Hokkaido Mathematical Journal Vol. 26 (1997) p. 421–434

An oscillation result for a certain linear differential equation of second order

Katsuya ISHIZAKI

(Received January 4, 1996; Revised May 21, 1996)

Abstract. We consider the second order equation $f'' + (e^{P_1(z)} + e^{P_2(z)} + Q(z))f = 0$, where $P_1(z) = \zeta_1 z^n + \ldots$, $P_2(z) = \zeta_2 z^n + \ldots$, are non-constant polynomials, $Q(z)$ is an entire function and the order of Q is less than n. Bank, Laine and Langley studied the cases when $Q(z)$ is a polynomial and ξ_2/ξ_1 is either non-real or real negative, while the author and Tohge studied the cases when $\xi_1 = \xi_2$ or ξ_2/ξ_1 is non-real. In this paper we treat the case when ζ_2/ζ_1 is real and positive.

Key words: complex oscillation theory, Nevanlinna theory, Value distribution.

1. Introduction

We are concerned with the zero distribution of solutions of some linear differential equations of second order

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function. We assume that the reader is familiar with the standard notation in Nevanlinna theory (see e.g. [8], [10], [11]). Let f be a meromorphic function. As usual, $m(r, f)$, $N(r, f)$, and $T(r, f)$ denote the proximity function, the counting function, and the characteristic function of f, respectively. We denote by $S(r, f)$ any quantity of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. We use the symbols $\sigma(f)$ to denote the order of f, and $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of f. The studies and problems on complex oscillation theory are found in, for instance, Laine [10, Chapter 3–8] and Yang, Wen, Li and Chiang [14, pp. 357–358].

This note is devoted to the study of the equation (1.1) in the case

---

1991 Mathematics Subject Classification : Primary 34A20, 30D35; Secondary 34C10, 34A30.

This work was supported in part by a Grant-in-Aid for General Scientific Research from the Ministry of Education, Science and Culture 07740127, 08740117 and by a Grant from NIPPON Institute of Technology 102.

$A(z) = e^{P_1(z)} + e^{P_2(z)} + Q(z)$, i.e.,

$$f'' + (e^{P_1(z)} + e^{P_2(z)} + Q(z))f = 0, \tag{1.2}$$

where P_1, P_2 are non-constant polynomials:

$$\begin{cases} P_1(z) = \zeta_1 z^n + \cdots & \zeta_1 \cdot \zeta_2 \neq 0, \quad n, m \in \mathbb{N} \\ P_2(z) = \zeta_2 z^m + \cdots \end{cases} \tag{1.3}$$

and $Q(z)$ is an entire function of order less than $\max\{n, m\}$. Further we assume that e^{P_1} and e^{P_2} are linearly independent.

Bank, Laine and Langley [4 Theorem 4.1, Corollary 4.2, Theorem 4.3] obtained the results which imply the following conclusions when $Q(z)$ is a polynomial and $n = m$: (i) if ζ_2/ζ_1 is non-real then any non-trivial solution f satisfies $\lambda(f) = \infty$, (ii) if ζ_2/ζ_1 is real and negative then any non-trivial solution f satisfies $\lambda(f) = \infty$.

Tohge and the author [9] proved

Theorem A

(i) *If $n \neq m$ in (1.2), then for any non-trivial solution of (1.1) we have $\lambda(f) = \infty$.*

(ii) *If $n = m$ and $\zeta_1 = \zeta_2$ in (1.2), then for any non-trivial solution of (1.2) we have $\lambda(f) \geqq n$.*

(iii) *Suppose that $n = m$ and $\zeta_1 \neq \zeta_2$ in (1.2). If ζ_1/ζ_2 is non-real, then for any non-trivial solution of (1.2) we have $\lambda(f) = \infty$.*

In this note we will treat the case when $n = m$ and $\rho := \zeta_2/\zeta_1$ is real and positive. Without loss of generality, we may assume that $0 < \rho < 1$.

Theorem 1 *Consider equation (1.2) when $n = m$ and $\rho > 0$.*

(i) *If $0 < \rho < 1/2$, then for any non-trivial solution of (1.2) we have $\lambda(f) \geqq n$.*

(ii) *Suppose that $Q(z) \equiv 0$ in (1.2). If $3/4 < \rho < 1$, then for any non-trivial solution of (1.2) we have $\lambda(f) \geqq n$.*

Concerning Theorem 1 (i), $\rho = 1/2$ is impossible to get the same conclusion which is shown by the following example:

Example 1. We consider the differential equation below having $\rho = 1/2$.

$$f'' + (e^{4iz+\log 4} + e^{2iz+\log 4})f = 0. \tag{1.4}$$

The function $f(z) = \exp(e^{2iz})$, which is zero free, satisfies the equation (1.4).

