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Stability of symmetric systems under hyperbolic perturbations

(Dedicated to Professor Rentaro Agemi on his sixtieth birthday)

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Abstract. Let $L(x)$ be the symbol of a $m\times m$ symmetric first order hyperbolic system with real constant coefficients. The range of $L(x)$ is a subspace, containing a positive definite $L(\theta)$, in the linear space of dimension $d(m)=m(m+1)/2$ of all $m\times m$ real symmetric matrices. We study a hyperbolic perturbation $\tilde{L}(x)=L(x)+R(x)$ of $L(x)$, that is $R(x)$ is $O(|x|^2)$ $(x\to 0)$ which is real analytic and all eigenvalues λ of $\tilde{L}(x+\lambda\theta)$ are real near the origin. We prove that if the dimension of the range of $L(x)$ is greater than $d(m)-m+2$, then generically, every such hyperbolic perturbation is trivial, namely there are real analytic $A(x)$, $B(x)$ near the origin with $A(0)B(0)=I$ such that $A(x)\tilde{L}(x)B(x)$ becomes symmetric. When $m=3$, the same conclusion holds if the range is greater than 3.

Key words: hyperbolic perturbation, symmetric system, non-degenerate.

1. Introduction

Let

$$\mathcal{L}(x)=\sum_{j=1}^{n}A_jx_j,\quad x=(x_1,\dots,x_n),$$

where A_j are real symmetric $m\times m$ matrices which are linearly independent. Since we are interested in hyperbolic systems we assume that $\mathcal{L}(\Theta)$ is positive definite with some $\Theta\in\mathbf{R}^n$. We may suppose that $\mathcal{L}(\Theta)=I$ considering $\mathcal{L}(\Theta)^{-1/2}\mathcal{L}(x)\mathcal{L}(\Theta)^{-1/2}$. The range $\mathcal{L}=\{\mathcal{L}(x)\mid x\in\mathbf{R}^n\}$ of $\mathcal{L}(x)$ is a linear subspace in $M^s(m,\mathbf{R})$, the space of all real symmetric $m\times m$ matrices. Note that the range contains the identity I and of n dimensional because A_j are linearly independent.

We study the symbol $\mathcal{P}(x)$ of a hyperbolic system which is *close* to $\mathcal{L}(x)$ near $x=0$;

$$\mathcal{P}(x)=\mathcal{L}(x)+R(x)$$

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where $R(x) = O(|x|^2)$ as $x \to 0$ which is real analytic near the origin and all eigenvalues λ of $\mathcal{P}(x + \lambda\Theta)$ are real near $x = 0$.

By Theorem 4.2 in [9], every hyperbolic perturbation is trivial if the dimension of the range $\mathcal{L}$ is maximal, that is $n = m(m+1)/2 = d(m)$ in the sense that there are real analytic $A(x)$, $B(x)$ defined near the origin with $A(0)B(0) = I$ such that $A(x)\mathcal{P}(x)B(x)$ becomes symmetric. Our aim in this note is to study symmetric systems $\mathcal{L}(x)$ whose range have dimension less than $d(m)$.

Theorem 1.1 *Assume $d(m) - m + 3 \le n \le d(m)$. Then in the $(d(m) - n)(n - 1)$ dimensional Grassmannian of n dimensional subspaces of $M^s(m, \mathbf{R})$ containing the identity, the subset for which hyperbolic perturbations are trivial is an open and dense subset.*

Here we have identified a symmetric matrix $\mathcal{L}(x)$ with its range $\mathcal{L}$ because the assertion is independent of linear changes of coordinates x.

In Section 2, reexamining the proof and the hypotheses of the above mentioned result in [9] we show that: Let us denote by $S_{\mathcal{L}}(x)$ the linear map sending a $H \in M^s(m, \mathbf{R})$ with zero diagonal elements to an anti-symmetric $[\mathcal{L}(x), H]$. Let

$$\det S_{\mathcal{L}}(x) = \prod_{j=1}^{s} g_j(x)^{r_j}$$

be the irreducible factorization of $\det S_{\mathcal{L}}(x)$ in $\mathbf{R}[x]$. Then assuming that

$$\{x | g_j(x) = 0\},\ 1 \le j \le s, \text{ contains a regular point} \tag{1.2}$$

and that every characteristic of order less than m of $\mathcal{L}(x)$ is *non-degenerate* (see Definition 2.1) we can conclude that all hyperbolic perturbations are trivial (Theorem 2.1).

To check these two conditions, in Section 3, we study characteristics of $\mathcal{L}(x)$ and we prove that, in the Grassmannian of n dimensional subspaces of $M^s(m, \mathbf{R})$ containing the identity, the subset for which every characteristic of order less than m is non-degenerate is an open and dense subset (Proposition 3.3).

In Section 4, in this Grassmannian of n dimensional subspaces, we show that the set for which the condition (1.2) is fulfilled is an open and dense subset if $n \ge d(m) - m + 3$ (Proposition 4.1).

The last restriction on n comes from purely technical reasons in proving

Proposition 4.1 and it is plausible that it could be weakened. Indeed, if $m = 3$, Theorem 1.1 holds for $n \geq 4$:

Theorem 1.2 *Assume that $m = 3$ and $4 \leq n \leq 6 = d(3)$. Then in the $(6-n)(n-1)$-dimensional Grassmannian of n dimensional subspaces of $M^s(3, \mathbf{R})$ containing the identity, the subset for which hyperbolic perturbations are trivial is an open and dense subset.*

The proof will be given in Section 5. We can find detailed studies on the structure of 6-dimensional Grassmannian of 4-dimensional subspaces of $M^s(3, \mathbf{R})$ containing the identity in Theorems 3.5 and 3.6 in [4].

2. Non-degenerate characteristics

We first make precise the notion of non-degenerate characteristics of order greater than two (see [8], [9]). Let $\mathcal{P}(x)$ be a real analytic function with values in $M(m, \mathbf{R})$, the set of all real matrices of order m, defined near the origin of $\mathbf{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$. Let $x = \bar{x}$ be a characteristic of $\mathcal{P}(x)$, that is $\bar{x}$ is a zero of $\det \mathcal{P}(x)$. Assume that

$$\operatorname{Ker} \mathcal{P}(\bar{x}) \cap \operatorname{Im} \mathcal{P}(\bar{x}) = \{0\}. \tag{2.1}$$

