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Extrinsic shape of circles and the standard imbedding of a Cayley projective plane

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Abstract. The main purpose of this paper is to give a characterization of the parallel imbedding of a Cayley projective plane $P_{Cay}(c)$ into a real space form in terms of the extrinsic shape of particular circles on $P_{Cay}(c)$.

Key words: cayley projective plane, parallel imbedding, cayley circle, totally real circle.

1. Introduction

To what extent can we determine the properties of a submanifold by observing the extrinsic shape of geodesics or circles of a submanifold? As typical cases, we recall that a submanifold is totally geodesic (resp. totally umbilic with parallel mean curvature vector) if and only if *all* geodesics (resp. circles) of the submanifold are geodesics (resp. circles) in the ambient space ([7]).

On the other hand, it is well-known that a sphere is the only surface in E^3 all of whose geodesics are circles in E^3. This result is generalized as follows: A submanifold of a real space form is isotropic and parallel if and only if all geodesics of the submanifold are circles in the ambient space ([4], [9]).

Then, what is the extrinsic shape of circles of an isotropic parallel submanifold of a real space form? An isotropic parallel submanifold of a real space form is locally equivalent either to the first standard imbedding of one of the compact symmetric spaces of rank one or to the second standard imbedding of a sphere. It is proved in [3] that the image of a circle under the first standard imbedding of a real projective space or the second standard imbedding of a sphere is *never* a circle in the ambient space. On the contrary, *some* circles of a complex projective space or a quaternionic projective space are mapped to circles in the ambient space under the first standard imbedding ([1]).

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Our purpose of this paper is to prove that *some* circles of a Cayley projective plane are mapped to circles in the ambient space under the first standard imbedding and to give some characterizations of the first standard imbeddings of a Cayley projective plane by observing the extrinsic shape of particular circles.

2. Cayley circles

We first review the definition of circles. A curve $\gamma = \gamma(s)$, parametrized by arclength s, in a Riemannian manifold M is called a *circle* if there exist a field $Y = Y(s)$ of unit vectors along γ and a positive constant k which satisfy

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = kY \\ \nabla_{\dot\gamma}Y = -k\dot\gamma, \end{cases} \tag{2.1}$$

where $\dot\gamma$ denotes the unit tangent vector of γ and ∇ the covariant differentiation. The constant k is called the *curvature* of the circle. For an arbitrary point x, an arbitrary orthonormal pair (u, v) of vectors at x and an arbitrary positive number k, there exists a unique circle $\gamma = \gamma(s)$ with initial condition $\gamma(0) = x, \dot\gamma(0) = u$ and $Y(0) = v$. For detail, see [7].

It follows from (2.1) that the sectional curvature $K(\dot\gamma, Y)$ given by the plane spanned by $\dot\gamma$ and Y is constant along γ if M is locally symmetric. Therefore, in a Cayley projective plane $P_{Cay}(c)$ of maximal sectional curvature c, we define a *Cayley* circle as a circle γ which satisfies $K(\dot\gamma, Y) = c$. The extrinsic shape of Cayley circles through the first standard minimal imbedding of a Cayley projective plane will be studied in section 4.

3. Isotropic immersions

First of all, we recall the notion of isotropic immersions. Let M and $\widetilde{M}$ be Riemannian manifolds and $f : M \longrightarrow \widetilde{M}$ be an isometric immersion. We denote by ∇ and $\widetilde\nabla$ the Riemannian connections of M and $\widetilde{M}$, respectively, and by σ the second fundamental form of f. Then the Gauss formula is given by

$$\widetilde\nabla_X Z = \nabla_X Z + \sigma(X, Z) \tag{3.1}$$

and the Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = \nabla^{\perp}_X \xi - A_\xi X, \tag{3.2}$$

where $\nabla^{\perp}$ denotes the covariant differentiation in the normal bundle and A_ξ the shape operator in the direction of ξ so that $\langle A_\xi X, Z\rangle = \langle \sigma(X,Z), \xi\rangle$.

The immersion f is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2$ is constant on the tangent space $T_x(M)$ of M at x. If the immersion is isotropic at every point, then the immersion is said to be isotropic. Note that a totally umbilic immersion is isotropic, but not *vice versa.*

The following is well-known ([8]).

