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On some generalized difference sequence spaces and related matrix transformations

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Abstract. In this paper we introduce β-duals and γ-duals of the sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, $(m \in \mathbf{N})$ where for instance $l_\infty(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in l_\infty\}$, and we characterize some matrix classes related with these sequence spaces. This study generalizes some results of Kızmaz [4] in special cases.

Key words: difference sequences, matrix transformations, β-dual, γ-dual.

1. Introduction

Let l_∞, c, and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$||x||_\infty = \sup_k |x_k|$$

where $k \in \mathbf{N} = \{1, 2, \ldots\}$, the set of positive integers.

Kızmaz [4] defined the sequence spaces

$$\begin{aligned} l_\infty(\Delta) &= \{x = (x_k) : \Delta x \in l_\infty\}, \\ c(\Delta) &= \{x = (x_k) : \Delta x \in c\}, \\ c_0(\Delta) &= \{x = (x_k) : \Delta x \in c_0\} \end{aligned}$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaces with norm

$$||x|| = |x_1| + ||\Delta x||_\infty.$$

After then Et [1] defined the sequence spaces

$$\begin{aligned} l_\infty(\Delta^2) &= \{x = (x_k) : \Delta^2 x \in l_\infty\}, \\ c(\Delta^2) &= \{x = (x_k) : \Delta^2 x \in c\}, \\ c_0(\Delta^2) &= \{x = (x_k) : \Delta^2 x \in c_0\} \end{aligned}$$

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where $\Delta^2 x = (\Delta^2 x_k) = (\Delta x_k - \Delta x_{k+1})$, and showed that these are Banach spaces with norm

$$||x||_1 = |x_1| + |x_2| + ||\Delta^2 x||_\infty .$$

Recently Et and Çolak [2] defined the sequence spaces

$$\begin{aligned} l_\infty(\Delta^m) &= \{x = (x_k) : \Delta^m x \in l_\infty\}, \\ c(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c\}, \\ c_0(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c_0\} \end{aligned}$$

where $m \in \mathbf{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so that

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$$

and showed that these sequence spaces are Banach spaces with norm

$$||x||_\Delta = \sum_{i=1}^{m} |x_i| + ||\Delta^m x||_\infty .$$

Further the inclusions $c_0(\Delta^m) \subset c_0(\Delta^{m+1})$, $c(\Delta^m) \subset c(\Delta^{m+1})$, $l_\infty(\Delta^m) \subset l_\infty(\Delta^{m+1})$, and $c_0(\Delta^m) \subset c(\Delta^m) \subset l_\infty(\Delta^m)$ are satisfied and strict.

The operator

$$D : l_\infty(\Delta^m) \to l_\infty(\Delta^m)$$

defined by $Dx = (0, 0, \ldots, x_{m+1}, x_{m+2}, \ldots)$, where $x = (x_1, x_2, x_3, \ldots)$ is a bounded linear operator on $l_\infty(\Delta^m)$. Furthermore the set

$$\begin{aligned} D[l_\infty(\Delta^m)] &= Dl_\infty(\Delta^m) \\ &= \{x = (x_k) : x \in l_\infty(\Delta^m), x_1 = x_2 = \cdots = x_m = 0\} \end{aligned}$$

is a subspace of $l_\infty(\Delta^m)$, and $||x||_\Delta = ||\Delta^m x||_\infty$ in $Dl_\infty(\Delta^m)$.

Now let us define

$$\begin{aligned} &\Delta^m : Dl_\infty(\Delta^m) \to l_\infty, \\ &\Delta^m x = y = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}). \end{aligned} \tag{1.1}$$

It can be shown that Δ^m is a linear homeomorphism. Hence $Dl_\infty(\Delta^m)$ and l_∞ are equivalent as topological spaces. Also $Dc(\Delta^m)$ and c, $Dc_0(\Delta^m)$ and

c_0 are equivalent as topological spaces and $[Dc(\Delta^m)]' \cong [Dc_0(\Delta^m)]' \cong l_1$ in [2], where $[Dc(\Delta^m)]'$ and $[Dc_0(\Delta^m)]'$ denote the continuous duals of $Dc(\Delta^m)$ and $Dc_0(\Delta^m)$ respectively, and $l_1 = \{x = (x_k) : \Sigma_k |x_k| < \infty\}$.

2. Dual spaces

In this section we give β- and γ-duals of $l_\infty(\Delta^m)$ and $c(\Delta^m)$. Also we show that these spaces are not normal and not monotone spaces.