In this case we can define the localization $\mathcal{P}_{\bar{x}}(x)$ of $\mathcal{P}(x)$ at $\bar{x}$ as follows (see Definition 3.1 in [8], see also [10], [1]). The assumption (2.1) identifies $\operatorname{Coker} \mathcal{P}(\bar{x})$ and $\operatorname{Ker} \mathcal{P}(\bar{x})$. Since $d\mathcal{P}(x)$, the differential of $\mathcal{P}$ at $\bar{x}$, is a well defined map going from $\operatorname{Ker} \mathcal{P}(\bar{x})$ to $\operatorname{Coker} \mathcal{P}(\bar{x})$ then the map followed by the canonical map to $\operatorname{Coker} \mathcal{P}(\bar{x})$ is identified with a map $\operatorname{Ker} \mathcal{P}(\bar{x}) \to \operatorname{Ker} \mathcal{P}(\bar{x})$, which is the localization $\mathcal{P}_{\bar{x}}(x)$. For later references we give a representation of $\mathcal{P}_{\bar{x}}(x)$ in local coordinates. Set $s = \operatorname{dimKer} \mathcal{P}(\bar{x})$. Let $\{v_1, \ldots, v_s\}$ be a basis for $\operatorname{Ker} \mathcal{P}(\bar{x})$ and let $\{\phi_1, \ldots, \phi_s\}$, $\phi_i \in (\mathbf{C}^m)^*$ be linearly independent and vanish on $\operatorname{Im} \mathcal{P}(\bar{x})$ such that $(\langle \phi_i, v_j \rangle) = I_s$. Then $\mathcal{P}_{\bar{x}}(x)$ is given by

$$(\langle \phi_i, \mathcal{P}(\bar{x} + \mu x) v_j \rangle) = \mu(\mathcal{P}_{\bar{x}}(x) + O(\mu))$$

as $\mu \to 0$.

Definition 2.1 Let $x = \bar{x}$ be a characteristic of $\mathcal{P}(x)$. We say that $\bar{x}$ is non degenerate if the following conditions are verified;
(1) $\operatorname{Ker} \mathcal{P}(\bar{x}) \cap \operatorname{Im} \mathcal{P}(\bar{x}) = \{0\}$,
(2) $\dim\{\mathcal{P}_{\bar{x}}(x) \mid x \in \mathbf{R}^n\} = s(s+1)/2$ with $s = \operatorname{dimKer} \mathcal{P}(\bar{x})$,

(3) $\mathcal{P}_{\bar{x}}(x)$ is diagonalizable for every x.
We call s the order of the characteristic $\bar{x}$.

We return to $\mathcal{L}(x)$ mentioned in Introduction. By a linear change of coordinates x we may suppose that $\Theta = (1, 0, \dots, 0)$ so that

$$\mathcal{L}(x) = x_1 I + \sum_{j=2}^{n} F^j x_j = x_1 I + L(x') \tag{2.2}$$

where $F^j \in M^s(m, \mathbf{R})$, $x' = (x_2, \dots, x_n)$ and $\{F^2, \dots, F^n, I\}$ are linearly independent.

Theorem 2.1 *Assume that every characteristic of $\mathcal{L}(x)$ of order less than m is non degenerate. Suppose that* $\det S_{\mathcal{L}}(x)$ *satisfies* (1.2). *Then for every hyperbolic perturbation $\mathcal{P}(x) = \mathcal{L}(x) + R(x)$ of $\mathcal{L}(x)$ we can find real analytic $A(x)$, $B(x)$ defined near the origin with $A(0)B(0) = I$ so that*

$$A(x)\mathcal{P}(x)B(x)$$

becomes symmetric.

Proof. By a preparation theorem for systems proved in [3, Theorem 4.3], generalizing the Weierstrass preparation theorem, one can write

$$\mathcal{P}(x + \lambda\Theta) = C(x, \lambda)(\lambda I + \mathcal{Q}(x))$$

where $C(x, \lambda)$ is real analytic near $(0, 0)$, $\det C(0, 0) \neq 0$ and $\mathcal{Q}(x)$ is real analytic with values in $M(m, \mathbf{R})$, $\mathcal{Q}(0) = O$. Comparing the first order term in the Taylor expansion at $(0, 0)$ of both sides we see that $C(0, 0) = I$ and $\mathcal{Q}(x) = \mathcal{L}(x) + \tilde{R}(x)$ where $\tilde{R}(x) = O(|x|^2)$. Taking $x' = 0$, $\lambda = -x_1$ we get that $O = C(x_1, 0, -x_1)\tilde{R}(x_1, 0)$ and hence $\tilde{R}(x_1, 0, \dots, 0) = O$. Since

$$C(x, 0)^{-1}\mathcal{P}(x) = \mathcal{L}(x) + \tilde{R}(x)$$

it is enough to study a perturbation term $R(x)$ which verifies $R(x_1, 0, \dots, 0) = O$. We also note that $C(\epsilon\Theta, 0)^{-1}\mathcal{P}(\epsilon\Theta) = \epsilon I$ for small ϵ. We set

$$P(x', x_1) = L(x') + R(x_1, x'), \quad L(x') = \sum_{j=2}^{n} F^j x_j$$

where $S_L(x')$ verifies the assumption (1.2) because $\mathcal{L}(x) - L(x') = x_1 I$. Introducing the polar coordinates $x' = r\omega$, we blow up $P(x', x_1)$ at $x' = 0$ so that $r^{-1}P(r\omega, x_1)$ will be studied. We first show that, for every fixed

$\omega \neq 0$, there is a real analytic positive definite $H_\omega(r,\theta,x_1)$ with diagonal elements 1 defined near $(0,\omega,0)$ such that

$$P(r\theta,x_1)H_\omega(r,\theta,x_1) = H_\omega(r,\theta,x_1)^t P(r\theta,x_1). \tag{2.3}$$

To prove the above assertion we can follow the same proof of Proposition 4.3 in [9] except for that of Lemma 4.7 in [9] which was proved assuming that $x=0$ is non-degenerate. We examine that the assertion of Lemma 4.7 holds under the assumptions of Theorem 2.1. We fix $\omega \neq 0$ and take an orthogonal T_0 so that $T_0^{-1}L(\omega)T_0 = \bigoplus_{i=1}^p \lambda_i I_{s_i}$ just as in the proof of Proposition 4.3. Set $\tilde{L}(\theta) = T_0^{-1}L(\theta)T_0 = (\tilde{L}_{ij}(\theta))_{1\le i,j\le p}$ and

$$\tilde{F}^j = T_0^{-1}F^jT_0 = (\tilde{F}^j_{kl})_{1\le k,l\le p}, \quad \tilde{L}_{ii}(\theta) = \sum_{j=2}^n \tilde{F}^j_{ii}\theta_j$$

where the block decomposition corresponds to that of $\oplus\lambda_i I_{s_i}$. Then it is easy to see that to prove the assertion of Lemma 4.7 it is enough to show the following. □