Lemma 1 *Let $f : M \longrightarrow \widetilde{M}$ be an isometric immersion. Then f is isotropic at $x \in M$ if and only if $\langle \sigma(X,X), \sigma(X,Y)\rangle = 0$ for an arbitrary orthgonal pair $X, Y \in T_x(M)$, or equivalently, $A_{\sigma(X,X)}X$ is proportional to X for an arbitrary $X \in T_x(M)$.*

Lemma 2 *Let $f : M \longrightarrow \widetilde{M}$ be an isotropic parallel immersion and γ be a circle on M. Then $f(\gamma)$ is a circle on $\widetilde{M}$ if and only if $\sigma(\dot\gamma(0), Y(0)) = 0$.*

Proof. Let γ be a circle of curvature k on M. Put $\lambda = \|\sigma(\dot\gamma,\dot\gamma)\|$. Then λ is constant, since the second fundamental form is parallel and isotropic (see, Lemma 1). It follows from Lemma 1 that $A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma = \lambda\dot\gamma$. Since σ is parallel, we get from (2.1) that

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \sqrt{k^2+\lambda^2}\,\widetilde{Y} \tag{3.3}$$

and

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{Y} = -\sqrt{k^2+\lambda^2}\,\dot\gamma + \frac{3k}{\sqrt{k^2+\lambda^2}}\sigma(\dot\gamma, Y), \tag{3.4}$$

where

$$\widetilde{Y} = \frac{1}{\sqrt{k^2+\lambda^2}}\left\{kY + \sigma(\dot\gamma,\dot\gamma)\right\}.$$

It follows from (2.1) and Lemma 1 that $\|\sigma(\dot\gamma, Y)\|$ is constant along γ so that $\sigma(\dot\gamma, Y) = 0$ along γ. Therefore (3.4) reduces to

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{Y} = -\sqrt{k^2+\lambda^2}\,\dot\gamma. \tag{3.5}$$

(3.3) and (3.5) tell us that $f(\gamma)$ is a circle on $\widetilde{M}$. □

4. Extrinsic shape of Cayley circles via first standard minimal imbedding

It is known that the parallel imbedding of a Cayley projective plane $P_{Cay}(c)$ of maximal sectional curvature c into a real space form $\widetilde{M}^{16+p}(\tilde{c})$ of curvature $\tilde{c}$ is nothing but the first standard minimal imbedding $f : P_{Cay}(c) \longrightarrow S^{25}(\frac{3c}{4})$ followed by a totally umbilic imbedding into $\widetilde{M}^{16+p}(\tilde{c})$ ([4, 9]). As for the extrinsic shape of circles on $P_{Cay}(c)$ *via* f, we have the following.

Proposition 1 *The first standard minimal imbedding of $P_{Cay}(c)$ into $S^{25}(\frac{3c}{4})$ maps a Cayley circle of curvature k to a circle of curvature $\sqrt{k^2 + c/4}$.*

Proof. Let $f : P_{Cay}(c) \longrightarrow S^{25}(\frac{3c}{4})$ be the first standard minimal imbedding and let γ be a Cayley circle of curvature k on $P_{Cay}(c)$. Then the equation of Gauss yields

$$\begin{aligned} c &= \langle R(\dot{\gamma}, Y)Y, \dot{\gamma}\rangle \\ &= \frac{3c}{4} + \langle \sigma(\dot{\gamma}, \dot{\gamma}), \sigma(Y, Y)\rangle - \|\sigma(\dot{\gamma}, Y)\|^2, \end{aligned}$$

that is,

$$\|\sigma(\dot{\gamma}, Y)\|^2 = \langle \sigma(\dot{\gamma}, \dot{\gamma}), \sigma(Y, Y)\rangle - \frac{c}{4}. \tag{4.1}$$

On the other hand, since f is isotropic and it satisfies $\|\sigma(X, X)\|/\|X\|^2 = \sqrt{c}/2$ ([4]), we have

$$\begin{aligned} &\langle \sigma(X, Y), \sigma(Z, W)\rangle + \langle \sigma(X, Z), \sigma(Y, W)\rangle + \langle \sigma(X, W), \sigma(Y, Z)\rangle \\ &\quad = \frac{c}{4}(\langle X, Y\rangle\langle Z, W\rangle + \langle X, Z\rangle\langle Y, W\rangle + \langle X, W\rangle\langle Y, Z\rangle) \end{aligned}$$

for arbitrary X, Y, Z, W. Then, in particular, we get

$$2\|\sigma(\dot{\gamma}, Y)\|^2 + \langle \sigma(\dot{\gamma}, \dot{\gamma}), \sigma(Y, Y)\rangle = \frac{c}{4}. \tag{4.2}$$

Since we have $\sigma(\dot{\gamma}, Y) = 0$ from (4.1) and (4.2), our Proposition 1 follows from Lemma 2. □

5. Characterization of standard imbedding of Cayley projective plane by observing extrinsic shape of Cayley circles

We consider converses of Proposition 1 to obtain a characterization of the first standard minimal imbedding of a Cayley projective plane. First we prove the following.