Throughout the paper we write Σ_k for $\sum_{k=1}^{\infty}$ and $\lim_n$ for $\lim_{n\to\infty}$.

Lemma 2.1 ([4]). *Let (p_n) be a sequence of positive numbers increasing monotonically to infinity.*

i) *If* $\sup_n |\sum_{v=1}^{n} p_v a_v| < \infty$, *then* $\sup_n |p_n \sum_{k=n+1}^{\infty} a_k| < \infty$,

ii) *If $\Sigma_k p_k a_k$ is convergent, then* $\lim_n p_n \sum_{k=n+1}^{\infty} a_k = 0$.

Lemma 2.2 ([2]). $x \in l_\infty(\Delta^m)$ *implies* $\sup_k k^{-m}|x_k| < \infty$.

Definition 2.3 ([3]). Let X be a sequence space and define

$$X^\alpha = \{a = (a_k) : \Sigma_k |a_k x_k| < \infty, \quad \text{for all} \quad x \in X\},$$

$$X^\beta = \{a = (a_k) : \Sigma_k a_k x_k \ \text{ is convergent, for all} \ \ x \in X\},$$

$$X^\gamma = \left\{a = (a_k) : \sup_n \left|\sum_{k=1}^{n} a_k x_k\right| < \infty, \quad \text{for all} \quad x \in X\right\}.$$

Then X^α, X^β, X^γ are called α-, β-, γ- dual spaces of X, respectively. It is easy to show that $\phi \subset X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha, \beta, \gamma$.

Definition 2.4 ([3]). Let X be a sequence space. Then X is called

i) Perfect if $X = X^{\alpha\alpha}$,

ii) Normal if $y \in X$ whenever $|y_k| \le |x_k|$, $k \ge 1$, for some $x \in X$,

iii) Monotone provided X contains the canonical preimages of all its stepspaces.

Lemma 2.5 ([3]). *Let X be a sequence space. Then we have*

i) *X is perfect $\Longrightarrow$ X is normal $\Longrightarrow$ X is monotone,*

ii) *X is normal $\Longrightarrow$ $X^\alpha = X^\gamma$,*

iii) *X is monotone* $\Longrightarrow X^\alpha = X^\beta$.

Lemma 2.6 i) $[Dl_\infty(\Delta^m)]^\beta = \{a = (a_k) : \Sigma_k k^m a_k$ *is convergent*, $\Sigma_k k^{m-1}|R_k| < \infty\}$,

ii) $[Dl_\infty(\Delta^m)]^\gamma = \{a = (a_k) : \sup_n |\sum_{k=1}^n k^m a_k| < \infty, \Sigma_k k^{m-1}|R_k| < \infty\}$,

where $R_k = \sum_{v=k+1}^\infty a_v$.

Proof. i) Let $U = \{a = (a_k) : \Sigma_k k^m a_k$ is convergent, $\Sigma_k k^{m-1}|R_k| < \infty\}$. If $x \in Dl_\infty(\Delta^m)$ then there exists one and only one $y = (y_k) \in l_\infty$ such that

$$\begin{aligned} x_k &= \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v \\ &= \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \\ & \qquad\qquad\qquad y_{1-m} = y_{2-m} = \cdots = y_0 = 0 \end{aligned}$$

for sufficiently large k, for instance $k > 2m$ by (1.1). Let $a \in U$, and suppose that $\binom{-1}{-1} = 1$ (in some literature it is assumed that $\binom{r}{k} = 0$ for $k < 0$). Then we may write

$$\begin{aligned} \sum_{k=1}^n a_k x_k &= \sum_{k=1}^n a_k \left(\sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v \right) \\ &= (-1)^m \sum_{k=1}^{n-m} (k+m-1)^{m-1} R_{k+m-1} \\ & \qquad \left(\frac{1}{(k+m-1)^{m-1}} \sum_{v=1}^{k} \binom{k+m-v-2}{m-2} y_v \right) \\ & \qquad - n^m R_n n^{-m} x_n. \end{aligned} \tag{2.1}$$

Since $\Sigma_k k^{m-1}|R_k| < \infty$, the series $\Sigma_k (k+m-1)^{m-1} R_{k+m-1} z_k$ is absolutely convergent, where

$$z = (z_k) = \left(\frac{1}{(k+m-1)^{m-1}} \sum_{v=1}^{k} \binom{k+m-v-2}{m-2} y_v \right).$$

Moreover we have $R_n n^m \to 0$ as $n \to \infty$ (Lemma 2.1), $\sup_n n^{-m}|x_n| < \infty$ (Lemma 2.2), hence $\Sigma_k a_k x_k$ is convergent for all $x \in Dl_\infty(\Delta^m)$, so

$a \in [Dl_\infty(\Delta^m)]^\beta$.