This example was given in Bank and Laine [2] as a zero-free solution of the equation (1.1) when $A(z)$ is periodic. The case when $Q(z)$ is not identically zero in Theorem (ii) is treated in the forthcoming paper Tohge [13]. He gives a counter example for the case when $Q(z) \not\equiv 0$ and $\rho = 3/4$:

Example 2. The function $f(z) = \exp(\frac{1}{2}e^{2z} + ie^z - \frac{1}{2}z)$ solves the equation

$$f'' + \left(e^{4z+\log(-1)} + e^{3z+\log(-2i)} - \frac{1}{4}\right) f = 0. \tag{1.5}$$

At the end of this section, we pose a question: is it possible that we can replace "$\lambda(f) \geqq n$" with "$\lambda(f) = \infty$" in the conclusions of Theorem 1.

2. Preliminary Lemmas

We prepare some notations for the proof of Theorem 1. Let $P(z)$ be a polynomial of degree $n \geqq 1$: $P(z) = (\alpha + \beta i)z^n + \ldots$, α, $\beta \in \mathbb{R}$. Define for $\theta \in [0, 2\pi)$

$$\delta(P,\theta) = \alpha\cos n\theta - \beta\sin n\theta, \quad \tilde{\delta}(P,\theta) = \beta\cos n\theta + \alpha\sin n\theta.$$

We write $\zeta_j = \alpha_j + i\beta_j$, α_j, $\beta_j \in \mathbb{R}$, $j = 1, 2$. Set

$$S_j^+ = \{\theta | \delta(P_j,\theta) > 0\}, \quad S_j^- = \{\theta | \delta(P_j,\theta) < 0\}, \quad j = 1, 2.$$

We see that S_j^+ and S_j^- have n components S_{jk}^+ and S_{jk}^-, $k = 1, 2, \ldots, n$, respectively. Hence we can write

$$S_j^+ = \bigcup_{k=1}^{n} S_{jk}^+, \quad S_j^- = \bigcup_{k=1}^{n} S_{jk}^-, \quad j = 1, 2.$$

To prove Theorem 1 (i), we recall some lemmas below. Lemma B is given in Bank and Langley [5, Lemma 3]. We also need Lemma C in Gundersen [7, Corollary 1 to Theorem 2].

Lemma B *Let $P(z)$ be a polynomial of degree $n \geqq 1$, and let $\varepsilon > 0$ be a given constant. Then we have*

(1) *If $\delta(P,\theta) > 0$, then there exists an $r(\theta) > 0$ such that for any $r \geqq r(\theta)$,*

$$|e^{P(re^{i\theta})}| \geqq \exp((1-\varepsilon)\delta(P,\theta)r^n).$$

(2) *If* $\delta(P,\theta)<0$, *then there exists an* $r(\theta)>0$ *such that for any* $r\geqq r(\theta)$,

$$|e^{P(re^{i\theta})}|\leqq \exp((1-\varepsilon)\delta(P,\theta)r^n).$$

Lemma C *Let* f *be a meromorphic function of finite order* ρ, *let* $\varepsilon>0$ *be a given constant and let* $k>j\geqq 0$ *be integers. Then there exists a set* $E_0\subset[0,2\pi)$ *of linear measure zero, such that if* $\theta_0\in[0,2\pi)\setminus E_0$, *then there is a constant* $R_0=R_0(\theta_0)>1$ *such that*

$$\left|\frac{f^{(k)}(re^{i\theta_0})}{f^{(j)}(re^{i\theta_0})}\right|\leqq r^{(k-j)(\rho-1+\varepsilon)}$$

for all $r\geqq R_0$.

Lemma D is the well-known Phragmén-Lindelöf type theorem. We refer to Titchmarsh [12, p.177]. Later we state Lemma 2.3, which is a slightly modified form of Lemma D.

Lemma D *Let* $f(z)$ *be an analytic function of* $z=re^{i\theta}$, *regular in the region* D *between two straight lines making an angle* π/α *at the origin, and on the lines themselves. Suppose that* $|f(z)|\leqq M$ *on the lines, and that, as* $r\to\infty$ $|f(z)|=O(e^{r^\beta})$, *where* $\beta<\alpha$, *uniformly in the angle. Then actually the inequality* $|f(z)|\leqq M$ *holds throughout the region* D.

We need Lemma E in Tohge and the author [9, Theorem 2.1] to prove (ii).

Lemma E *Let* $A(z)$ *be a transcendental entire function of order* $\sigma(A)$. *Suppose that*

$$K\overline{N}\left(r,\frac{1}{A}\right)\leqq T(r,A)+S(r,A),\quad r\notin E$$

holds for a $K>4$ *and an exceptional set* E *of finite linear measure. Then any non-trivial solution* f *of the equation* (1.1) *satisfies* $\lambda(f)\geqq\sigma(A)$.

