2t, x; 0, z) \leq \frac{C^2}{\nu\delta t} e^{2(1+\delta)Q(t)+\alpha\cdot(x-z)}. \tag{2.7}$$

In (2.7) we put $\alpha = -\mu(x-z)/t$ for any positive μ, then we have

$$\begin{aligned} &2(1+\delta)Q(t) + \alpha\cdot(x-z) \\ &\quad = \{2\nu(1+\delta)\mu^2 - \mu\} \cdot \frac{|x-z|^2}{t} + 4(1+\delta)M\mu \cdot \frac{|x-z|}{\sqrt{t}}. \end{aligned} \tag{2.8}$$

Furthermore for any positive ε we have

$$4(1+\delta)M\mu \cdot \frac{|x-z|}{\sqrt{t}} \leq \frac{4(1+\delta)^2M^2\mu}{\varepsilon} + \varepsilon\mu \cdot \frac{|x-z|^2}{t}.$$

Here we put $\mu = 1/(2\sqrt{N}\nu(1+\delta))$ and $\varepsilon = 1 - 2/\sqrt{N}$ for any $N > 4$. Then we obtain

$$2(1+\delta)Q(t) + \alpha\cdot(x-z) \leq \frac{-|x-z|^2}{2N\nu(1+\delta)t} + \frac{2(1+\delta)}{\nu(\sqrt{N}-2)} \cdot M^2$$

Hence by (2.7) we conclude

$$\Gamma_u(t,x;0,z) \leq \frac{2C^2 e^{K_1 M^2}}{\nu\delta t} e^{-K_2|x-z|^2/(\nu t)},$$

where the constants K_1 and K_2 are as follows

$$K_1 = \frac{2(1+\delta)}{\nu(\sqrt{N}-2)} \quad \text{and} \quad K_2 = \frac{1}{N(1+\delta)}.$$

This proves our theorem 1.

Remark. From the proof we have

$$\Gamma_u(t,x;s,y) \leq \frac{C}{\nu\delta(t-s)} \cdot \exp\left\{ -\frac{1}{\nu(1+\delta)} \left(\frac{1}{\sqrt{N}} - \frac{1}{N} \right) \cdot \frac{|x-y|^2}{t-s} + \frac{2^{3/2}M}{\sqrt{N}} \cdot \frac{|x-y|^2}{\sqrt{t-s}} \right\}$$

for any $N > 0$. This follows from (2.3) and (2.8) and $\mu = 1/(2\sqrt{N}\nu(1+\delta))$.

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