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Seminormal composition operators induced by affine transformations

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Abstract. A class of composition operators on $L^2(\mu)$-spaces induced by nonsingular affine transformations of d-dimensional Euclidean space is investigated. Criteria for their boundedness and estimates for their spectral radii (from above as well as from below) are established. The question of the existence of seminormal composition operators in this class is studied. Cohyponormal composition operators with nontrivial translation part are indicated.

Key words: composition operator, spectral radius, seminormal operator.

Introduction

The foundations of the theory of composition operators in abstract L^2-spaces are well developed. In particular boundedness, subnormality, hyponormality etc. of such operators are completely characterized (cf. [3, 4, 8, 10, 11, 13, 18]). However, if we try to apply directly general theory to concrete classes of composition operators, we get results which are far from being definitive. An attempt to overcome this problem has been done by Mlak in [12] and later by the second-named author in [16].

The present paper, which is an extension and continuation of [16], deals with composition operators on $L^2(\mathbb{R}^d, \mu)$ induced by affine transformations T of $\mathbb{R}^d$, where μ is a positive Borel measure having a radially symmetric density function. Our aim here is to find criteria for their boundedness and to calculate their spectral radii. It turns out that the boundedness of C_T depends only on T and the specific behaviour of μ at infinity (see Theorem 2.2). In general, it is not easy to calculate explicitly the norm of C_T in terms of T and the density function of μ. This is only the case for very particular choices of μ (see Corollary 2.5 and Theorem 5.4). Fortunately, in most cases, we can estimate the norm of C_T and consequently, we can find explicit

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estimates for the spectral radius of C_T (see Theorem 3.4). This enables us to answer the question under what circumstances C_T is seminormal. Except few cases there are no seminormal composition operators induced by T having nontrivial translation part (see Theorems 4.4, 4.6, 4.7 and Propositions 4.9 and 4.10). But the exceptional cases (see Theorem 5.4) permit the existence of cosubnormal composition operators. This will be investigated in the forthcoming paper.

1. Preliminaries

Given a bounded linear operator B on a (real or complex) Hilbert space H, we denote by $\mathcal{N}(B)$ and $\mathcal{R}(B)$ the kernel and the range of B, respectively. B is said to be *hyponormal* (resp. *cohyponormal*) if $BB^* \leq B^*B$ (resp. $B^*B \leq BB^*$). An operator which is either hyponormal or cohyponormal is called *seminormal.* If B is a positive operator on $\mathbb{R}^d$, the real d-dimensional Euclidean space, then $B|_{\mathcal{R}(B)}$ is an invertible operator on $\mathcal{R}(B)$. Set $B^{-1} := (B|_{\mathcal{R}(B)})^{-1}$.

Let us consider a positive Borel measure μ on $\mathbb{R}^d$ given by the formula $\mathrm{d}\mu(x) = \varphi(\|x\|^2)^{-1}\mathrm{d}x$, where $\varphi : [0, +\infty) \longrightarrow (0, +\infty)$ is an arbitrary Borel function, $\mathrm{d}x$ is the d-dimensional Lebesgue measure and $\|x\|^2 = \sum_{k=1}^d {x_k}^2$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, is the canonical norm on $\mathbb{R}^d$ ($d \geq 1$). Assume that we are given an invertible linear transformation A of $\mathbb{R}^d$ and a vector $a \in \mathbb{R}^d$. Define the affine transformation T by $Tx = Ax + a$, $x \in \mathbb{R}^d$, and denote by C_T the composition operator on $L^2(\mu)$ induced by T:

$$C_T f = f \circ T, \quad f \in L^2(\mu).$$

It is easy to see that C_T is a densely defined closed linear operator in $L^2(\mu)$. Arguing similarly to [16] one can show that C_T is bounded if and only if $\operatorname{ess\,sup}_x \varphi(\|Tx\|^2)/\varphi(\|x\|^2) < +\infty$; if C_T is bounded, then

$$\|C_T\|^2 = \frac{1}{|\det A|} \operatorname{ess\,sup}_x \frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)}. \tag{1.1}$$

In order to find more useful criteria for the boundedness of C_T we concentrate on functions φ which are "smoothly" increasing at infinity. More precisely let $\mathcal{E}_\tau$ ($\tau \geq 0$) stands for the set of all continuous functions $\varphi : [0, +\infty) \longrightarrow (0, +\infty)$ such that φ is continuously differentiable on $[\tau, +\infty)$, $\varphi' \geq 0$ on $[\tau, +\infty)$ and φ' is monotonically increasing on $[\tau, +\infty)$.