Moreover, we need the lemmas below.

Lemma 2.1 *Let* $P(z)$ *be a polynomial with* $\delta(P,\theta)<0$ *for a fixed* θ. *Then we have for all* r *sufficiently large*

$$|1+e^{P(re^{i\theta})}|>\frac{1}{2}. \tag{2.1}$$

Further for a set of θ*, say* $G \subset [0, 2\pi)$*, if* $\delta(P, \theta) < 0$*,* $\theta \in G$ *and there exists the* $\max_{\theta\in G} \delta(P, \theta) = \delta_m < 0$*, then we find* $R = R(G)$ *such that* (2.1) *holds for* $r \geqq R$ *and* $\theta \in G$*.*

Proof of Lemma 2.1 Write

$$P(z) = (\alpha + \beta i)z^n + B(z) = (\alpha + \beta i)z^n(1 + D(z)),$$

where α, $\beta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $B(z)$ is a polynomial with $\deg B \leqq n - 1$ and $D(z) = B(z)/((\alpha + \beta i)z^n)$. If we write

$$D(re^{i\theta}) = p(r, \theta)e^{i\varphi(r,\theta)},$$

then we see that $p(r, \theta) \to 0$ as $r \to \infty$ since $\deg B \leqq n - 1$. For the sake of brevity we write $p(r, \theta) = p$ and $\varphi(r, \theta) = \varphi$ respectively. By a simple computation we get

$$P(re^{i\theta}) = r^n(\Delta_1 + \Delta_2 i),$$

where

$$\begin{aligned} \Delta_1 &= \Delta_1(r, \theta) = \delta(P, \theta)(1 + p \cos \varphi) - p\tilde{\delta}(P, \theta) \sin \varphi, \\ \Delta_2 &= \Delta_2(r, \theta) = \tilde{\delta}(P, \theta)(1 + p \cos \varphi) + p\delta(P, \theta) \sin \varphi. \end{aligned}$$

It gives that

$$\begin{aligned} |1 + e^{P(re^{i\theta})}| &= |1 + e^{r^n\Delta_1} \cos(r^n\Delta_2) + e^{r^n\Delta_1} \sin(r^n\Delta_2)i| \\ &= \sqrt{1 + 2e^{r^n\Delta_1} \cos(r^n\Delta_2) + e^{2r^n\Delta_1}} \end{aligned} \tag{2.2}$$

Since $\delta(P, \theta) < 0$ and $p \to 0$ as $r \to \infty$, we have that $\Delta_1(r, \theta) < 0$ for all r large enough. This implies that $e^{r^n\Delta_1} \to 0$ as $r \to \infty$. Hence by (2.2) we get (2.1) immediately.

We consider the latter part of Lemma 2.1. We have that $\max\{|\delta(P, \theta)|, |\tilde{\delta}(P, \theta)|\} \leqq \sqrt{\alpha^2 + \beta^2}$. Hence we can find an $R = R(G)$ such that for $r \geqq R$

$$\begin{aligned} \Delta_1 &\leqq \delta(P, \theta) + 2p\sqrt{\alpha^2 + \beta^2} \\ &\leqq \delta_m + 2p\sqrt{\alpha^2 + \beta^2} \leqq \frac{1}{2}\delta_m < 0, \quad \theta \in G. \end{aligned}$$

As in the same arguments above, the latter part of the assertion of Lemma 2.1 is proved. □

Lemma 2.2 *Let $P_1(z)$ and $P_2(z)$ be polynomials:*

$$P_1(z) = \zeta z^n + B_1(z), \quad P_2(z) = \rho\zeta z^n + B_2(z), \quad n \geqq 1$$

where $\zeta = \alpha + \beta i$, α, $\beta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $\rho \in \mathbb{R}$, $0 < \rho < 1$, $B_1(z)$ and $B_2(z)$ are polynomials with degree at most $n - 1$. Then for any $\varepsilon > 0$, we have

$$m(r, e^{P_1} + e^{P_2}) \geqq (1 - \varepsilon)m(r, e^{P_1}) + O(r^\xi), \quad \text{as } r \to \infty \tag{2.3}$$

where $n - 1 < \xi < n$.