Let $a \in [Dl_\infty(\Delta^m)]^\beta$. Then $\Sigma_k a_k x_k$ is convergent for each $x \in Dl_\infty(\Delta^m)$. For the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 0, & k \leq m \\ k^m, & k > m \end{cases}$$

we may write

$$\Sigma_k k^m a_k = \sum_{k=1}^{m} k^m a_k + \Sigma_k a_k x_k.$$

Thus the series $\Sigma_k k^m a_k$ is convergent. This implies that $R_n n^m = o(1)$ by Lemma 2.1 (ii).

Now let $a \in [Dl_\infty(\Delta^m)]^\beta - U$. Then $\Sigma_k k^{m-1}|R_k|$ is divergent, that is, $\Sigma_k k^{m-1}|R_k| = \infty$. We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \leq m \\ \sum_{v=1}^{k-1} v^{m-1} \operatorname{sgn} R_v, & k > m \end{cases}$$

where $a_k > 0$ for all k or $a_k < 0$ for all k. Since $|\Delta^m(x)| = (m-1)!$ for $k > m$, it is trivial that $x = (x_k) \in Dl_\infty(\Delta^m)$. Then we may write for $n > m$

$$\begin{aligned} \sum_{k=1}^{n} a_k x_k &= -\sum_{k=1}^{m} R_{k-1} \Delta x_{k-1} \\ &\quad - \sum_{k=1}^{n-m} R_{k+m-1} \Delta x_{k+m-1} - n^m R_n n^{-m} x_n. \end{aligned}$$

Now letting $n \to \infty$ we get

$$\begin{aligned} \Sigma_k a_k x_k &= -\Sigma_k R_{k+m-1} \Delta x_{k+m-1} \\ &= \Sigma_k (k+m-1)^{m-1} |R_{k+m-1}| = \infty. \end{aligned}$$

This contradicts to $a \in [Dl_\infty(\Delta^m)]^\beta$. Hence $a \in U$.

ii) can be proved by the same way as above, using lemma 2.1 (i). This completes the proof. □

Lemma 2.7 $[Dl_\infty(\Delta^m)]^\eta = [Dc(\Delta^m)]^\eta$ *for* $\eta = \beta$ *or* γ

Proof is trivial.

Lemma 2.8 i) $[l_\infty(\Delta^m)]^\eta = [Dl_\infty(\Delta^m)]^\eta$
ii) $[c(\Delta^m)]^\eta = [Dc(\Delta^m)]^\eta$
for $\eta = \beta$ *or* γ.

Proof. i) We give the proof for $\eta = \beta$ only. It can be proved in a similar way for $\eta = \gamma$. Since $Dl_\infty(\Delta^m) \subset l_\infty(\Delta^m)$, then $[l_\infty(\Delta^m)]^\beta \subset [Dl_\infty(\Delta^m)]^\beta$. Let $a \in [Dl_\infty(\Delta^m)]^\beta$. If $x = (x_k) \in l_\infty(\Delta^m)$,

$$x_k = \begin{cases} x_k, & k \leq m \\ x'_k, & k > m \end{cases} \tag{2.2}$$

where $x' = (x'_k) \in Dl_\infty(\Delta^m)$, then we may write for $n > m$

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{m} a_k x_k + \sum_{k=1}^{n} a_k x'_k.$$

Now letting $n \to \infty$, we get the series in the same way as the proof of Lemma 2.6. i),

$$\Sigma_k a_k x_k = \sum_{k=1}^{m} a_k x_k + (-1)^m \Sigma_k (k+m-1)^{m-1} R_{k+m-1} z_k$$

is convergent. This implies that $a \in [l_\infty(\Delta^m)]^\beta$.
ii) can be proved by the same way as above. □

Theorem 2.9 ([2]). *Let* X *stand for* l_∞ *or* c. *Then*

$$[X(\Delta^m)]^\alpha = \{a = (a_k) : \Sigma_k k^m |a_k| < \infty\}.$$

Now we give the main result.

Theorem 2.10 *Let* X *stand for* l_∞ *or* c. *Then*
i) $[X(\Delta^m)]^\beta = \{a = (a_k) : \Sigma_k k^m a_k$ *is convergent*, $\Sigma_k k^{m-1}|R_k| < \infty\}$,
ii) $[X(\Delta^m)]^\gamma = \{a = (a_k) : \sup_n |\sum_{k=1}^n k^m a_k| < \infty, \Sigma_k k^{m-1}|R_k| < \infty\}$,
where $R_k = \sum_{v=k+1}^\infty a_v$.