Put $\mathcal{E} := \bigcup_{\tau \geq 0} \mathcal{E}_\tau$. Roughly speaking, $\mathcal{E}$ is composed of continuous functions which are continuously differentiable, monotonically increasing and convex at $+\infty$. Following [16] we denote by $\mathcal{H}_0$ the class of all nonconstant entire functions φ such that $\varphi(0) > 0$ and $\mathrm{d}^n\varphi/\mathrm{d}z^n(0) \geq 0$ for every $n \geq 1$. It is clear that $\mathcal{H}_0 \subseteq \mathcal{E}$.

The following three lemmas will help us to estimate the norm and the spectral radius of C_T.

Lemma 1.1 *If $\varphi \in \mathcal{E}_\tau$ $(\tau \geq 0)$, then the following conditions are equivalent*
(i) $\limsup_{t\to+\infty} \varphi'(t)/\varphi(t) < +\infty$,
(ii) *for every* $v > 0$, $\sup_{t\geq 0} \varphi(t+v)/\varphi(t) < +\infty$,
(iii) *there exists* $v > 0$ *such that* $\sup_{t\geq 0} \varphi(t+v)/\varphi(t) < +\infty$.
Moreover, if $\sigma \geq \tau$, then

$$\varphi(t) \leq \varphi(\sigma) \exp(\overline{L}_\sigma (t-\sigma)), \quad t \geq \sigma, \tag{1.2}$$

$$\varphi(t) \geq \varphi(\sigma) \exp(\underline{L}_\sigma (t-\sigma)), \quad t \geq \sigma, \tag{1.3}$$

where $\overline{L}_\sigma = \sup_{t\geq\sigma} \varphi'(t)/\varphi(t)$ *and* $\underline{L}_\sigma = \inf_{t\geq\sigma} \varphi'(t)/\varphi(t)$.

Proof. (i) $\Rightarrow$ (ii) Since $\limsup_{t\to+\infty} \varphi'(t)/\varphi(t) < +\infty$, there exists $\sigma \geq \tau$ such that $\overline{L}_\sigma < +\infty$. Thus we have

$$\frac{\varphi(t+v)}{\varphi(t)} = \exp\left(\int_t^{t+v} \frac{\varphi'(s)}{\varphi(s)}\,\mathrm{d}s\right) \leq \exp(\overline{L}_\sigma v), \quad t \geq \sigma. \tag{1.4}$$

On the other hand $\sup_{t\in[0,\sigma]} \varphi(t+v)/\varphi(t) < +\infty$, because φ is continuous.

(iii) $\Rightarrow$ (i). By the Lagrange theorem, for any $t \geq \tau$ there exists $\theta \in (0,1)$ such that

$$+\infty > \sup_{s\geq 0} \frac{\varphi(s+v)}{\varphi(s)} \geq \frac{\varphi(t+v)}{\varphi(t)} = 1 + \frac{v\varphi'(t+\theta\, v)}{\varphi(t)} \geq \frac{v\varphi'(t)}{\varphi(t)}.$$

Inequality (1.2) follows from (1.4); the proof of (1.3) is similar. □

Lemma 1.2 *If $\varphi \in \mathcal{E}_\tau$ $(\tau \geq 0)$, then the following conditions are equivalent*
(i) $\limsup_{t\to+\infty} \sqrt{t}\,\varphi'(t)/\varphi(t) < +\infty$,
(ii) *for every* $v > 0$, $\sup_{t\geq 0} \varphi(t^2+vt)/\varphi(t^2) < +\infty$,
(iii) *there exists* $v > 0$ *such that* $\sup_{t\geq 0} \varphi(t^2+vt)/\varphi(t^2) < +\infty$.

Moreover, if $\sigma \geq \tau$, then

$$\varphi(t) \leq \varphi(\sigma) \exp(2\overline{M}_\sigma (t^{1/2} - \sigma^{1/2})), \quad t \geq \sigma, \tag{1.5}$$

$$\varphi(t) \geq \varphi(\sigma) \exp(2\underline{M}_\sigma (t^{1/2} - \sigma^{1/2})), \quad t \geq \sigma, \tag{1.6}$$

where $\overline{M}_\sigma = \sup_{t\geq\sigma} \sqrt{t}\,\varphi'(t)/\varphi(t)$ *and* $\underline{M}_\sigma = \inf_{t\geq\sigma} \sqrt{t}\,\varphi'(t)/\varphi(t)$.