Proof of Lemma 2.2 We denote by θ_0, $0 \leqq \theta_0 < 2\pi/n$ the angle that satisfies $\delta(P_1, \theta_0) = 0$ and $\theta_k = \theta_0 + \frac{\pi k}{n}$, $k = 0, 1, \ldots$. Let $0 < \eta < \pi/2n$, be a real number. We define a set

$$S(\eta) = S^+ \bigcap \Big([0, 2\pi) \setminus \bigcup_{k=0}^{2n-1} \Big[\theta_k - \frac{\eta}{n}, \theta_k + \frac{\eta}{n} \Big] \Big).$$

We define $\sin^+ \theta = \max\{\sin\theta, 0\}$, for $\theta \in [0, 2\pi)$. Then we have

$$\begin{aligned}
&\int_0^{2\pi} \log^+ |e^{\zeta(re^{i\theta})^n}| d\theta \\
&= \int_0^{2\pi} \log^+ |e^{r^n(\delta(P_1,\theta) + i\tilde{\delta}(P_1,\theta))}| d\theta \\
&= r^n \int_0^{2\pi} \log^+ |e^{\delta(P_1,\theta)}| d\theta = r^n \int_{S^+} \delta(P_1, \theta) d\theta \\
&= r^n \sqrt{\alpha^2 + \beta^2} \int_{S^+} \sin(n\theta_0 - n\theta) d\theta \\
&= r^n \sqrt{\alpha^2 + \beta^2} \sum_{k=0}^{2n-1} \int_{\theta_k}^{\theta_{k+1}} \sin^+(n\theta_0 - n\theta) d\theta = 2r^n \sqrt{\alpha^2 + \beta^2}
\end{aligned}$$

While we compute

$$\begin{aligned}
&\int_{S(\eta)} \log^+ |e^{\zeta(re^{i\theta})^n}| d\theta \\
&= r^n \sqrt{\alpha^2 + \beta^2} \int_{S(\eta)} \sin(n\theta_0 - n\theta) d\theta \\
&= r^n \sqrt{\alpha^2 + \beta^2} \sum_{k=0}^{2n-1} \int_{\theta_k + \frac{\eta}{n}}^{\theta_{k+1} - \frac{\eta}{n}} \sin^+(n\theta_0 - n\theta) d\theta \\
&= 2r^n \cos\eta \sqrt{\alpha^2 + \beta^2}.
\end{aligned}$$

Hence setting η being small enough so that $\cos\eta > 1-\varepsilon$, we get

$$(1-\varepsilon)m(r, e^{\zeta z^n}) \leqq \frac{1}{2\pi}\int_{S(\eta)} \log^+ |e^{\zeta(re^{i\theta})^n}|d\theta. \tag{2.4}$$

We have

$$m(r, e^{\zeta z^n}) - m(r, e^{-B_1}) \leqq m(r, e^{P_1}) \leqq m(r, e^{\zeta z^n}) + m(r, e^{B_1}),$$

which implies that

$$m(r, e^{P_1}) = m(r, e^{\zeta z^n}) + O(r^{\xi}), \quad \text{as } r \to \infty. \tag{2.5}$$

We put $P_3(z) = (\rho-1)\zeta z^n + B_2(z) - B_1(z)$. Then we have $\max_{\theta\in S(\eta)} \delta(\theta, P_3) < 0$, By Lemma 2.1, we get

$$\begin{aligned} &m(r, e^{P_1} + e^{P_2}) \\ &\geqq \frac{1}{2\pi}\int_{S(\eta)} \log^+ |e^{P_1(re^{i\theta})} + e^{P_2(re^{i\theta})}|d\theta \\ &\geqq \frac{1}{2\pi}\int_{S(\eta)} \log^+ |e^{\zeta(re^{i\theta})^n}||1 + e^{P_3(re^{i\theta})}|d\theta - O(r^{\xi}) \\ &\geqq \frac{1}{2\pi}\int_{S(\eta)} \log^+ |e^{\zeta(re^{i\theta})^n}|d\theta - O(r^{\xi}), \quad \text{as } r\to\infty. \end{aligned} \tag{2.6}$$

It follows from (2.4), (2.5) and (2.6) that we obtain the assertion (2.3).

□

Lemma 2.3 *Let $U(z)$ be an analytic function of $z = re^{i\theta}$, regular in the region S between two straight lines* $\arg z = \theta_1$ *and* $\arg z = \theta_2$ *making an angle π/α at the origin, and on the lines themselves. Suppose that $|U(z)| \leqq O(r^N)$, $N \in \mathbb{N}$ on the line* $\arg z = \theta_1$ *and $|U(z)| \leqq O(e^{r^{\xi_0}})$ on the line* $\arg z = \theta_2$*, and that, $|U(z)| = O(e^{r^{\beta}})$, as $r \to \infty$ uniformly in the angle where $0 < \xi_0 < \xi < \beta < \alpha$. Then actually the inequality*

$$|U(z)| \leqq O(e^{r^{\xi}}) \tag{2.7}$$

holds throughout the region S.