Lemma 2.2 *$\{I_{s_i}, \tilde{F}^j_{ii}\}$ span $M^s(s_i,\mathbf{R})$.*

Proof. Let $\tilde{\mathcal{L}}(x) = T_0^{-1}\mathcal{L}(x)T_0$. Since $(x_1,x') = (-\lambda_i,\omega)$ is a characteristic of $\tilde{\mathcal{L}}(x)$ of order less than m it is non-degenerate by assumption. It is clear that the localization of $\tilde{\mathcal{L}}(x)$ at $(-\lambda_i,\omega)$ is

$$\tilde{\mathcal{L}}_{(-\lambda_i,\omega)}(x) = x_1 I_{s_i} + \sum_{j=2}^n \tilde{F}^j_{ii}x_j$$

because $\tilde{\mathcal{L}}(-\lambda_i,\omega)$ is diagonal. Noting that the non-degeneracy of characteristics is invariant under changes of basis for $\mathbf{C}^m$ we conclude that the matrices $\{I_{s_i}, \tilde{F}^j_{ii}\}$ span $M^s(s_i,\mathbf{R})$ since the image $\tilde{\mathcal{L}}_{(-\lambda_i,\omega)}$ is s_i-dimensional. This proves the assertion. □

Thus we get $H_\omega(r,\theta,x_1)$ near every $\omega \neq 0$ verifying (2.3) with diagonal elements 1. Since $\det S_L(\theta) \neq 0$ on a dense subset then H_ω can be continued analytically to a neighborhood of $\{0\}\times S^{n-2}\times\{0\}$ yielding $H(r,\theta,x_1)$ which verifies (2.3) there (see Lemma 4.8 in [9]). We then show that there is a real analytic $G(x',x_1)$ defined near the origin such that

$$H(r,\theta,x_1) = G(r\theta,x_1), \quad G(0) = I \tag{2.4}$$

which proves that $T(x)^{-1}P(x)T(x)$ becomes symmetric with $T(x) = G(x)^{1/2}$.

Taking $A(x) = T(x)^{-1}C(x,0)^{-1}$, $B(x) = T(x)$ we obtain Theorem 2.1. Here we note that $A(\epsilon\Theta)\mathcal{P}(\epsilon\Theta)B(\epsilon\Theta) = \epsilon I$ for small ϵ. To see (2.4) we make the following observation. Let $f(\theta)$, $g(\theta)$ be homogeneous polynomials in θ of degree p, q respectively where $p \geq q$. Let

$$g(\theta) = \prod_{j=1}^{s} g_j(\theta)^{r_j}$$

be the irreducible factorization of $g(\theta)$ in $\mathbf{R}[\theta]$. We assume that $f(\theta)/g(\theta)$ is C^∞ apart from the origin and that $V_j = \{\theta | g_j(\theta) = 0\}$, $1 \leq j \leq s$ contains a regular point. Then applying Lemma 2.5 in [6] repeatedly, we conclude that $f(\theta)/g(\theta)$ is a homogeneous polynomial in θ of degree $p - q$.

Then, in the proof of Proposition 4.5 in [9], replacing Lemma 4.9 by the assumption (1.2) and the argument applying Lemma 2.5 in [6] by the above observation, we conclude (2.4) easily.

Since the non-degeneracy of characteristics is invariant under orthogonal changes of basis for $\mathbf{C}^m$ we have

Corollary 2.3 *Assume that every characteristic of $\mathcal{L}(x)$ of order less than m is non-degenerate and there is an orthogonal $T \in O(m)$ such that* $\det S_{T^{-1}\mathcal{L}T}(x)$ *verifies* (1.2). *Then the same conclusion as in* Theorem 2.1 *holds.*

Remark. The condition (1.2) is not invariant under orthogonal changes of basis for $\mathbf{C}^m$. Let

$$\mathcal{L}(x) = x_1 I_2 + \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix}.$$

Then it is obvious that $\det S_{\mathcal{L}}(x) = 0$. But it is easy to see that there is an orthogonal $T \in O(2)$ so that $\det S_{T^{-1}\mathcal{L}T}(x)$ verifies (1.2).

We remark here that the definition of non-degenerate characteristics given here is equivalent to that used in the previous papers [4], [2] for double characteristics. Let

$$\mathcal{L}(x) = x_1 I + L(x'), \quad x' = (x_2, \ldots, x_n),$$

where $L(x')$ is real analytic with values in $M(m, \mathbf{R})$ defined near $x' = \bar{x}'$ which is not necessarily linear in x'.

Lemma 2.4 *Assume that all eigenvalues of $L(x')$ are real near $x' = \bar{x}'$.*

Let $\bar{x} = (\bar{x}_1, \bar{x}')$ be a double characteristic of $\mathcal{L}(x)$. Then $\bar{x}$ is non degenerate if and only if

$$\operatorname{dimKer} \mathcal{L}(\bar{x}) = 2 \text{ and } \operatorname{rankHess} h(\bar{x}) = 3$$

where $h(x) = \det \mathcal{L}(x)$.

Proof. Take a constant matrix T so that

$$T^{-1}\mathcal{L}(\bar{x})T = \begin{pmatrix} A & O \\ O & G \end{pmatrix}$$

where G is a non singular matrix of order $m-2$ and the two eigenvalues of A are zero. Assume that $\operatorname{dimKer} \mathcal{L}(\bar{x}) = 2$ and $\operatorname{rankHess} h(\bar{x}) = 3$. Then it follows that $A = O$ and hence $\operatorname{Ker} \mathcal{L}(\bar{x}) \cap \operatorname{Im} \mathcal{L}(\bar{x}) = \{0\}$. Let $\mathcal{L}_{\bar{x}}(x)$ be the localization of $\mathcal{L}(x)$ at $\bar{x}$. Denoting $T^{-1}\mathcal{L}(x)T = (L_{ij}(x))_{1\le i,j\le 2}$ we get $L_{11}(\bar{x}+\mu x) = \mu(\mathcal{L}_{\bar{x}}(x) + O(\mu))$ as $\mu \to 0$. Then it follows that

$$h(\bar{x}+x) = \det \mathcal{L}(\bar{x}+x) = (\det G) \det \mathcal{L}_{\bar{x}}(x) + O(|x|^3) \tag{2.5}$$

as $x \to 0$. Since $\mathcal{L}_{\bar{x}}(x)$ is a 2×2 hyperbolic system and $\operatorname{rankHess}\ \det \mathcal{L}_{\bar{x}}(0) = 3$ by (2.5) then it can be symmetrized by a constant matrix by Lemma 4.1 in [7]. In particular $\mathcal{L}_{\bar{x}}(x)$ is diagonalizable for every x and $\dim\{\mathcal{L}_{\bar{x}}(x) \mid x \in \mathbf{R}^n\} = 3$. Conversely we assume that $\bar{x}$ is non degenerate in the sense of Definition 2.1. From $\operatorname{Ker} \mathcal{L}(\bar{x}) \cap \operatorname{Im} \mathcal{L}(\bar{x}) = \{0\}$ it follows that $A = O$ and hence $\operatorname{dimKer} \mathcal{L}(\bar{x}) = 2$. Since $\mathcal{L}_{\bar{x}}(x)$ is diagonalizable and $\dim \mathcal{L}_{\bar{x}} = 3$ then $\mathcal{L}_{\bar{x}}(x)$ is symmetrizable (see [2]). Thus $\operatorname{rankHess} \det \mathcal{L}_{\bar{x}}(0) = 3$ and hence $\operatorname{rankHess} h(\bar{x}) = 3$ by (2.5). □