Proof. (i) $\Rightarrow$ (ii) Since $\limsup_{t\to+\infty} \sqrt{t}\,\varphi'(t)/\varphi(t) < +\infty$, there exists $\sigma \geq \tau$ such that $\overline{M}_\sigma < +\infty$. Hence

$$\begin{aligned} \frac{\varphi(t^2+vt)}{\varphi(t^2)} &\leq \frac{\varphi((t+\frac{1}{2}v)^2)}{\varphi(t^2)} \\ &= \exp\left(2\int_t^{t+\frac{1}{2}v} \frac{s\varphi'(s^2)}{\varphi(s^2)}\,\mathrm{d}s\right) \\ &\leq \exp(\overline{M}_\sigma v), \qquad t \geq \sigma^{1/2}. \end{aligned} \tag{1.7}$$

Since φ is continuous, we have $\sup_{t^2\in[0,\sigma]} \varphi(t^2+vt)/\varphi(t^2) < +\infty$.

(iii) $\Rightarrow$ (i). Again, by the Lagrange theorem, for any $t \geq \tau^{1/2}$ there exists $\theta \in (0,1)$ such that

$$\begin{aligned} +\infty > \sup_{s\geq 0} \frac{\varphi(s^2+vs)}{\varphi(s^2)} &\geq \frac{\varphi(t^2+vt)}{\varphi(t^2)} \\ &= 1 + \frac{vt\varphi'(t^2+\theta\,vt)}{\varphi(t^2)} \geq \frac{vt\varphi'(t^2)}{\varphi(t^2)}. \end{aligned}$$

Inequality (1.5) follows from (1.7). The proof of (1.6) is similar. □

Lemma 1.3 *If $\varphi \in \mathcal{E}_\tau$ $(\tau \geq 0)$, then the following conditions are equivalent*

(i) $\limsup_{t\to+\infty} t\varphi'(t)/\varphi(t) < +\infty$,

(ii) *for every* $v > 1$, $\sup_{t\geq 0} \varphi(vt)/\varphi(t) < +\infty$,

(iii) *there exists* $v > 1$ *such that* $\sup_{t\geq 0} \varphi(vt)/\varphi(t) < +\infty$.

Moreover, if $\sigma \geq \tau \geq 1$, then

$$\varphi(t) \leq \varphi(\sigma) \left(\frac{t}{\sigma}\right)^{\overline{N}_\sigma}, \quad t \geq \sigma,$$

$$\varphi(t) \geq \varphi(\sigma) \left(\frac{t}{\sigma}\right)^{\underline{N}_\sigma}, \quad t \geq \sigma,$$

where $\overline{N}_\sigma = \sup_{t \geq \sigma} t\varphi'(t)/\varphi(t)$ *and* $\underline{N}_\sigma = \inf_{t \geq \sigma} t\varphi'(t)/\varphi(t)$.

Proof. Applying Lemma 1.1 to the function $\varphi \circ \exp \in \mathcal{E}_{\ln \tau}$ we get the conclusion. $\square$

2. Boundedness

In this section we present necessary and sufficient conditions for the composition operator C_T to be bounded. We first investigate the behaviour of the quadratic form $\Delta(x) := \|Tx\|^2 - \|x\|^2$, $x \in \mathbb{R}^d$, at infinity.

Lemma 2.1 (i) $\limsup_{\|x\| \to +\infty} \Delta(x)/\|x\|^2 = \|A\|^2 - 1$,
(ii) $\limsup_{\|x\| \to +\infty} \Delta(x)/\|x\| < +\infty \Leftrightarrow \|A\| \leq 1$,
(iii) $\limsup_{\|x\| \to +\infty} \Delta(x) < +\infty \Leftrightarrow \|A\| \leq 1$ *and* $a \in \mathcal{R}(I - AA^*)$.
Moreover, if $\limsup_{\|x\| \to +\infty} \Delta(x) < +\infty$, *then*

$$\sup_x \Delta(x) = \max_x \Delta(x) = \Delta((I - A^*A)^{-1}A^*a) = ((I - AA^*)^{-1}a, a).$$

Proof. The proof of (i) is left to the reader.

(ii) If $\|A\| \leq 1$, then for any $v > 2\|A^*a\|$ there exists $t_0 > 0$ such that $\Delta(x) = \|Ax\|^2 - \|x\|^2 + 2(x, A^*a) + \|a\|^2 \leq 2\|A^*a\|\|x\| + \|a\|^2 \leq v\|x\|$ for $\|x\| \geq t_0$, so $\limsup_{\|x\| \to +\infty} \Delta(x)/\|x\| < +\infty$. The converse implication follows from (i).