Proof of Lemma 2.3 Set $g(z) = U(z)/\exp((ze^{-\frac{\theta_1+\theta_2}{2}i})^{\xi})$. Then $g(z)$ is regular in the region between two lines, $\arg z = \theta_1$, $\arg z = \theta_2$. We infer that

$\cos(\arg((ze^{-\frac{\theta_1+\theta_2}{2}i})^\xi)) \geqq \kappa$ for some $\kappa > 0$. In fact

$$-\frac{\pi}{2} < -\frac{\pi\xi}{2\alpha} \leqq -\xi\left(\frac{\theta_2-\theta_1}{2}\right) \leqq \arg((ze^{-\frac{\theta_1+\theta_2}{2}i})^\xi)$$
$$\leqq \xi\left(\frac{\theta_2-\theta_1}{2}\right) \leqq \frac{\pi\xi}{2\alpha} < \frac{\pi}{2}.$$

Hence for $\theta_1 < \theta < \theta_2$

$$|g(re^{i\theta})| \leqq \left|\frac{U(re^{i\theta})}{e^{\kappa r^\xi}}\right| \leqq O(e^{r^\beta}).$$

It follows from the assumption for some $M > 0$

$$|g(re^{i\theta})| \leqq \frac{O(r^N)}{e^{\kappa r^\xi}} \leqq M, \quad \text{on the line } \arg z = \theta_1.$$

and

$$|g(re^{i\theta})| \leqq \frac{O(e^{r^{\xi_0}})}{e^{\kappa r^\xi}} \leqq M, \quad \text{on the line } \arg z = \theta_2$$

By means of Lemma D, we conclude that for any θ (2.7) holds. □

3. Proof of Theorem 1

We will follow the reasoning in Bank and Langley [5], Chiang, Laine and Wang [6] and Ishizaki and Tohge [9] to prove Theorem 1.

Proof of Theorem 1. (i) Suppose that (1.2) possesses a non-trivial solution f such that $\lambda(f) < n$. Write $f = \pi e^h$, where π is the canonical product from zeros of f and h is an entire function. From our hypothesis $\sigma(\pi) = \lambda(\pi) < n$. From (1.2) we get

$$(h')^2 = -h'' - 2\frac{\pi'}{\pi}h' - \frac{\pi''}{\pi} - e^{P_1} - e^{P_2} - Q. \tag{3.1}$$

Eliminating e^{P_1} from (3.1), we have

$$2Uh' = -Q' - h''' + \left(P_1' - 2\frac{\pi''}{\pi}\right)h'' + 2\left(P_1'\frac{\pi'}{\pi} - \left(\frac{\pi'}{\pi}\right)'\right)h'$$
$$+ P_1'\frac{\pi''}{\pi} - \left(\frac{\pi''}{\pi}\right)' + (P_1' - P_2')e^{P_2}, \tag{3.2}$$

where

$$U = h'' - \frac{1}{2}P_1'h'. \tag{3.3}$$

From (3.2) and (3.3), we get

$$C_1(z)h' = C_0(z), \tag{3.4}$$

where

$$\begin{aligned} C_0 &= (P_1' - P_2')e^{P_2} - Q' + \frac{UP_1'}{2} - 2U\frac{\pi''}{\pi} - U' \\ &\qquad + P_1'\frac{\pi''}{\pi} + \frac{\pi'\pi''}{\pi^2} - \frac{\pi'''}{\pi}, \end{aligned} \tag{3.5}$$

$$C_1 = 2U - 2P_1'\frac{\pi'}{\pi} + P_1'\frac{\pi''}{\pi} - 2\left(\frac{\pi'}{\pi}\right)^2 + 2\frac{\pi''}{\pi} - \frac{(P_1')^2}{4} + \frac{P_1''}{2} \tag{3.6}$$

If we suppose that $C_0(z) \not\equiv 0$ and $C_1(z) \not\equiv 0$ in (3.4), then we have by the first fundamental theorem

$$T(r, h') \leqq T(r, C_0) + T(r, C_1) + O(1). \tag{3.7}$$

We estimate $T(r, h')$, $T(r, C_0)$ and $T(r, C_1)$ in (3.7) respectively.

We set $\max\{\sigma(Q), \lambda(f)\} < \xi_1 < \xi_2 < \xi < n$. First we consider $T(r, h')$. We see that

$$T(r, Q) = m(r, Q) \leqq O(r^{\xi_1}), \quad \text{as } r \to \infty.$$

By applying the Clunie Lemma to (3.1), we obtain

$$\begin{aligned} T(r, h') &\leqq m(r, Q) + m\left(r, \frac{\pi''}{\pi}\right) \\ &\qquad + m\left(r, \frac{\pi'}{\pi}\right) + m(r, e^{P_1} + e^{P_2}) + S(r, h') \\ &\leqq O(r^{n+\varepsilon_0}) + S(r, h'), \quad \text{for any } \varepsilon_0 > 0 \end{aligned}$$

which implies that $\sigma(h') \leqq n$. Hence, from (3.1) and the theorem on the logarithmic derivatives

$$\begin{aligned} m(r, e^{P_1} + e^{P_2}) &\leqq 2m(r, h') + m(r, Q) \\ &\qquad + m\left(r, \frac{h''}{h'}\right) + m\left(r, \frac{\pi'}{\pi}\right) + m\left(r, \frac{\pi''}{\pi}\right) \\ &\leqq 2T(r, h') + O(r^{\xi}) + O(\log r), \quad \text{as } r \to \infty. \end{aligned}$$