3. Non-degenerate characteristics for symmetric systems

For symmetric systems with constant coefficients the description of non degeneracy of characteristics becomes simple. Let $\mathcal{L}(x)$ be

$$\mathcal{L}(x) = \sum_{j=1}^{n} A_j x_j$$

where $A_j \in M^s(m, \mathbf{R})$. We denote by $M^s_k(m, \mathbf{R})$ the set of all $A \in M^s(m, \mathbf{R})$ with rank $m-k$. Then we have

Lemma 3.1 *Let $\bar{x}$ be a characteristic of $\mathcal{L}(x)$ of order k. Then $\bar{x}$ is non-degenerate if and only if the range $\mathcal{L}$ intersects $M^s_k(m, \mathbf{R})$ at $\mathcal{L}(\bar{x})$ transversally.*

Proof. Since $\mathcal{L}(\bar{x})$ and $\mathcal{L}_{\bar{x}}(x)$ are symmetric, the conditions (1) and (3) in Definition 2.1 are automatically satisfied. Without restrictions we may assume that $\bar{x} = (0, \ldots, 0, 1)$. Then A_n is of rank $m - k$. We can make an orthogonal transformation of the matrices to attain that with a block matrix notation

$$A_n = \begin{pmatrix} O & O \\ O & G \end{pmatrix}$$

where G is a $(m-k) \times (m-k)$ non-singular matrix. The tangent space of $M_k^s(m, \mathbf{R})$ at A_n consists of matrices of the form

$$\begin{pmatrix} O & * \\ * & * \end{pmatrix} \tag{3.1}$$

with the corresponding block decomposition. On the other hand, with the same block decomposition of $\mathcal{L}(x)$

$$\mathcal{L}(x) = \begin{pmatrix} L_{11}(x) & L_{12}(x) \\ L_{21}(x) & L_{22}(x) \end{pmatrix}$$

it is clear that $\mathcal{L}_{\bar{x}}(x) = L_{11}(x)$. Thus the transversality of intersection means that $\dim L_{11} = d(k)$ that is, $\dim \mathcal{L}_{\bar{x}} = d(k)$ and hence $\bar{x}$ is non-degenerate. The converse follows in the same way. □

Taking Lemma 2.4 into account one sees that Lemma 3.1 generalizes Lemma 3.2 in [4].

We continue to study non-degenerate characteristics for $\mathcal{L}(x)$ in (2.2). We start with the special case that $\dim \mathcal{L} = d(m) - 1$. Since $\mathcal{L}$ has codimension one in $M^s(m, \mathbf{R})$ then $\mathcal{L}$ is defined by

$$\mathcal{L} : \ \mathrm{tr}\,(AX) = 0, \quad X = (x_{ij}), \quad x_{ij} = x_{ji} \tag{3.2}$$

with some $A \in M^s(m, \mathbf{R})$. Note that $\mathrm{tr}\, A = 0$ because $\mathcal{L}$ contains the identity. Now we have

Proposition 3.2 *Assume that $\mathcal{L}$ is given by (3.2) with $A \in M^s(m, \mathbf{R})$ and that the rank of A is greater than k. Then every characteristic of order k of $\mathcal{L}(x)$ is non-degenerate.*

Proof. Let $\bar{x}$ be a characteristic of order k of $\mathcal{L}(x)$ and hence $H = \mathcal{L}(\bar{x}) \in \mathcal{L} \cap M_k^s(m, \mathbf{R})$. Here we note that $\dim T_H M_k^s(m, \mathbf{R}) = d(m) - d(k)$ which is seen by the proof of Lemma 3.1. To show $\bar{x}$ is non-degenerate it suffices to prove that

$$\dim(\mathcal{L} \cap T_H M_k^s(m, \mathbf{R})) = d(m) - d(k) - 1 \tag{3.3}$$

by Lemma 3.1. As in the proof of Lemma 3.1, considering $T^{-1}\mathcal{L}T$ with a suitable $T \in O(m)$ we may assume that

$$H = \begin{pmatrix} O & O \\ O & G \end{pmatrix} \tag{3.4}$$

where G is a $(m-k) \times (m-k)$ non-singular matrix. Recalling that the tangent space $T_H M_k^s(m, \mathbf{R})$ is spanned by matrices of the form (3.1) we see that $\mathcal{L} \cap T_H M_k^s(m, \mathbf{R})$ consists of matrices of the form

$$X = \begin{pmatrix} O & x_{ij} \\ x_{ij} & x_{ij} \end{pmatrix}, \quad \mathrm{tr}\,(AX) = \sum_{k+1 \le j, i \le j} (2 - \delta_{ij}) a_{ij} x_{ij} = 0$$

where $A = (a_{ij})$ and δ_{ij} is the Kronecker's delta. Since A is symmetric and the rank of A is greater than k by assumption then it follows that $(a_{ij})_{k+1 \le j, i \le j} \ne O$. This proves (3.3) and hence the assertion. □

We turn to the general case that $1 \le \dim \mathcal{L} \le d(m) - 1$.

Proposition 3.3 *In the Grassmannian $G^n_{d(m),I}$ of n dimensional subspaces of $M^s(m, \mathbf{R})$ containing the identity I, the subset for which every characteristic of order less than m is non-degenerate is an open and dense subset.*

Let $\mathbf{P}^N(\mathbf{R})$ be the N dimensional real projective space and let $X \subset \mathbf{P}^N(\mathbf{R})$ be a non-singular algebraic manifold of dimension r and assume that $x_0 \notin T_x X$ for all $x \in X$. Let us denote

$$\tilde{G}^s_{N,x_0} = \{W \subset \mathbf{P}^N(\mathbf{R}) \mid W;\ \text{linear space},\ \dim W = s, x_0 \in W\}$$

and set $s' = N - s$. Then we have

Lemma 3.4 *A generic $W \in \tilde{G}^s_{N,x_0}$ intersects X transversally.*

Proof. [1] Let $Y = \{(x, W) \in X \times \tilde{G}^s_{N,x_0} \mid x \in W\}$ and denote by p_1, p_2 the projections onto X and $\tilde{G}^s_{N,x_0}$ respectively. Note that $\dim Y = s's - s' + r$ and $\dim \tilde{G}^s_{N,x_0} = s's$. Then if $r < s'$ a generic $W \in \tilde{G}^s_{N,x_0}$ does not intersect

[1]The author owes this simple proof to A.Gyoja

X and hence the result. Thus it is enough to study the case $r \geq s'$. Let us set

$$Z = \{(x, W) \in Y \mid \dim(T_x X + W) \leq N - 1\}.$$

It is not difficult to see that

$$\dim(p_1|Z)^{-1}(x) = ss' - r - 1, \quad x \in X$$

so that $\dim Z = ss' - 1 = \dim \tilde{G}^s_{N,x_0} - 1$. Thus for every W belonging to the open dense subset $\tilde{G}^s_{N,x_0} \setminus \overline{p_2(Z)}$, W intersects X transversally. This proves the assertion. □