(iii) For abbreviation we put $W := I - A^*A$ and $V := I - AA^*$. Let us assume that $\limsup_{\|x\| \to +\infty} \Delta(x) < +\infty$. Then, by (i), we have $\|A\| \leq 1$. If $x \in \mathcal{N}(W)$, then $\sup_{t \in \mathbb{R}}(2t(x, A^*a) + \|a\|^2) = \sup_{t \in \mathbb{R}} \Delta(tx) < +\infty$, so $(x, A^*a) = 0$. Thus we have shown that $\mathcal{N}(W) \subseteq (A^*a)^\perp$, which implies that $a \in \mathcal{R}(V)$.

Suppose now that $\|A\| \leq 1$ and $a \in \mathcal{R}(V)$. Since $a \in \mathcal{R}(V)$ is equivalent to $A^*a \in \mathcal{R}(W)$, we can define $c := W^{-1}A^*a \in \mathcal{R}(W)$. Then

$$A^*a = Wc. \tag{2.1}$$

The last equality yields

$$\begin{aligned} \Delta(x) &= -(Wx, x) + 2(x, Wc) + \|a\|^2 \\ &= -(W(x - c), x - c) + (Wc, c) + \|a\|^2. \end{aligned}$$

According to our assumptions $W \geq 0$, so

$$\sup_x \Delta(x) = \max_x \Delta(x) = \Delta(c) = (Wc, c) + \|a\|^2,$$

which shows that $\limsup_{\|x\| \to +\infty} \Delta(x) < +\infty$.

Now we prove the equality $(Wc, c) + \|a\|^2 = (V^{-1}a, a)$. By (2.1) we have

$$a = A^{*-1}Wc = VA^{*-1}c$$

and consequently

$$A^{*-1}c = V^{-1}a.$$

This and (2.1) imply

$$\begin{aligned}(Wc, c) + \|a\|^2 &= (A^*a, c) + \|a\|^2 \\ &= (AA^*a, A^{*-1}c) + \|a\|^2 \\ &= (AA^*a, V^{-1}a) + \|a\|^2 \\ &= (a, (AA^* + V)V^{-1}a) \\ &= (a, V^{-1}a),\end{aligned}$$

which completes the proof. □

Define the quantity τ_T (depending on $\tau \geq 0$ and T) as the maximum of τ and $\sup\{\|x\|^2 : \|Tx\|^2 \leq \tau\}$. Notice that if $\|x\|^2 > \tau_T$, then $\|x\|^2 > \tau$ and $\|Tx\|^2 > \tau$.

Theorem 2.2 *If $\varphi \in \mathcal{E}$ and $a \neq 0$, then C_T is bounded if and only if one of the following conditions holds*
(i) $\|A\| < 1$,
(ii) $\|A\| = 1$, $a \in \mathcal{R}(I - AA^*)$ *and* $\limsup_{t \to +\infty} \varphi'(t)/\varphi(t) < +\infty$,
(iii) $\|A\| = 1$, $a \notin \mathcal{R}(I - AA^*)$ *and* $\limsup_{t \to +\infty} \sqrt{t}\,\varphi'(t)/\varphi(t) < +\infty$,
(iv) $\limsup_{t \to +\infty} t\varphi'(t)/\varphi(t) < +\infty$.

Proof. Let $\tau \geq 0$ be such that $\varphi \in \mathcal{E}_\tau$.

Sufficiency. Notice that the boundedness of C_T will be proved once we show that $\sup_{\|x\|^2 > t_0} \varphi(\|Tx\|^2)/\varphi(\|x\|^2)$ is finite for some $t_0 \geq 0$.

If $\|A\| < 1$, then by Lemma 2.1 (i) there exists $t_0 \geq \tau_T$ such that

$\Delta(x) \leq 0$ for $\|x\|^2 \geq t_0$. This and the monotonicity of φ in $[\tau, +\infty)$ imply

$$\frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} = \frac{\varphi(\|x\|^2 + \Delta(x))}{\varphi(\|x\|^2)} \leq 1, \quad \|x\|^2 > t_0.$$

If $\|A\| = 1$, $a \in \mathcal{R}(I - AA^*)$ and $\limsup_{t\to+\infty} \varphi'(t)/\varphi(t) < +\infty$, then by Lemma 2.1 (iii) there exist $t_0 \geq \tau_T$ and $v > 0$ such that $\|Tx\|^2 \leq \|x\|^2 + v$ for $\|x\|^2 \geq t_0$. This and Lemma 1.1 (ii) imply

$$\frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} \leq \frac{\varphi(\|x\|^2 + v)}{\varphi(\|x\|^2)} \leq \sup_{t\geq 0} \frac{\varphi(t+v)}{\varphi(t)} < +\infty, \quad \|x\|^2 > t_0.$$