Proof. Proof follows from Lemma 2.6, Lemma 2.7 and Lemma 2.8. □

It is known that $l_\infty(\Delta^m)$, $c(\Delta^m)$ are not perfect in [2]. Combining

Lemma 2.5, Theorem 2.9 and Theorem 2.10, we get:

Corollary 2.11 i) $l_\infty(\Delta^m)$, $c(\Delta^m)$ *are not normal,*
ii) $l_\infty(\Delta^m)$, $c(\Delta^m)$ *are not monotone.*

If we take $m = 1$ in Theorem 2.9 and in Theorem 2.10, then we obtain the following result.

Corollary 2.12 ([4]). i) $[X(\Delta)]^\alpha = \{a = (a_k) : \Sigma_k k|a_k| < \infty\}$,
ii) $[X(\Delta)]^\beta = \{a = (a_k) : \Sigma_k ka_k$ *is convergent,* $\Sigma_k|R_k| < \infty\}$,
iii) $[X(\Delta)]^\gamma = \{a = (a_k) : \sup_n |\sum_{k=1}^n ka_k| < \infty, \Sigma_k|R_k| < \infty\}$,
where $R_k = \sum_{v=k+1}^\infty a_v$.

3. Matrix transformations

In this section we characterize some matrix classes. Let G denote one of the sequence spaces l_∞ and c, and H denote l_∞. Let us consider $G(\Delta^m) = \{x = (x_k) : \Delta^m x \in G\}$. We denote the set of all matrices from sequence space X to sequence space Y by (X, Y).

Theorem 3.1 $A = (a_{nk}) \in (G(\Delta^m), H)$ *if and only if*
i) $(a_{nj})_n \in H$ $(j = 1, 2, \ldots, m)$ *and* $(A_n(\mathbf{k}^m)) \in H$,
ii) $R_m = (k^{m-1}r_{nk}) \in (G, H)$,
where $A_n(\mathbf{k}^m) = \Sigma_k k^m a_{nk}$ *and* $r_{nk} = \sum_{v=k+1}^\infty a_{nv}$.

Proof. Let $A \in (G(\Delta^m), H)$, then the series $A_n(x) = \Sigma_k a_{nk}x_k$ is convergent for each $n \in \mathbf{N}$ and $(A_n(x)) \in H$ for all $x \in G(\Delta^m)$. □

If we take $x = (x_k) = (0, 0, \ldots, 0, 1$ (j.th place), $0, \ldots)$ $(1 \le j \le m)$ and $x = (x_k) = (k^m)$, then we get the necessity of (i). If $R_m = (k^{m-1}r_{nk}) \notin (G, H)$, then there exist subsequences (n_i) and (k_i) of positive integers such that

$$\sum_{k=1}^{k_i} k^{m-1}|r_{n_i k}| \to \infty \quad \text{as} \quad i \to \infty. \tag{3.1}$$

From Theorem 2.10 we have

$$\Sigma_k k^{m-1}|r_{nk}| < \infty, \tag{3.2}$$

for each $n \in \mathbf{N}$. By (3.2), there exists $M > 0$ such that

$$k^{m-1}|r_{nk}| < M, \tag{3.3}$$

for all k and for all n. By (3.1), choose $n = n_1$ and $k = s_1$ such that

$$\sum_{k=1}^{s_1} k^{m-1}|r_{n_1 k}| > 1. \tag{3.4}$$

Having fixed n_1, by (3.2), choose $k_1 > s_1$ such that

$$\sum_{k=k_1+1}^{\infty} k^{m-1}|r_{n_1 k}| < \varepsilon \tag{3.5}$$

If we take, for all n

$$x_k = \sum_{v=1}^{k-1} v^{m-1}\operatorname{sgn} r_{nv}, \text{ for } 1 \leq k \leq k_1 \text{ and } k_{i-1} < k \leq k_i,$$
$$(i = 2, 3, \ldots), \quad x_1 = 0, \tag{3.6}$$

where $a_{nk} > 0$ for all n, k (or $a_{nk} < 0$ for all n, k), then we have $x \in G(\Delta^m)$.