Theorem 1 *Let M be an open set of $P_{Cay}(c)$ which is isometrically immersed into a real space form $\widetilde{M}^{16+p}(\tilde{c})$. If there exists $k > 0$ and all Cayley circles of curvature k on M are circles in $\widetilde{M}^{16+p}(\tilde{c})$, then M is locally congruent to a Cayley projective plane imbedded into $S^{25}(\frac{3c}{4})$ in $\widetilde{M}^{16+p}(\tilde{c})$ through the first standard minimal imbedding.*

Proof. Let $\gamma = \gamma(s)$ be a Cayley circle of curvature k on M so that it satisfies (2.1). Then, since γ is a circle as a curve in $\widetilde{M}^{16+p}(\tilde{c})$, it satisfies

$$\begin{cases} \widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \widetilde{k}\widetilde{Y} \\ \widetilde{\nabla}_{\dot{\gamma}}\widetilde{Y} = -\widetilde{k}\dot{\gamma}, \end{cases} \tag{5.1}$$

for some positive constant $\widetilde{k}$ and some field $\widetilde{Y}$ of unit vectors, where $\widetilde{\nabla}$ denotes the covariant differentiation on $\widetilde{M}^{16+p}(\tilde{c})$. Equations (2.1) and (5.1), together with the formulae of Gauss and Weingarten, yield

$$A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} = (\widetilde{k}^2 - k^2)\dot{\gamma} \tag{5.2}$$

and

$$\nabla^{\perp}_{\dot{\gamma}}\sigma(\dot{\gamma},\dot{\gamma}) + k\sigma(\dot{\gamma}, Y) = 0. \tag{5.3}$$

It follows from (5.2) that

$$\langle A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}, Z\rangle = 0$$

or equivalently

$$\langle \sigma(\dot{\gamma},\dot{\gamma}), \sigma(\dot{\gamma}, Z)\rangle = 0$$

for all Z orthogonal to $\dot{\gamma}$. Since $\dot{\gamma}$ is arbitrary, it follows from Lemma 1 that M is isotropic. Defining the covariant derivative $\nabla'_X\sigma$ of σ by

$$(\nabla'_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

we get from (5.3) that

$$(\nabla'_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma) + 3k\sigma(\dot\gamma, Y) = 0. \tag{5.4}$$

Consider another Cayley circle γ_1 of curvature k with $\gamma_1(0) = \gamma(0), \dot\gamma_1(0) = \dot\gamma(0)$ and $Y_1(0) = -Y(0)$. Then we obtain

$$(\nabla'_{\dot\gamma_1}\sigma)(\dot\gamma_1,\dot\gamma_1) + 3k\sigma(\dot\gamma_1, Y_1) = 0. \tag{5.5}$$

Therefore it follows from (5.4) and (5.5) that

$$(\nabla'_{\dot\gamma(0)}\sigma)(\dot\gamma(0),\dot\gamma(0)) = 0. \tag{5.6}$$

Since γ is arbitrary so that $\dot\gamma(0)$ is arbitrary, thanks to the equation of Codazzi $\nabla'_X\sigma(Y,Z) = \nabla'_Y\sigma(X,Z)$, we get $\nabla'\sigma = 0$.

Thus our assertion follows from the results of [7] and [9]. □

6. Totally real circles

By Proposition 1 in section 4, we know the extrinsic shape of Cayley circles on $P_{Cay}(c)$ *via* the first standard minimal imbedding $f : P_{Cay}(c) \longrightarrow S^{25}(\frac{3c}{4})$. Then, what can we say about the extrinsic shape of circles on $P_{Cay}(c)$ which are not Cayley? In particular, we consider circles which are as far from being Cayley as possible. A circle γ on $P_{Cay}(c)$ is said to be *totally real* if it satisfies $K(\dot\gamma, Y) = \frac{c}{4}$. We consider the problem: *What does a totally real circle on $P_{Cay}(c)$ look like in $S^{25}(\frac{3c}{4})$?* To answer this problem, we first prove the following.