Proof of Proposition 3.3 Take X and $\tilde{G}^s_{N,x_0}$ as the projective spaces $M^s_k(m, \mathbf{R})^{pr}$ and $(G^{s+1}_{d(m),I})^{pr}$ based on $M^s_k(m, \mathbf{R})$ and $G^{s+1}_{d(m),I}$ respectively. Applying Lemma 3.4 with $N = d(m) - 1$, $r = N - d(k)$, $x_0 = I$ we get the desired result. □

4. Condition (1.2)

As mentioned in Introduction we study $S_{\mathcal{L}}(x)$ for symmetric $\mathcal{L}(x)$ when $\dim \mathcal{L} = d(m) - \nu$ where $1 \leq \nu \leq m - 3$. We first examine a matrix representation of $S_{\mathcal{L}}(x)$. Let

$$F_m = \{H = (h_{ij}) \in M^s(m, \mathbf{R}) \mid h_{ii} = 0\}$$

then $S_{\mathcal{L}}(x)$ is defined as the linear map between two $d(m-1)$-dimensional linear subspaces F_m and $M^{as}(m, \mathbf{R})$

$$F_m \ni H \mapsto [\mathcal{L}(x), H] = K \in M^{as}(m, \mathbf{R})$$

where $M^{as}(m, \mathbf{R})$ denotes the set of all real anti-symmetric matrices of order m. Let us write

$$\mathcal{L}(x) = (\phi^i_j(x))_{1 \leq i,j \leq m}, \quad \phi^i_j(x) = \phi^j_i(x). \tag{4.1}$$

For $H \in F_m$ we write $\check{H} = {}^t(h_{12}, h_{13}, h_{23}, h_{14}, h_{24}, h_{34}, \ldots, h_{m-1m}) \in \mathbf{R}^{d(m-1)}$. Then the equation $[\mathcal{L}(x), H] = K$ can be written as

$$S_{\mathcal{L}}(x)\check{H} = \check{K}$$

where $S_{\mathcal{L}}(x)$ is a $d(m-1)\times d(m-1)$ matrix. For instance when $m=3$ we have

$$S_{\mathcal{L}}(x)=\begin{pmatrix}\phi_1^1(x)-\phi_2^2(x) & -\phi_3^2(x') & \phi_3^1(x')\\ -\phi_3^2(x') & \phi_1^1(x)-\phi_3^3(x) & \phi_2^1(x')\\ -\phi_3^1(x') & \phi_2^1(x') & \phi_2^2(x)-\phi_3^3(x)\end{pmatrix}. \tag{4.2}$$

We turn to the case $\mathcal{L}(x)$ is a $m\times m$ matrix. Let

$$\mathcal{L}(x)=\begin{pmatrix}L(x) & l(x')\\ {}^tl(x') & \phi_m^m(x)\end{pmatrix}$$

where $l(x') = {}^t(\phi_m^1(x'),\dots,\phi_m^{m-1}(x'))$ and $L(x)$ stands for $\mathcal{L}(x)$ in (4.1) with $m-1$. For $H\in F_m$ and $K\in M^{as}(m,\mathbf{R})$ we write

$$H=\begin{pmatrix}H_1 & h\\ {}^th & 0\end{pmatrix},\quad K=\begin{pmatrix}K_1 & k\\ {}^tk & 0\end{pmatrix}$$

with $H_1\in F_{m-1}$, $K_1\in M^{as}(m-1,\mathbf{R})$ and $h={}^t(h_{1m},\dots,h_{m-1m})$. Then it is easy to see that the equation $[\mathcal{L}(x),H]=K$ is written as

$$\begin{pmatrix}S_L(x) & c(l)\\ c'(l) & L(x)-\phi_m^mI\end{pmatrix}\begin{pmatrix}\check{H}_1\\ h\end{pmatrix}=\begin{pmatrix}\check{K}_1\\ k\end{pmatrix}=\check{K}$$

and hence we get

$$S_{\mathcal{L}}(x)=\begin{pmatrix}S_L(x) & c(l)\\ c'(l) & L(x)-\phi_m^mI\end{pmatrix}. \tag{4.3}$$

Our aim in this section is to prove

Proposition 4.1 *Assume that $1\le\nu\le m-3$. Then in the Grassmannian $G_{d(m),I}^{d(m)-\nu}$, the subset of $\mathcal{L}$ for which the condition (1.2) is fulfilled for $T^{-1}\mathcal{L}T$ with some $T\in O(m)$ is an open and dense subset.*

We first give a parametrization of the Grassmannian $G_{d(m),I}^n$ of n dimensional subspaces of $M^s(m,\mathbf{R})$ containing the identity. Take a map

$$\sigma:\{1,\dots,\nu\}\to\{(i,j)\mid 1\le i\le j\le m,(i,j)\ne(m,m)\}$$

which is injective. Denote by U_σ the set of all ν-tuples of $m\times m$ symmetric matrices $A=(A_1,\dots,A_\nu)$ such that $\operatorname{tr}A_j=0$ and the element $\sigma(k)$ of A_j

is zero unless $k = j$ and the element $\sigma(j)$ of A_j is 1. Let

$$\phi_\sigma : U_\sigma \ni A \mapsto \mathcal{L},$$
$$\mathcal{L} = \{X \in M^s(m, \mathbf{R}) \mid \mathrm{tr}\,(A_j X) = 0, 1 \le j \le \nu\}$$

and set $\Omega_\sigma = \phi_\sigma(U_\sigma)$ then with all such injective σ, $(\phi_\sigma^{-1}, \Omega_\sigma)$ give charts of the Grassmannian $G^n_{d(m),I}$. We set $\triangle = \{(i,i) \mid 1 \le i \le m\}$ and let $1 \le k \le m-1$. We first remark that

Lemma 4.2 *Assume that $1 \le k \le m-1$. Then there are finitely many $S_1, \ldots, S_N \in O(m)$ such that for any $\mathcal{L} \in G^{d(m)-k}_{d(m),I}$ one can find S_i so that $S_i^{-1}\mathcal{L}S_i \in \Omega_\sigma$ with some σ verifying $\sigma(\{1, \ldots, k\}) \cap \triangle = \emptyset$.*

