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On defect groups of the Mackey algebras for finite groups

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Abstract. In this paper, we introduce a new Mackey functor $\mathcal{T}$ and give a relation of ordinary defect group and defect group of the Mackey algebra of a finite group.

Key words: Mackey algebra, Mackey functor, group representation, block, defect group.

1. Introduction

The Mackey algebra $\mu_R(G)$ of a finite group G over a commutative ring R introduced by J. Thévenaz and P.J. Webb [TW] for studying the structure of Mackey functors. This is an algebra of finite rank over R with the property that the category of Mackey functors of G over R is equivalent to the category of left $\mu_R(G)$-modules. So Thévenaz and Webb studied the blocks of Mackey functors in terms of the simple Mackey functors. In [TW] they determined the division of the simple Mackey functors into blocks of Mackey functors.

On the other hand, Yoshida introduced the span ring of the category of finite G-sets and gave the formula of the centrally primitive idempotents of the span ring [Yo]. It is interesting that the Mackey algebra $\mu_R(G)$ is isomorphic to the span ring of the category of finite G-sets. We can apply the formula of the span ring to the Mackey algebra $\mu_R(G)$. A centrally primitive idempotent of the span ring is indexed by the p-perfect subgroup J and the p-block of $N_G(J)/J$. In particular, we consider that the p-blocks of the group algebra of G is the corresponding centrally primitive idempotents of the span ring indexed by the trivial subgroup and p-blocks of $N_G(1)/1 = G$.

In this paper, we consider a defect group of the blocks of Mackey functors of G like as the ordinary block theory. The word "blocks of Mackey functors" means two-sided direct summands of $\mu_R(G)$ or the corresponding centrally primitive idempotents of $\mu_R(G)$. We introduce a Mackey functor $\mathcal{T}$ for the sake of the definition of a defect groups of blocks of Mackey functors. The inductions of $\mathcal{T}$ are generalization of the trace maps of the group

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algebra as a G-algebra. We can give the definition a defect group of $\mu_R(G)$ using a Mackey functor $\mathcal{T}$ and study the relation of the group algebra.

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2. Mackey functor $\mathcal{T}$

In this section, we introduce a new Mackey functor $\mathcal{T}$ for a finite group G. Let R be a commutative ring and $\mu_R(G)$ a Mackey algebra over R. Let α be an R-algebra homomorphism

$$\alpha : \mu_R(H) \longrightarrow \mu_R(G)$$

which is terms of symbols are $\alpha(I_K^J) = I_K^J$, $\alpha(R_K^J) = R_K^J$ and $\alpha(c_h^K) = c_h^K$ for all subgroups $K \leq J \leq H \leq G$, and $h \in H$. In general, α is not injective.

For a subgroup H of G, we put

$$\mu_R(G)_H = \{x \in \mu_R(G) \mid \alpha(\theta)x = x\alpha(\theta) \text{ for all } \theta \in \mu_R(H)\}.$$

Moreover, we put

$$\mathcal{T}(H) := \mu_R(G)^H := 1_H \mu_R(G)_H$$

where

$$1_H = \sum_{L \leq H} I_L^L \in \mu_R(H).$$

In particular, $\mathcal{T}(G)$ is the center of $\mu_R(G)$.

For all subgroups $K \leq H \leq G$ and $g \in G$ we define the R-homomorphisms $\mathcal{I}_K^H$, $\mathcal{R}_K^H$, $\mathcal{C}_g^H$ as follows:

$$\mathcal{I}_K^H : \mathcal{T}(K) \to \mathcal{T}(H) : \theta \mapsto \sum_{L \leq H} \sum_{h \in [L \backslash H / K]} I_{L \cap {}^h K}^L c_h^{L^h \cap K} \theta c_{h^{-1}}^{L \cap {}^h K} R_{L \cap {}^h K}^L,$$

$$\mathcal{R}_K^H : \mathcal{T}(H) \to \mathcal{T}(K) : \theta \mapsto 1_K \theta,$$

$$\mathcal{C}_g^H : \mathcal{T}(H) \to \mathcal{T}({}^g H) : \theta \mapsto \sum_{L \leq {}^g H} c_g^L \theta c_{g^{-1}}^{{}^g L}.$$

Proposition 1 *Let $\mathcal{T}$ be as above notation with morphisms $\mathcal{I}$, $\mathcal{R}$, $\mathcal{C}$. Then $\mathcal{T}$ is the multiplicative Mackey functor (Green functor) for G.*

Proof. We only check the Mackey decomposition formula. For an element θ of $\mu_R(G)^K$, we have

$$\begin{aligned}\mathcal{R}^H_J\mathcal{I}^H_K(\theta) &= \sum_{E\leq J} I^E_E \sum_{L\leq H}\sum_{g\in[L\backslash H/K]} I^L_{L\cap{}^gK} c_g^{L^g\cap K}\theta c_{g^{-1}}^{L\cap^gK} R^L_{L\cap^gK}\\ &= \sum_{E\leq J}\sum_{g\in[E\backslash H/K]} I^E_{E\cap^gK} c_g^{E^g\cap K}\theta c_{g^{-1}}^{E\cap^gK} R^E_{E\cap^gK}.\end{aligned}$$