By means of Lemma 2.2, for $1-2\rho>\varepsilon>0$,

$$m(r,e^{P_1}+e^{P_2})\geqq(1-\varepsilon)T(r,e^{P_1})+O(r^{\xi}),$$

hence we have

$$T(r,h')\geqq\frac{1}{2}(1-\varepsilon)T(r,e^{P_1})+O(r^{\xi}). \tag{3.8}$$

Secondly we estimate $T(r,C_0)$ and $T(r,C_1)$. To do this, we first estimate the growth of $|U(re^{i\theta})|$. Since $\zeta_2/\zeta_1=\rho$ is real and positive, we have $\delta(P_2,\theta)=\rho\delta(P_1,\theta)$ which implies that S_{1k}^{+}, $k=0,1,\dots,n$ coincide with S_{2k}^{+} and also S_{1k}^{-}, $k=0,1,\dots,n$ coincide with S_{2k}^{-}. Thus for the sake of simplicity we write $S_1^{+}=S_2^{+}=S^{+}$ and $S_1^{-}=S_2^{-}=S^{-}$. We assert that for any θ, we have

$$|U(re^{i\theta})|\leqq O(e^{r^{\xi}}),\quad \text{as } r\to\infty. \tag{3.9}$$

We show (3.9) dividing the proof into two cases when $\theta\in S^{+}$ and $\theta\in S^{-}$.

Assume that $\theta\in S^{-}\setminus E_0$, where E_0 is of linear measure zero. In the case $|h'(re^{i\theta})|<1$, from (3.3) we have

$$|U(re^{i\theta})|\leqq\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right|+\frac{1}{2}|P_1'(re^{i\theta})|. \tag{3.10}$$

If $|h'(re^{i\theta})|\geqq 1$, then from (3.2),

$$\begin{aligned}|2U(re^{i\theta})|\ \leqq\ &\left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right|+\left(|P_1'(re^{i\theta})|+2\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right|\\&+2\left(|P_1'(re^{i\theta})|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right)\\&+|P_1'(re^{i\theta})|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|+\left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right|\\&+\left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right|+(|P_1'(re^{i\theta})|\\&+|P_2'(re^{i\theta})|)|e^{P_2(re^{i\theta})}|+\frac{|Q'(re^{i\theta})|}{|Q(re^{i\theta})|}|Q(re^{i\theta})|.\end{aligned} \tag{3.11}$$

We note that for any fixed θ we have that $|Q(re^{i\theta})|\leqq e^{r^{\xi_1}}$ for all r sufficiently large. Since Q and h' are of finite order, by means of Lemma C, (3.10) and

(3.11), we obtain

$$|U(re^{i\theta})| \leqq O(e^{r^{\varepsilon_2}}), \quad \text{as } r \to \infty. \tag{3.12}$$

Next we treat the case $\theta \in S^+ \setminus E_0$. We write $\delta(P_1, \theta)$ as δ_1 for the simplicity and set $\rho\delta_1 < \sigma_2 < \sigma_1 < \delta_1$, $0 < \varepsilon_1 < 1 - \sigma_1/\delta_1$, $0 < \varepsilon_2 < (\sigma_2/2 - \rho\delta_1)/(\rho\delta_1)$. In view of Lemma B, we have

$$\begin{aligned} &|e^{P_1(re^{i\theta})} + e^{P_2(re^{i\theta})} + Q(re^{i\theta})| \\ &\geqq |e^{P_1(re^{i\theta})}| \left| 1 - |e^{P_2(re^{i\theta}) - P_1(re^{i\theta})}| - \frac{|Q(re^{i\theta})|}{|e^{P_1(re^{i\theta})}|} \right| \\ &\geqq e^{(1-\varepsilon_1)\delta_1 r^n}(1 - o(1)) \\ &\geqq e^{\sigma_1 r^n}(1 - o(1)), \quad \text{as } r \to \infty. \end{aligned} \tag{3.13}$$