On the other hand, if we consider Lemma 2.1 and Lemma 2.2, then we have

$$\sum_{k=1}^{\infty} a_{nk}x_k = -\sum_{k=1}^{\infty} r_{nk}\Delta x_k, \quad x_1 = 0. \tag{3.7}$$

Hence

$$|A_{n_1}(x)| \geq \sum_{k=1}^{k_1} k^{m-1}|r_{n_1 k}| - \sum_{k=k_1+1}^{\infty} k^{m-1}|r_{n_1 k}| > 1 - \varepsilon$$

using (3.4), (3.5), (3.6) and (3.7).

From (3.3), we have for all n,

$$\sum_{k=1}^{k_i} k^{m-1}|r_{n_i k}| < \sum_{k=1}^{k_i} M = k_i M = C_{k_i}. \tag{3.8}$$

By (3.1), choose $n = n_2 > n_1$ and $s_2 > k_1$ such that

$$\sum_{k=k_1+1}^{s_2} k^{m-1}|r_{n_2 k}| > 2 + C_{k_1} \tag{3.9}$$

Having fixed n_2, by (3.2), choose $k_2 > s_2$ such that

$$\sum_{k=k_2+1}^{\infty} k^{m-1}|r_{n_2 k}| < \varepsilon. \tag{3.10}$$

Then we have

$$|A_{n_2}(x)| \geq \sum_{k=k_1+1}^{k_2} k^{m-1}|r_{n_2k}| - \sum_{k=1}^{k_1} k^{m-1}|r_{n_2k}| - \sum_{k=k_2+1}^{\infty} k^{m-1}|r_{n_2k}| > 2 - \varepsilon$$

using (3.6), (3.7), (3.8), (3.9) and (3.10).

Proceeding like this, by (3.1), we can choose $n_i > n_{i-1}$ and $s_i > k_{i-1}$ (so it is clear that $s_1 < k_1 < s_2 < k_2 < \cdots < s_{i-1} < k_{i-1} < s_i < k_i \ldots$) such that

$$\sum_{k=k_{i+1}+1}^{s_i} k^{m-1}|r_{n_ik}| > i + C_{k_{i-1}}. \tag{3.11}$$

Having fixed n_i, by (3.2), choose $k_i > s_i$ such that

$$\sum_{k=k_i+1}^{\infty} k^{m-1}|r_{n_ik}| < \varepsilon. \tag{3.12}$$

We can show as above $|A_{n_i}(x)| > i - \varepsilon$. Since ε is arbitrary, $|A_{n_i}(x)| \to \infty$ as $i \to \infty$. Hence $(A_n(x)) \notin H$. This is a contradiction to $A \in (G(\Delta^m), H)$. Hence $R_m = (k^{m-1}r_{nk}) \in (G, H)$.

Now suppose that i) and ii) hold. We define the sequence $x = (x_k) \in G(\Delta^m)$ by

$$x_k = \begin{cases} x_k, & k \leq m \\ x'_k, & k > m \end{cases}$$

where $x' = (x'_k) \in DG(\Delta^m)$. Then we may write for $m < t$ in the same way as the proof of Lemma 2.6,

$$\begin{aligned} A_n(t, m, x) &= \sum_{k=1}^{t} a_{nk}x_k \\ &= \sum_{k=1}^{m} a_{nk}x_k + (-1)^m \sum_{k=1}^{t-m} (k+m-1)^{m-1} r_{n,k+m-1} z_k \\ &\quad - t^m r_{nt} t^{-m} x'_t \end{aligned}$$

where $z = (z_k) = \left(\frac{1}{(k+m-1)^{m-1}} \sum_{v=1}^{k} \binom{k+m-v-2}{m-2} y_v \right)$ and $y \in G$.

If we consider Lemma 2.1 and Lemma 2.2, then we have

$$\begin{aligned}\lim_t A_n(t,m,x) &= A_n(x) \\ &= \sum_{k=1}^{m} a_{nk}x_k + (-1)^m \Sigma_k (k+m-1)^{m-1} r_{n,k+m-1} z_k\end{aligned}$$

for the sequence $x = (x_k) \in G(\Delta^m)$. This implies that $(A_n(x)) \in H$ for each $x \in G(\Delta^m)$, and $A \in (G(\Delta^m), H)$.

If we take $m = 1$ in Theorem 3.1, then we obtain the following result.

Corollary 3.2 ([4]). $A = (a_{nk}) \in (G(\Delta), H)$ *if and only if*

i) $(a_{n1}) \in H$ *and* $(A_n(\mathbf{k})) \in H$,

ii) $R \in (G, H)$,

where $A_n(\mathbf{k}) = \Sigma_k k a_{nk}$ *and* $R = (r_{nk}) = (\sum_{v=k+1}^{\infty} a_{nv})$.

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