If $\|A\| = 1$ and $\limsup_{t\to+\infty} \sqrt{t}\,\varphi'(t)/\varphi(t) < +\infty$, then by Lemma 2.1 (ii) there exist $t_0 \geq \tau_T$ and $v > 0$ such that $\|Tx\|^2 \leq \|x\|^2 + v\|x\|$ for $\|x\|^2 \geq t_0$. This and Lemma 1.2 (ii) yield

$$\frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} \leq \frac{\varphi(\|x\|^2 + v\|x\|)}{\varphi(\|x\|^2)} \leq \sup_{t\geq 0} \frac{\varphi(t^2+vt)}{\varphi(t^2)} < +\infty, \quad \|x\|^2 > t_0.$$

Assume that $\limsup_{t\to+\infty} t\varphi'(t)/\varphi(t) < +\infty$. Applying Lemma 2.1 (i) we can find $t_0 \geq \tau_T$ and $v > 1$ such that $\|Tx\|^2 \leq v\|x\|^2$ for $\|x\|^2 \geq t_0$. This and Lemma 1.3 (ii) imply

$$\frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} \leq \frac{\varphi(v\|x\|^2)}{\varphi(\|x\|^2)} \leq \sup_{t\geq 0} \frac{\varphi(vt)}{\varphi(t)} < +\infty, \quad \|x\|^2 > t_0.$$

Necessity. Suppose that C_T is bounded. We can assume that $\|A\| \geq 1$.

If $\|A\| > 1$, then taking a normalized vector x_0 such that $\|A(x_0)\| = \|A\|$ and fixing $v \in (1, \|A\|^2)$ we can find $t_0 \geq \tau$ such that

$$\|T(\sqrt{t}\,x_0)\|^2 = t\|A\|^2 + 2\sqrt{t}\,(x_0, A^*a) + \|a\|^2 \geq vt, \quad t \geq t_0.$$

This yields

$$\frac{\varphi(vt)}{\varphi(t)} \leq \frac{\varphi(\|T(\sqrt{t}\,x_0)\|^2)}{\varphi(\|\sqrt{t}\,x_0\|^2)} \leq \sup_{x} \frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} < +\infty, \quad t > t_0.$$

Applying Lemma 1.3 we get (iv).

Assume that $\|A\| = 1$. Then $W := I - A^*A \geq 0$ and $\mathcal{N}(W) \neq \{0\}$. Notice that $a \notin \mathcal{R}(I - AA^*)$ if and only if there exists $x_0 \in \mathcal{N}(W)$ such that $(x_0, A^*a) > 0$. Indeed: $a \in \mathcal{R}(I - AA^*) \Leftrightarrow A^*a \in \mathcal{R}(W) \Leftrightarrow A^*a \in \mathcal{N}(W)^\perp$ $\Leftrightarrow \mathcal{N}(W) \subseteq (A^*a)^\perp \Leftrightarrow (x, A^*a) = 0$ for each $x \in \mathcal{N}(W)$.

If $a \in \mathcal{R}(I - AA^*)$, then taking $x_0 \in \mathcal{N}(W)$ such that $\|x_0\| = 1$ we have $\|T(\sqrt{t}\, x_0)\|^2 = t + \|a\|^2$ for $t \geq 0$, so

$$\frac{\varphi(t + \|a\|^2)}{\varphi(t)} = \frac{\varphi(\|T(\sqrt{t}\, x_0)\|^2)}{\varphi(\|\sqrt{t} x_0\|^2)} \leq \sup_x \frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} < +\infty, \quad t \geq 0,$$

which in virtue of Lemma 1.1 implies (ii).

If $a \notin \mathcal{R}(I - AA^*)$, then there exists a normalized vector $x_0 \in \mathcal{N}(W)$ such that $(x_0, A^*a) > 0$. Taking any $v \in (0, 2(x_0, A^*a))$ we can find $t_0 \geq \sqrt{\tau}$ such that

$$\|T(tx_0)\|^2 = t^2 + 2t(x_0, A^*a) + \|a\|^2 \geq t^2 + vt, \quad t \geq t_0.$$

Therefore

$$\frac{\varphi(t^2 + vt)}{\varphi(t^2)} \leq \frac{\varphi(\|T(tx_0)\|^2)}{\varphi(\|tx_0\|^2)} \leq \sup_x \frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} < +\infty, \quad t > t_0,$$

so (iii) follows from Lemma 1.2. This completes the proof. □

The case $a = 0$, not included in Theorem 2.2, is much simpler. Namely we have the following criterion.