Suppose that there exists an unbounded sequence $\{r_q\}$ such that $0 < |h'(r_q e^{i\theta})| \leqq 1$. From (3.1), (3.13) and by Lemma C, we get for an N_1

$$\begin{aligned} e^{\sigma_1 r_q^n}(1 + o(1)) &\leqq 1 + \left| \frac{h''(r_q e^{i\theta})}{h'(r_q e^{i\theta})} \right| + 2\left| \frac{\pi'(r_q e^{i\theta})}{\pi(r_q e^{i\theta})} \right| + \left| \frac{\pi''(r_q e^{i\theta})}{\pi(r_q e^{i\theta})} \right| \\ &\leqq O(r_q^{N_1}), \quad \text{as } q \to \infty, \end{aligned}$$

which is absurd. Hence we may assume that $|h'(re^{i\theta})| \geqq 1$ for all sufficiently large r. It follows from (3.1) and Lemma C, for an N_2,

$$\begin{aligned} &|e^{P_1(re^{i\theta})} + e^{P_2(re^{i\theta})} + Q(re^{i\theta})| \\ &\leqq |h'(re^{i\theta})|^2 \left(1 + \left| \frac{h''(re^{i\theta})}{h'(re^{i\theta})} \right| + 2\left| \frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})} \right| + \left| \frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})} \right| \right) \\ &\leqq |h'(re^{i\theta})|^2 (1 + O(r^{N_2})), \quad \text{as } r \to \infty. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we get for all r sufficiently large

$$|h'(re^{i\theta})|^2 \geqq \frac{1 - o(1)}{1 + O(r^{N_2})} e^{\sigma_1 r^n} \geqq e^{\sigma_2 r^n},$$

thus we obtain for all r large enough

$$|h'(re^{i\theta})| \geqq e^{\frac{1}{2}\sigma_2 r^n}. \tag{3.15}$$

It follows from (3.2) and (3.15) that

$$
\begin{aligned}
&|2U(re^{i\theta})| \\
&\leqq \left|\frac{Q(re^{i\theta})}{h'(re^{i\theta})}\right| + \left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right| + \left(|P_1'(re^{i\theta})| + 2\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| \\
&\quad + 2\left(|P_1'(re^{i\theta})|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right) \\
&\quad + |P_1'(re^{i\theta})|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right| \\
&\quad + (|P_1'(re^{i\theta})| + |P_2'(re^{i\theta})|)\left|\frac{e^{P_2(re^{i\theta})}}{h'(re^{i\theta})}\right| \\
&\leqq O(r^{N_2}) + (1+o(1))\exp\left(\left(\rho\delta_1(1+\varepsilon_2) - \frac{\sigma_2}{2}\right)r^n\right), \quad \text{as } r \to \infty.
\end{aligned}
$$

Since $\rho\delta_1(1+\varepsilon_2) - \sigma_2/2 < 0$, it gives that for an N_3

$$|U(re^{i\theta})| \leqq O(r^{N_3}), \quad \text{as } r \to \infty. \tag{3.16}$$

Now we fix a $\gamma(=\gamma_k) \in S_k^+ \setminus E_0$, $k = 1, 2, \ldots, n$. Then we find γ_1, $\gamma_2 \in S^- \setminus E_0$, $\gamma_1 < \gamma < \gamma_2$ such that $\gamma - \gamma_1 < \pi/n$, $\gamma_2 - \gamma < \pi/n$. Write $\gamma - \gamma_1 = \pi/(n+\tau)$, $\tau > 0$. From (3.12) on $\arg z = \gamma_1$, we have that $|U(z)| \leqq O(e^{r^{\xi_2}})$, as $r \to \infty$, $\xi_2 < n + \tau$. While from (3.16) on $\arg z = \gamma$ we have $|U(z)| \leqq O(r^{N_3})$. By Lemma 2.3, we obtain (3.9). Similarly, we see that (3.9) holds for $\gamma < \theta < \gamma_2$. Hence we conclude that for any θ (3.9) holds.

By our assumption $\lambda(f) < \xi < n$, we have $\overline{N}(r, 1/\pi) \leqq O(r^{\xi})$, as $r \to \infty$. From (3.5), (3.9) and by the theorem on the logarithmic derivatives

$$
\begin{aligned}
T(r, C_0) &\leqq 3\overline{N}\left(r, \frac{1}{\pi}\right) + 3m(r, U) + O(\log r) \\
&\qquad + 2m(r, Q) + m(r, e^{\rho\zeta_1 z^n}) + O(r^{\xi}) \\
&\leqq \rho T(r, e^{P_1}) + O(r^{\xi}), \quad \text{as } r \to \infty.
\end{aligned} \tag{3.17}
$$

Similarly from (3.6) and (3.9) we get

$$
\begin{aligned}
T(r, C_1) &\leqq 2\overline{N}\left(r, \frac{1}{\pi}\right) + m(r, U) + O(\log r) \\
&\leqq O(r^{\xi}), \quad \text{as } r \to \infty.
\end{aligned} \tag{3.18}
$$