Proposition 2 *Let $g : P^2_R(\frac{c}{4}) \longrightarrow S^4(\frac{3c}{4})$ be the first standard minimal imbedding of real projective plane $P^2_R(\frac{c}{4})$ of curvature $\frac{c}{4}$ into a 4-dimensional sphere $S^4(\frac{3c}{4})$ of curvature $\frac{3c}{4}$. Then*

(i) *g maps each geodesic to a circle of curvature $\frac{\sqrt{c}}{2}$.*

(ii) *g maps each circle of curvature $\frac{\sqrt{c}}{2\sqrt{2}}$ to a helix of order 3 of curvatures $\frac{\sqrt{3c}}{2\sqrt{2}}, \frac{\sqrt{3c}}{2}$.*

(iii) *g maps each circle of curvature $k \neq \frac{\sqrt{c}}{2\sqrt{2}}$ to a helix of order 4 of curvatures $\frac{\sqrt{4k^2+c}}{2}, \frac{3k\sqrt{c}}{\sqrt{4k^2+c}}, \frac{|8k^2-c|}{2\sqrt{4k^2+c}}$.*

Proof. Note that g is a $\frac{\sqrt{c}}{2}$-isotropic parallel imbedding and it satisfies (cf. [4])

$$\langle\sigma(X,Y),\sigma(Z,W)\rangle \\ = -\frac{c}{4}(\langle X,Y\rangle\langle Z,W\rangle - \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle). \tag{5.7}$$

Let γ be a geodesic of $P_R^2(\frac{c}{4})$. Then the argument similar to Lemma 2, combined with (5.7), proves (i).

Let γ be a circle of curvature k in $P_R^2(\frac{c}{4})$ so that it satisfies $\nabla_{\dot\gamma}\dot\gamma = kY$ and $\nabla_{\dot\gamma}Y = -k\dot\gamma$. We denote by $\widetilde\nabla$ the covariant differentiation of $S^4(\frac{3c}{4})$. Then the Gauss formula gives

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = k_1\xi_2, \tag{5.8}$$

where

$$k_1 = \frac{\sqrt{4k^2+c}}{2} \tag{5.9}$$

and

$$\xi_2 = \frac{2}{\sqrt{4k^2+c}}\left(kY + \sigma(\dot\gamma,\dot\gamma)\right). \tag{5.10}$$

Differentiating (5.10), we obtain

$$\widetilde\nabla_{\dot\gamma}\xi_2 = -k_1\dot\gamma + \frac{6k}{\sqrt{4k^2+c}}\sigma(\dot\gamma,Y).$$

Therefore, if we put

$$k_2 = \frac{3k\sqrt{c}}{\sqrt{4k^2+c}} \tag{5.11}$$

and

$$\xi_3 = \frac{2}{\sqrt{c}}\sigma(\dot\gamma,Y), \tag{5.12}$$

then we have

$$\widetilde\nabla_{\dot\gamma}\xi_2 = -k_1\dot\gamma + k_2\xi_3. \tag{5.13}$$

Similarly, differentiating (5.12), we obtain

$$\widetilde\nabla_{\dot\gamma}\xi_3 = -k_2\xi_2 + k_3\xi_4, \tag{5.14}$$

where

$$k_3 = \frac{|8k^2 - c|}{2\sqrt{4k^2 + c}} \tag{5.15}$$

and

$$\xi_4 = \frac{\sqrt{c}}{\sqrt{4k^2+c}}Y + \frac{8k(c-2k^2)}{(8k^2-c)\sqrt{c(4k^2+c)}}\sigma(\dot{\gamma},\dot{\gamma}) + \frac{4k\sqrt{4k^2+c}}{(8k^2-c)\sqrt{c}}\sigma(Y,Y).$$

From (5.8), (5.9), (5.11), (5.13), (5.14) and (5.15) we get (ii) and (iii). □

We see from Remark 2.2 in [6] that every circle of $P_{Cay}(c)$ is contained in some totally geodesic $P^2_C(c)$. This, combined with Proposition 2 in [2], implies that every totally real circle of $P_{Cay}(c)$ is contained in some totally geodesic $P^2_R(c/4)$.

$$\begin{array}{ccc} P^2_R(\frac{c}{4}) & \xrightarrow{\ g\ } & S^4(\frac{3c}{4}) \\ {\scriptstyle t.g.}\downarrow & & \downarrow{\scriptstyle t.g.} \\ P_{Cay}(c) & \xrightarrow[\ f\ ]{} & S^{25}(\frac{3c}{4}) \end{array}$$

Therefore our Proposition 2 yields

Theorem 2 *The first standard minimal imbedding of $P_{Cay}(c)$ into $S^{25}(\frac{3c}{4})$ maps a totally real circle to a helix of order* 3 *or* 4.

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