Proof. In this proof we denote $|C| = \max_{i,j} |c_{ij}|$ for a matrix $C = (c_{ij})$. Let $T_{pq}(\epsilon)$ be the orthogonal matrix obtained replacing p-th and q-th, $p < q$, rows of the identity matrix by

$$(0, \ldots, 0, f(\epsilon), 0, \ldots, 0, \epsilon, 0, \ldots, 0),$$
$$(0, \ldots, 0, -\epsilon, 0, \ldots, 0, f(\epsilon), 0, \ldots, 0)$$

where $\epsilon^2 + f(\epsilon)^2 = 1$. We show that it is enough to take $\{S_i\}$ as the set of all m times compositions of I and $T_{pq}(\epsilon_i)$, $\epsilon_i = (C_i m^{2^{i-1}})^{-1}$, $i = 1, \ldots, m$, where $C_1 < C_2 < \cdots < C_m$ will be chosen suitably. Let $\mathcal{L} \in G^{d(m)-k}_{d(m),I}$ and let $A_1, \ldots, A_k$ define $\mathcal{L}$ so that $\mathcal{L}$ consists of all $X \in M^s(m, \mathbf{R})$ such that $\mathrm{tr}\,(A_j X) = 0$, $1 \le j \le k$ where A_j are linearly independent and $\mathrm{tr}\, A_j = 0$. We first note that we may assume $(H)_\mu$: there is an injective $\tau : \{1, \ldots, \mu\} \to \{(i,j) \mid 1 \le i < j \le m\}$ such that the element $\tau(i)$ of A_j is zero unless $i = j$, the element $\tau(j)$ of A_j is 1, $|A_j| \le a_\mu m^{2^{\mu-1}}$ for $1 \le j \le \mu$ and $A_{\mu+1}, \ldots, A_k$ are diagonal where $a_1 = 1$, $a_{\mu+1} = B a_\mu C_\mu$ with a fixed large B. In fact if some A_j has a non-zero off diagonal element we may assume that the off diagonal element $\tau(1)$ of A_1 is 1 and $|A_1| \le 1$. Replacing A_j by $A_j - \alpha_j A_1$, $j \ne 1$, with suitable α_j one can assume that the element $\tau(1)$ of A_j is zero if $j \ne 1$. A repetition of this argument gives the assertion. If $\mu = k$ then $\tau(\{1, \ldots, k\}) \cap \triangle = \emptyset$ and there is nothing to prove. Then we may assume that $\mu \le k-1$. Let $A_{\mu+1} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_m)$. Since $\mathrm{tr}\, A_{\mu+1} = 0$ it is easy to see that there are at least $m-1$ pairs (i,j), $i < j$ such that

$$3|\lambda_i - \lambda_j| \ge |\lambda_r|, \quad r = 1, \ldots, m.$$

Since $\mu \le m-2$ there exists such a (p,q) with $(p,q) \notin \tau(\{1,\dots,\mu\})$. Let us set

$$A_j(\epsilon_\mu) = T_{pq}(\epsilon_\mu)^{-1} A_j T_{pq}(\epsilon_\mu), \quad 1 \le j \le k$$

and note that $|A_j(\epsilon_\mu) - A_j| \le B_1 a_\mu C_\mu^{-1}$, $1 \le j \le \mu$. Choose C_μ so that $a_\mu C_\mu^{-1}$ is small enough then taking $\tilde{A}_j(\epsilon_\mu) = \sum_{i=1}^{\mu} c_{ji} A_i(\epsilon_\mu)$, $1 \le j \le \mu$, with a non-singular $C = (c_{ji})$ we may suppose that the element $\tau(i)$ of $\tilde{A}_j(\epsilon_\mu)$ is zero unless $i = j$ and the element $\tau(j)$ of $\tilde{A}_j(\epsilon_\mu)$ is 1 and $|\tilde{A}_j(\epsilon_\mu)| \le 2|A_j|$. Note that the off diagonal elements of $A_{\mu+1}(\epsilon_\mu)$ are zero except for (p,q), (q,p) elements which are $\epsilon_\mu f(\epsilon_\mu)(\lambda_q - \lambda_p)$. Set

$$\tilde{A}_{\mu+1}(\epsilon_\mu) = \{\epsilon_\mu f(\epsilon_\mu)(\lambda_q - \lambda_p)\}^{-1} A_{\mu+1}(\epsilon_\mu)$$

and hence $|\tilde{A}_{\mu+1}(\epsilon_\mu)| \le B_2 C_\mu m^{2^{\mu-1}}$. Replacing $\tilde{A}_j(\epsilon_\mu)$ by $\tilde{A}_j(\epsilon_\mu) - \alpha_j \tilde{A}_{\mu+1}(\epsilon_\mu)$ with suitable α_j we can attain that the element $\tau(\mu+1) = (p,q)$ of $\tilde{A}_j(\epsilon_\mu)$ is zero for $1 \le j \le \mu$ and $|\tilde{A}_j(\epsilon_\mu)| \le a_{\mu+1} m^{2^\mu}$, $1 \le j \le \mu+1$. By subtraction again we may suppose that $A_j(\epsilon_\mu)$, $j \ge \mu+2$ are diagonal and then we get to $(H)_{\mu+1}$. The rest of the proof is clear. □

Proof of Proposition 4.1 We first assume that $\mathcal{L} \in \Omega_\tau$ with $\tau(\{1,\dots,\nu\}) \cap \triangle = \emptyset$ and let $A = (A_1,\dots,A_\nu) \in U_\tau$ be the coordinate of $\mathcal{L}$. Let us denote

$$\mathcal{L}(x) = \sum_{j=1}^{n} K_j x_j = (\phi_j^i(x))$$

where K_j, $1 \le j \le n = d(m) - \nu$, is a basis for $\mathcal{L}$ and set $g(x) = \det S_{\mathcal{L}}(x)$. Let $J_\tau = \{(i,j) \mid 1 \le i \le j \le m\} \setminus \tau(\{1,\dots,\nu\})$ and note that $\phi_j^i(x)$, $(i,j) \in J_\tau$ are linearly independent and $\triangle \subset J_\tau$. With $A_k = (a_{ij}^{(k)})$ it is clear that the equations $\phi_j^i(x) = 0$, $(i,j) \in J_\tau \setminus \triangle$ and $\operatorname{tr}(A_k \mathcal{L}(x)) = 0$ define a plane

$$\sum_{j=1}^{m} a_{jj}^{(k)} \phi_j^j(x) = \sum_{j=1}^{m-1} a_{jj}^{(k)} (\phi_j^j(x) - \phi_m^m(x)) = 0, \quad 1 \le k \le \nu \tag{4.4}$$

and $S_{\mathcal{L}}(x)$ is diagonal on the plane with the determinant

$$g(x) = \prod_{1 \le i < j \le m} (\phi_i^i(x) - \phi_j^j(x)). \tag{4.5}$$

We show that there is a polynomial $\pi(A)$ in $a_{jj}^{(k)}$, $1 \le k \le \nu$, $1 \le j \le m-1$

such that if $\pi(A) \neq 0$ then no two $\phi_i^i(x) - \phi_j^j(x)$, $i < j$ are proportional on the plane (4.4). To simplify notation we write y_i for $\phi_i^i(x) - \phi_m^m(x)$ so that