Combining (3.7), (3.8), (3.17) and (3.18), we obtain

$$\begin{aligned}\frac{1}{2}(1-\varepsilon)T(r,e^{P_1})+O(r^{\xi}) &\leqq T(r,h')\\ &\leqq \rho T(r,e^{P_1})+O(r^{\xi}), \quad \text{as } r\to\infty\end{aligned}$$

which implies that

$$\left(\frac{1}{2}(1-\varepsilon)-\rho-o(1)\right)T(r,e^{P_1})\leqq 0, \quad \text{as } r\to\infty.$$

This yields a contradiction when $0<\rho<1/2$. Hence we conclude that $C_0(z)\equiv C_1(z)\equiv 0$. It follows from (3.5) that

$$\begin{aligned}T(r,e^{P_2}) &\leqq 3\overline{N}\left(r,\frac{1}{\pi}\right)+3m(r,U)+O(\log r)+2m(r,Q)+O(r^{\xi})\\ &\leqq O(r^{\xi}), \quad \text{as } r\to\infty,\end{aligned}$$

which implies $\sigma(e^{P_2})<\xi<n$. This is a contradiction. Hence we have proved (i).

Now we shall prove (ii). Write

$$\begin{aligned}A(z) &:= e^{P_1(z)}+e^{P_2(z)}=e^{\zeta_1 z^n+B_1(z)}+e^{\rho\zeta_1 z^n+B_2(z)}\\ &= e^{\rho\zeta_1 z^n}\left(e^{(1-\rho)\zeta_1 z^n+B_1(z)}+e^{B_2(z)}\right).\end{aligned}$$

In view of Lemma 2.2, setting $0<\varepsilon<4\rho-3$, $0<\xi<n$, we get

$$\begin{aligned}T(r,A) &\geqq (1-\varepsilon)T(r,e^{P_1})+O(r^{\xi})\\ &\geqq (1-\varepsilon)T(r,e^{\zeta_1 z^n})+O(r^{\xi}), \quad \text{as } r\to\infty.\end{aligned} \tag{3.19}$$

We have

$$N(r,1/A)\leqq(1-\rho)T(r,e^{\zeta_1 z^n})+O(r^{\xi}), \quad \text{as } r\to\infty. \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\frac{1-\varepsilon}{1-\rho}N\left(r,\frac{1}{A}\right)\leqq T(r,A)+S(r,A), \quad 4<\frac{1-\varepsilon}{1-\rho}.$$

By Lemma E, we obtain $\lambda(f)\geqq n$. Theorem 1 is thus proved. □

Acknowledgment The author would like to acknowledge the referee who gave many suggestions. The author also would like to thank Professor Dr. N. Yanagihara for helpful discussions and suggestions.

References

[1] Bank S.B. and Laine I., *On the oscillation theory of $f'' + Af = 0$ where A is entire.* Trans. Amer. Math. **273** (1982), 351–363.

[2] Bank S.B. and Laine I., *Representations of solutions of periodic second order linear differential equations.* J. Reine Angew. Math. **344** (1983), 1–21.

[3] Bank S.B, Laine I. and Langley J.K., *On the frequency of zeros of solutions of second order linear differential equations.* Resultate Math. **10** (1986), 8–24.

[4] Bank S.B, Laine I. and Langley J.K., *Oscillation results for solutions of linear differential equations in the complex domain.* Resultate Math. **16** (1989), 3–15.

[5] Bank S.B. and Langley J.K., *On the oscillation of certain linear differential equation in the complex domain.* Proc. Edinburgh Math. Soc. (2) **30** (1987), 455–469.

[6] Chiang Y.-K., Laine I. and Wang S.-P., *Oscillation results for some linear differential equations.* Math. Scand. **77** (1995), 209–224.

[7] Gundersen G.G., *Estimates for the logarithmic derivatives of a meromorphic function, plus similar estimates.* J. London Math. Soc. (2) 37, (1988), 88–104.

[8] Hayman H.K., *Meromorphic functions.* Clarendon Press, Oxford, 1964.

[9] Ishizaki K. and Tohge K., *On the complex oscillation of some linear differential equations.* J. Math. Anal. Appl. (to appear).

[10] Laine I., *Nevanlinna theory and complex differential equations.* W. Gruyter, Berlin-New York, 1992.

[11] Nevanlinna R., *Analytic Functions.* Springer Verlag, Berlin-Heidelberg-New York, 1970.

[12] Titchmarsh E.C., *The theory of functions.* Oxford, 1986.

[13] Tohge K., *Remarks on the $\frac{1}{16}$-Theorem.* Preprint.

[14] Yang C.-C., Wen G.-C., Li K.-Y. and Chiang Y.-M., *Complex analysis and its applications.* Pitman Research Notes in Mathematics Series 305, Longman UK (copublished with Wiley New York), 1994.

Department of Mathematics
NIPPON Institute of Technology
4-1 Gakuendai Miyashiro, Minamisaitama
Saitama 345, Japan
E-mail: ishi@nit.ac.jp