Proposition 2.3 *If $\varphi \in \mathcal{E}$ and $a = 0$, then C_T is bounded if and only if $\|A\| \leq 1$ or $\limsup_{t \to +\infty} t\varphi'(t)/\varphi(t) < +\infty$.*

Proof. It is enough to modify appropriate parts of the proof of Theorem 2.2. □

Corollary 2.4 *Let $\varphi \in \mathcal{H}_0$. If $a \neq 0$, then C_T is bounded if and only if one of the following conditions holds*

(i) $\|A\| < 1$,

(ii) $\|A\| = 1$, $a \in \mathcal{R}(I - AA^*)$ *and* $\limsup_{t \to +\infty} \varphi'(t)/\varphi(t) < +\infty$,

(iii) $\|A\| = 1$, $a \notin \mathcal{R}(I - AA^*)$ *and* $\limsup_{t \to +\infty} \sqrt{t}\, \varphi'(t)/\varphi(t) < +\infty$,

(iv) φ *is a polynomial.*

If $a = 0$, then C_T is bounded if and only if $\|A\| \leq 1$ or φ is a polynomial.

Proof. Apply Theorem 2.2, Proposition 2.3, Lemma 1.3 and the fact that any entire function of polynomial growth is a polynomial. □

In general, it seems to be hopeless to find more explicit formula for the norm of C_T. However it is possible in the particular case of the Gaussian measure μ.

Corollary 2.5 *If $\varphi = \exp$, then C_T is bounded if and only if $\|A\| \leq 1$ and $a \in \mathcal{R}(I - AA^*)$. Moreover*

$$\|C_T\|^2 = \frac{1}{|\det A|} \exp(((I - AA^*)^{-1}a, a)) = \prod_{j=1}^{n} \frac{1}{t_j} \exp\left(\frac{|(a, h_j)|^2}{1 - t_j^2}\right),$$

where $t_1, \ldots, t_n$ are all eigenvalues of $|A^*|$ which are less than 1, listed in an order taking account of their multiplicities and $h_1, \ldots, h_n$ are corresponding normalized eigenvectors which are pairwise orthogonal.

Proof. The first part of the conclusion follows from Theorem 2.2 and Proposition 2.3, while the other from Lemma 2.1 as

$$\begin{aligned} \|C_T\|^2 &= \frac{1}{|\det A|} \sup_x \exp(\Delta(x)) \\ &= \frac{1}{|\det A|} \exp(((I - AA^*)^{-1}a, a)) \\ &= \prod_{j=1}^{n} \frac{1}{t_j} \exp\left(\frac{|(a, h_j)|^2}{1 - t_j^2}\right). \end{aligned}$$

The proof of the last equality is left to the reader. □

3. Spectral Radius

In this section we will estimate the spectral radius $r(C_T)$ of C_T. We begin by proving some preliminary results concerning operators that come from iterations of a contraction. First, recall that any contraction B on a (real or complex) Hilbert space K possesses a unique orthogonal decomposition $B = B_u \oplus B_c$, where B_u is a unitary operator on $\mathcal{D}(B_u)$ and B_c is a completely nonunitary operator on $\mathcal{D}(B_c)$ (cf. [17]).

Lemma 3.1 *Assume that $\|A\| \leq 1$. If $V_n := I - A^n A^{*n}$ for $n \geq 0$, then*

(i) *$\{V_n\}_{n=0}^{\infty}$ is a monotonically increasing sequence of positive operators and $\{\mathcal{R}(V_n)\}_{n=0}^{\infty}$ is a monotonically increasing sequence of subspaces.*

(ii) *There exists n_0 such that $\mathcal{R}(V_n) = \mathcal{R}(V_{n_0}) = \mathcal{D}(A_c)$ for $n \geq n_0$.*

Proof. We have only to prove (ii). Since, by (i), the sequence of finite dimensional subspaces $\{\mathcal{N}(V_n)\}_{n=0}^{\infty}$ is monotonically decreasing, it must stabilize beginning from some n_0. Notice that $H := \bigcap_{n\geq 0} \mathcal{N}(V_n) = \mathcal{N}(V_{n_0})$ is invariant for A^* and $A^*|_H$ is an isometry. However H is finite dimensional, so H reduces A^* to a unitary operator, which completes the proof. $\square$

The following lemma describes the behaviour (at infinity) of the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively by $T^n x = A^n x + a_n$, $x \in \mathbb{R}^d$, $n \geq 1$.