$$g(y) = \prod_{1 \le i < j \le m-1} (y_i - y_j) y_1 \cdots y_{m-1}$$

provided that $y\tilde{A} = 0$ where $y = (y_1, \ldots, y_{m-1})$ and $\tilde{A} = (a_{jj}^{(k)})$ which is a $(m-1) \times \nu$ matrix. Suppose that some two $y_i - y_j$ are proportional on the plane $y\tilde{A} = 0$ and hence $(b, y) = 0$ with some $b \in \mathbf{R}^{m-1}$ for every y with $y\tilde{A} = 0$. Then it is clear that $\operatorname{rank}(\tilde{A}, b) = \operatorname{rank}\tilde{A}$. Note that at most two components of b are the constant of the proportionality c and the other components are either 0 or 1 (at most two 1 appear). Take a $(\nu + 1) \times (\nu + 1)$ submatrix of $(\tilde{A}, b)$ and expand the determinant with respect to the last column. Equating the determinant to zero we get a linear relation of ν-minors of $\tilde{A}$ with coefficients which are either 1 or the proportional constant c. Since $\nu + 1 \le m - 2$ we have at least $m - 1$ such linear relations. Elimination of c gives a quadratic equation in ν-minors of $\tilde{A}$. Denote this equation by $\pi(A) = 0$. Then we conclude that the rank of the matrix $(\tilde{A}, b)$ is $\nu + 1$ if $\pi(A) \neq 0$. This shows that no two $y_i - y_j$ are proportional if $\pi(A) \neq 0$.

Let $g(x) = \prod g_j(x)^{r_j}$ be the irreducible factorization in $\mathbf{R}[x]$. Without restrictions we may assume that the plane $y\tilde{A} = 0$ is given by $y_b = f(y_a)$, after a linear change of coordinates y if necessary, where $y = (y_a, y_b)$ is a partition of the coordinates y. Then we have

$$\prod g_j(y_a, f(y_a))^{r_j} = \prod p_i(y_a)$$

where $p_i(y_a)$ are linear in y_a and no two $p_i(y_a)$ are proportional if $\pi(A) \neq 0$. Then it follows that $r_j = 1$ and $g_j(y_a, f(y_a))$ is a product of some $p_i(y_a)$'s:

$$g_j(y_a, f(y_a)) = \prod_{i \in I_j} p_i(y_a).$$

From this it is obvious that $\{g_j(y_a, f(y_a)) = 0\}$ contains a regular point. Then it follows that $\{g_j(x) = 0\}$ contains a regular point. This shows that, in U_τ, the set of A such that $S_{\mathcal{L}}(x)$ does not verify (1.2) is contained in an algebraic set. We now study $\mathcal{L} \in \Omega_\sigma$ with $\sigma(\{1, \ldots, \nu\}) \cap \triangle \neq \emptyset$. By Lemma 4.2 there is $S_i \in O(m)$ such that $S_i^{-1}\mathcal{L}S_i \in \Omega_\tau$ with some τ verifying $\tau(\{1, \ldots, \nu\}) \cap \triangle = \emptyset$. Since $\{S_i\}$ is a finite set the proof is clear. $\square$

Proof of Theorem 1.1 Let $d(m) - m + 3 \le n \le d(m)$. Then Theorem 1.1

follows immediately from Propositions 3.3, 4.1 and Corollary 2.3. □

5. A special case

In this section we prove Theorem 1.2. Thus we assume $m = 3$ throughout the section. Let $\mathcal{L} \in G^n_{6,I}$ for $n = 4$ or 5. With a basis K_j for $\mathcal{L}$, $\mathcal{L}$ is the range of

$$\mathcal{L}(x) = \sum_{j=1}^{n} K_j x_j.$$

We first study the case $n = 5$.

Lemma 5.1 *In the Grassmannian $G^5_{6,I}$, the subset of $\mathcal{L}$ for which the condition* (1.2) *is fulfilled for $T^{-1}\mathcal{L}T$ with some $T \in O(m)$ is an open and dense subset.*

Proof. Let $A = A_1 \in U_\sigma$ be the coordinate of $\mathcal{L}$ and assume that $\sigma(1) \cap \triangle = \emptyset$ so that the diagonal elements of $\mathcal{L}(x)$ are linearly independent. Considering $T^{-1}\mathcal{L}(x)T$ with suitable permutation matrix T, if necessary, we may assume that $\sigma(1) = (1, 2)$ so that with $\mathcal{L}(x) = (\phi^i_j(x))$ we have from $\mathrm{tr}\,(A\mathcal{L}(x)) = 0$ that

$$-2\phi^1_2(x) = a_{11}(\phi^1_1 - \phi^3_3) + a_{22}(\phi^2_2 - \phi^3_3) + 2a_{13}\phi^1_3 + 2a_{23}\phi^2_3.$$

From (4.2), with simplified notations, it is enough to study

$$S(x, y) = \begin{pmatrix} x_1 - x_2 & -y_1 & y_2 \\ -y_1 & x_1 & \phi(x, y) \\ -y_2 & \phi(x, y) & x_2 \end{pmatrix}$$

where $\phi(x, y) = a_1 x_1 + a_2 x_2 + b_1 y_1 + b_2 y_2$. We show that if $a_1 + a_2 \neq 1$ and $4a_1 a_2 - 1 \neq 0$ then the condition (1.2) is fulfilled. We first assume that $x_1 x_2 - \phi(x, 0)^2$ is irreducible. Note that $g(x, y) = \det S(x)$ is then irreducible. Indeed if $g(x, y)$ were reducible so that $g(x, y) = h(x, y)k(x, y)$ then from $g(x, 0) = (x_1 - x_2)\psi(x)$ with $\psi(x) = x_1 x_2 - \phi(x, 0)^2$ we may suppose that

$$h(x, y) = \psi(x) + p(x, y), \quad k(x, y) = x_1 - x_2 + q(y)$$

where $p(x, 0) = 0$, $q(y) = \alpha y_1 + \beta y_2$. Equating the coefficients of y_j in both sides of $g(x, y) = h(x, y)k(x, y)$ we see that $\alpha\psi(x)$, $\beta\psi(x)$ have a factor

$x_1 - x_2$ which implies that $q = 0$. This gives $g(x,y) = h(x,y)(x_1 - x_2)$ which is a contradiction. Thus g is irreducible. It is clear that $\{g(x,0) = 0\}$ has a regular point and hence so does $\{g(x,y) = 0\}$. This proves the assertion.

Assume now that $\psi(x) = x_1x_2 - \phi(x,0)^2$ is reducible. From the assumption $4a_1a_2 - 1 \neq 0$ it follows that $\psi(x)$ has no multiple factor. Note that $a_1 + a_2 \neq \pm 1$ implies that $\psi(x)$ and $x_1 - x_2$ are relatively prime. The rest of the proof is a repetition of the last part of the proof of Proposition 4.1. $\square$

We turn to the case $n = 4$. We show that

Lemma 5.2 *Assume that $n = 4$ and every double characteristic of $\mathcal{L}(x)$ is non degenerate. Then the condition* (1.2) *is fulfilled for $T^{-1}\mathcal{L}(x)T$ with a suitable $T \in O(3)$.*

Proof. Following the proof of Theorems 3.5 and 3.6 in [4] we choose a specific basis for $\tilde{\mathcal{L}} = T^{-1}\mathcal{L}T$ with suitably chosen $T \in O(3)$ and show that (1.2) is fulfilled for $\tilde{\mathcal{L}}$ using this basis. From the proof of Theorem 3.3 in [4], if every double characteristic of $\mathcal{L}$ is non-degenerate, then only two cases occur, that is $\mathcal{L}$ has either four non-degenerate double characteristics or two non-degenerate double characteristics.