On the other hand, for an element θ of $\mu_R(G)^K$

$$\begin{aligned}&\sum_{x\in[J\backslash H/K]} \mathcal{I}^J_{J\cap^xK}\mathcal{C}_x^{J^x\cap K}\mathcal{R}^K_{J^x\cap K}(\theta)\\ &= \sum_{x\in[J\backslash H/K]} \mathcal{I}^J_{J\cap^xK}\left(\sum_{E\leq J^x\cap K} c_x^E\theta c_{x^{-1}}^{{}^xE}\right)\\ &= \sum_{x\in[J\backslash H/K]}\sum_{L\leq J}\sum_{g\in[L\backslash J/J\cap^xK]} I^L_{L\cap^g(J\cap^xK)} c_g c_x\theta c_{x^{-1}} c_{g^{-1}} R^L_{L\cap^g(J\cap^xK)}\\ &= \sum_{L\leq J}\sum_{gx\in[L\backslash H/K]} I^L_{L\cap^{gx}K} c_{gx}\theta c_{(gx)^{-1}} R^L_{L\cap^{gx}K}.\end{aligned}$$

□

In the next result we will see the fact that $\mathcal{T}$ is the generalization of fixed point functor of a group algebra RG. Let $H\leq G$, the $FP_{RG}(H)=RG^H$ is a fixed point set of H in RG, i.e.,

$$RG^H=\{x\in RG \mid hxh^{-1}=x,\ h\in H\}.$$

Restriction, induction, conjugation are

$$\begin{aligned}\mathrm{res}^H_K &: RG^H\hookrightarrow RG^K : \text{embedding},\\ \mathrm{ind}^H_K &: RG^K\to RG^H : x\mapsto \sum_{h\in[H/K]} hxh^{-1},\\ \mathrm{con}^H_g &: RG^H\to RG^{{}^gH} : x\mapsto gxg^{-1}\end{aligned}$$

where $K\leq H$, $g\in G$. Then we denote by FP_{RG} the *fixed point* functor of RG. We remark that for a subgroup H of G, there is a surjective homomorphism

$$\pi_H:\mu_R(G)^H\to RG^H:\theta\mapsto I^1_1\theta R^1_1.$$

Remark. Let $\mathcal{T}$ be as above notation. For subgroups $K\leq H$ and $g\in G$, we have

$$\mathrm{ind}_K^H \pi_K = \pi_H \mathcal{I}_K^H.$$

3. Defect group

In general, a Mackey functor M is projective relative to $\mathcal{X}$ if and only if the sum of inductions

$$\theta_{\mathcal{X}}(G) := \sum_{H \in \mathcal{X}} t_H^G : \bigoplus_{H \in \mathcal{X}} M(H) \longrightarrow M(G)$$

is surjective and split. However, Dress assert that if M is a Green functor, then we need only to see that $\theta_{\mathcal{X}}$ is surjective [Dr] [Th] (2.4).

If M is a Mackey functor for G then there exists a unique minimal subconjugacy closed set $\mathcal{X}$ of G such that M is projective relative to $\mathcal{X}$. Dress called it the *defect base* of M [Gr], [Dr], [Th].

Let A be a multiplicative Mackey functor such that each algebra $A(H)$ is associative and has identity element 1_H. Then the defect base of A is the union of the defect base of Mackey functor e_iAe_i $(1 \leq i \leq n)$ for G where

$$e_iAe_i(H) := R_H^G(e_i)A(H)R_H^G(e_i) \quad (H \leq G,\ 1_G = e_1 + \cdots + e_n),$$

e_i's are mutually orthogonal idempotents. If the e_i's are centrally primitive idempotents then the defect base of e_iAe_i is $\{Q_i\}$ (up to G-conjugacy), we say that Q_i is the defect group of e_iAe_i.

From the formula for the centrally primitive idempotent of the Mackey algebra $\mu_R(G)$ (the span ring $RSp(\mathcal{S}_f^G)$ [Yo] Lemma 3.4) we obtain the defect base of $\mathcal{T}$ and the defect group of $\mathcal{T}_{S,B} := E_{S,B}\mathcal{T}E_{S,B}$. We call it the *defect group* of block idempotent $E_{S,B}$ of the Mackey algebra $\mu_R(G)$.

Theorem 2 *Let B be a p-block of RG and D the defect group of B and let P be the defect group of $E_{1,B}$ (or $\mathcal{T}_{1,B}$). Then*

$$D \leq_G P.$$

Proof. Let e_B be the block idempotent of B. By the formula for the centrally primitive idempotent of the Mackey algebra [Yo] we can define the homomorphism π_G (resp. π_P) of $\mathcal{T}_{1,B}(G)$ (resp. $\mathcal{T}_{1,B}(P)$) to $FP_{e_BRG}(G)$ (resp. $FP_{e_BRG}(P)$). By *Remark* we can consider the next commutative square

$$\begin{array}{ccc} FP_{e_B RG(P)} & \xrightarrow{\mathrm{ind}_P^G} & FP_{e_B RG(G)} \\ \uparrow \pi_P & & \uparrow \pi_G \\ \mathcal{T}_{1,B}(P) & \xrightarrow{\mathcal{I}_P^G} & \mathcal{T}_{1,B}(G) \end{array}$$

where π_G (resp. π_P) is restriction of $\mu_R(G)^G$ (resp. $\mu_R(G)^P$) to $\mathcal{T}_{1,B}(G)$ (resp. $\mathcal{T}_{1,B}(P)$). By assumption, $\mathcal{I}_P^G$ is surjective and π_G so is, ind_P^G is surjective. Hence, $FP_{e_B RG}$ is projective relative to P from *Remark* and Dress's result [Dr], [Th] (2.4). Thus D is contained in P (up to conjugacy) by the minimality of the defect group of B. □

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