Lemma 3.2 (i) *If $\|A\| < 1$, then the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.*

(ii) *If $\|A\| = 1$ and $a \in \mathcal{R}(I - AA^*)$, then $a_n \in \mathcal{R}(I - A^n A^{*n})$ for every $n \geq 1$ and the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{((I - A^n A^{*n})^{-1} a_n, a_n)\}_{n=1}^{\infty}$ are bounded.*

(iii) *If $\|A\| \leq 1$, then $\lim_{n\to+\infty} a_n/n = Pa$, where P is the orthogonal projection of $\mathbb{R}^d$ onto $\mathcal{N}(I - A)$.*

Proof. It is easy to see that

$$a_n = A^{n-1}a + A^{n-2}a + \cdots + Aa + a, \quad n \geq 1. \tag{3.1}$$

(i) If $\|A\| < 1$, then by (3.1) we have

$$\|a_n\| \leq \frac{\|a\|}{1 - \|A\|}, \quad n \geq 1.$$

(ii) Suppose that $\|A\| = 1$ and $a \in \mathcal{R}(I - AA^*)$. We first show that

$$a_n \in \mathcal{R}(I - A^n A^{*n}), \quad n \geq 1. \tag{3.2}$$

We proceed by induction. Assume that $a_n = (I - A^n A^{*n})h$. Then

$$\begin{aligned} Aa_n &= Ah - A^{n+1}A^{*n}h \\ &= (AA^* - A^{n+1}A^{*(n+1)})A^{*-1}h \\ &= (AA^* - I)A^{*-1}h + (I - A^{n+1}A^{*(n+1)})A^{*-1}h \\ &\in \mathcal{R}(I - AA^*) + \mathcal{R}(I - A^{n+1}A^{*(n+1)}). \end{aligned}$$

By Lemma 3.1 (i), Aa_n is in $\mathcal{R}(I - A^{n+1}A^{*(n+1)})$ and consequently so is $a_{n+1} = Aa_n + a$.

It is well-known (cf. [17]) that 1 is not the eigenvalue of A_c, so $I - A_c$ is invertible. By Lemma 3.1 (ii) there exists $n_0 \geq 1$ such that

$$\mathcal{D}(A_c) = \mathcal{R}(V_{n_0}) = \mathcal{R}(V_n), \quad n \geq n_0, \tag{3.3}$$

so in virtue of (3.1) and (3.2) we conclude

$$(I - A_c)a_n = (I - A_c^n)a, \quad n \geq n_0.$$

Therefore

$$a_n = (I - A_c)^{-1}(I - A_c^n)a, \quad n \geq n_0. \tag{3.4}$$

This in turn implies the boundedness of $\{a_n\}_{n=1}^{\infty}$.

Notice that the bounded sequence $\{I - A_c^n A_c^{*n}\}_{n=0}^{\infty}$ of positive operators, being monotonically increasing, is norm-convergent. However, by (3.3), we have $\mathcal{N}(I - A_c^{n_0} A_c^{*n_0}) = \{0\}$, which implies $\|A_c^{n_0}\| = \|A_c^{*n_0}\| < 1$. Since the sequence $\{I - (A_c^{n_0})^n (A_c^{*n_0})^n\}_{n=0}^{\infty}$ is convergent to I, so is $\{I - A_c^n A_c^{*n}\}_{n=0}^{\infty}$. This in turn implies that the sequence $\{(I - A_c^n A_c^{*n})^{-1}\}_{n=0}^{\infty}$ is convergent to I. By (3.2) and (3.3) we have

$$(V_n^{-1} a_n, a_n) = ((I - A_c^n A_c^{*n})^{-1} a_n, a_n), \quad n \geq n_0,$$

which together with the boundedness of $\{a_n\}_{n=1}^{\infty}$ implies (ii).

(iii) By (3.1), this is exactly the mean ergodic theorem (cf. [14]). □

The following lemma will be exploited in the proof of Theorem 3.4. Note that it still holds if we replace T by an arbitrary homeomorphism of $\mathbb{R}^d$.

Lemma 3.3 *For every $r \geq 0$, $\sup_{\|x\| \leq r} \|Tx\| = \sup_{\|x\| = r} \|Tx\|$.*

Proof. Denote by K_r (resp. S_r) the closed ball (resp. the sphere) centered at 0 with radius r. Since T is a homeomorphism of $\mathbb{R}^d$, we have $T(S_r) = T(\partial K_r) = \partial(T(K_r))$, so $\sup_{x \in K_r} \|Tx\| = \sup_{y \in T(K_r)} \|y\| = \sup_{y \in \partial(T(K_r))} \|y\| = \sup_{y \in T(S_r)} \|y\| = \sup_{x \in S_r} \|Tx\|$, which completes the proof. □

We can now formulate the explicit estimates for the spectral radius of C_T.