We first treat the case that $\mathcal{L}$ has four non-degenerate characteristics. Choosing a suitable $T \in O(3)$ we see from [4] that $A^{\pm} = \alpha_{\pm} \otimes \alpha_{\pm}$ and $B^{\pm} = \beta_{\pm} \otimes \beta_{\pm}$ is a basis for $\tilde{\mathcal{L}} = T^{-1}\mathcal{L}T$ where $\alpha_{\pm} = (a, \pm a, 1)$, $\beta_{\pm} = (b, \pm b, 1)$ and $a \neq b$, $ab \neq 0$. Now we can write

$$\tilde{\mathcal{L}}(x) = A^{+}x_1 + A^{-}x_2 + B^{+}x_3 + B^{-}x_4.$$

With $X = x_1 + x_2$, $Y = x_1 - x_2$, $Z = x_3 + x_4$, $W = x_3 - x_4$ we have

$$\tilde{\mathcal{L}} = \begin{pmatrix} a^2X + b^2Z & a^2Y + b^2W & aX + bZ \\ a^2Y + b^2W & a^2X + b^2Z & aY + bW \\ aX + bZ & aY + bW & X + Z \end{pmatrix}. \tag{5.1}$$

Therefore it follows from (4.2) and (5.1) that

$$S_{\tilde{\mathcal{L}}} = \begin{pmatrix} 0 & -aY - bW & aX + bZ \\ -aY - bW & cX + dZ & a^2Y + b^2W \\ -aX - bZ & a^2Y + b^2W & cX + dZ \end{pmatrix}$$

where $c = a^2 - 1$, $d = b^2 - 1$. Let $\tilde{g} = \det S_{\tilde{\mathcal{L}}}$. On the plane $a^2Y + b^2W = 0$, that is, if $W = -a^2Y/b^2 = eY$ we get

$$\tilde{g} = (cX + dZ)(aX + bZ + (a + be)Y)(aX + bZ - (a + be)Y).$$

Note that $a + be \neq 0$ because $a \neq b$ and no two factors in the right-hand side are proportional. Now, as the end of the proof of Proposition 4.2, it is easy to conclude that $\tilde{g}$ satisfies (1.2).

We next study the case $\mathcal{L}$ has two non-degenerate double characteristics. With a suitable $T \in O(3)$ we see that $\tilde{\mathcal{L}} = T^{-1}\mathcal{L}T$ contains $K^{\pm} = \alpha_{\pm} \otimes \alpha_{\pm}$ with $\alpha_{\pm} = (a, \pm a, 1)$, $a \neq 0$, which are intersections with $M_2^s(3, \mathbf{R})$. Since $\tilde{\mathcal{L}}$ contains the identity, as the third basis element in $\tilde{\mathcal{L}}$, one can take K_3

$$K_3 = \begin{pmatrix} 0 & 0 & -2a \\ 0 & 0 & 0 \\ -2a & 0 & 2(a^2 - 1) \end{pmatrix}$$

because $K^+ + K^- + K_3 = 2a^2 I$. The fourth basis element in $\tilde{\mathcal{L}}$ can then be chosen of the form

$$K_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & \mu & \nu \end{pmatrix}.$$

Thus with $X = x_1 + x_2$, $Y = x_1 - x_2$, $Z = x_3$, $W = x_4$ and $c = a^2 - 1$ the matrix $K^+x_1 + K^-x_2 + K_3x_3 + K_4x_4$ can be written

$$\tilde{\mathcal{L}} = \begin{pmatrix} a^2X & a^2Y & aX - 2aZ \\ a^2Y & a^2X + \lambda W & aY + \mu W \\ aX - 2aZ & aY + \mu W & X + 2cZ + \nu W \end{pmatrix}. \tag{5.2}$$

We examine if there are other double characteristics, that is, if $\tilde{\mathcal{L}}$ is of rank 1 for some (X, Y, Z, W) with $Z^2 + W^2 \neq 0$. It is not difficult to see that six 2-minors of (5.2) vanish for such (X, Y, Z, W) if and only if the equation

$$4a^2Z^2 + 2(a^2 + 1)\lambda ZW + (\lambda\nu - \mu^2)W^2 = 0$$

has a real solution $(Z, W) \neq (0, 0)$. Thus in order that $\tilde{\mathcal{L}}$ has two non-degenerate double characteristics it is necessary and sufficient that

$$4a^2\lambda\nu > 4a^2\mu^2 + (a^2 + 1)^2\lambda^2. \tag{5.3}$$

In particular λ and ν have the same signs. From (5.2) and (4.2) it follows that

$$S_{\tilde{\mathcal{L}}} = \begin{pmatrix} -\lambda W & -aY - \mu W & aX - 2aZ \\ -aY - \mu W & cX - 2cZ - \nu W & a^2 Y \\ -aX + 2aZ & a^2 Y & cX - 2cZ + (\lambda - \nu)W \end{pmatrix}.$$

If $c \neq 0$ then we consider $\tilde{g} = \det S_{\tilde{\mathcal{L}}}$ on $W = 0$ so that

$$\tilde{g} = (cX - 2cZ)(aX - 2aZ + aY)(aX - 2aZ - aY).$$

The same argument as before proves that (1.2) is verified for $\tilde{g}$. If $c = 0$ and hence $a^2 = 1$ then

$$\begin{aligned} \tilde{g} &= W(-\nu(aX - 2aZ)^2 + \lambda(\nu^2 - \mu^2)\alpha^{-1}Y^2 \\ &\qquad + (\lambda - \nu)\alpha(W - a\mu\alpha^{-1}Y)^2) \\ &= Wh(X, Y, Z, W) \end{aligned}$$

where $\alpha = \lambda\nu - \mu^2$. From (5.3) it follows that $\alpha > 0$ and $\nu^2 - \mu^2 > 0$ because $\nu^2 + \lambda^2 \geq \lambda\nu > \mu^2 + \lambda^2$. Then the quadratic form h is indefinite and hence $\{h = 0\}$ contains a regular point. This proves the assertion. $\square$

Proof of Theorem 1.2 If $n = 6$ then the assertion follows from Theorem 4.2 in [9]. If $n = 5$, combining Proposition 3.3 and Lemma 5.1 we get the result by Corollary 2.3. Let $n = 4$. Then by virtue of Proposition 3.3 and Lemma 5.2 one can apply Corollary 2.3 to get the assertion. $\square$

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