Theorem 3.4 *Assume that $\varphi \in \mathcal{E}$ and C_T is bounded.*

(i) *If $\|A\| \leq 1$ and $a \in \mathcal{R}(I - AA^*)$, then*

$$r(C_T)\sqrt{|\det A|} = 1.$$

(ii) *If $\|A\| = 1$ and $a \notin \mathcal{R}(I - AA^*)$, then*

$$\exp(\underline{M}\,\|Pa\|) \leq r(C_T)\sqrt{|\det A|} \leq \exp(\overline{M}\,\|Pa\|),$$

where $\underline{M} := \liminf_{t \to +\infty} \sqrt{t}\,\varphi'(t)/\varphi(t)$, $\overline{M} := \limsup_{t \to +\infty} \sqrt{t}\,\varphi'(t)/\varphi(t)$

and P is the orthogonal projection of $\mathbb{R}^d$ onto $\mathcal{N}(I-A)$.

(iii) *If $\|A\|>1$, then*

$$\max\{1, r(A)\}^{\underline{N}} \leq r(C_T)\sqrt{|\det A|} \leq \max\{1, r(A)\}^{\overline{N}},$$

where $\underline{N} := \liminf_{t\to+\infty} t\varphi'(t)/\varphi(t)$ and $\overline{N} := \limsup_{t\to+\infty} t\varphi'(t)/\varphi(t)$.

Proof. Let $\tau \geq 1$ be such that $\varphi \in \mathcal{E}_\tau$. Set $\kappa_\sigma := \sup_{[0,\sigma]}\varphi/\inf_{[0,\sigma]}\varphi$ for $\sigma \geq 0$. Then the following inequality holds for $\sigma \geq \tau$ (with the convention $\sup\emptyset := 0$)

$$\sup_x \frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} \leq \kappa_\sigma \max\left\{1, \sup\left\{\frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} : \|Tx\|^2 \geq \|x\|^2 \geq \sigma\right\}\right\}. \tag{3.5}$$

Indeed, if $\|Tx\|^2 \leq \sigma$, then the monotonicity of φ implies $\varphi(\|Tx\|^2)/\varphi(\|x\|^2) \leq \kappa_\sigma$. The same reasoning applies to the case $\|x\|^2 \geq \sigma$ and $\|Tx\|^2 \geq \sigma$. If $\|x\|^2 \leq \sigma$ and $\|Tx\|^2 \geq \sigma$, then one can conclude from Lemma 3.3 that

$$\begin{aligned}\frac{\varphi(\|Tx\|^2)}{\varphi(\|x\|^2)} &\leq \frac{\varphi(\sup\{\|Ty\|^2 : \|y\|^2 = \sigma, \|Ty\|^2 \geq \sigma\})}{\varphi(\sigma)} \frac{\varphi(\sigma)}{\varphi(\|x\|^2)} \\ &\leq \kappa_\sigma \sup\left\{\frac{\varphi(\|Ty\|^2)}{\varphi(\|y\|^2)} : \|Ty\|^2 \geq \|y\|^2 \geq \sigma\right\},\end{aligned}$$

which proves (3.5).

Note now that the following estimate is always true

$$r(C_T)\sqrt{|\det A|} \geq 1. \tag{3.6}$$

Indeed, by (1.1), we have

$$\|C_T^n\|^2 \geq \frac{1}{|\det A|^n}\frac{\varphi(\|a_n\|^2)}{\varphi(0)} \geq \frac{1}{\kappa_\tau |\det A|^n}, \quad n \geq 1,$$

so, by the Gelfand formula (cf. [15]), $r(C_T) = \lim_{n\to+\infty}\|C_T^n\|^{1/n} \geq 1/\sqrt{|\det A|}$.

(i) Due to (3.6) we have only to prove that $r(C_T) \leq 1/\sqrt{|\det A|}$. If $\|A\|<1$ (or $\|A\|=1$ and $a=0$), then by Lemma 3.2 (i), the sequence $\{\|a_n\|\}_{n=1}^\infty$ is bounded by some $\alpha \geq 0$, so $\|T^n x\| \leq \|A^n x\| + \|a_n\| \leq$

$\|A\|\,\|x\| + \alpha$ for $n \geq 1$. Taking $\sigma := \max\{\tau, \alpha^2/(1 - \|A\|)^2\}$ we get $\|T^n x\|^2 \leq \|x\|^2$ for $\|x\|^2 \geq \sigma$. Applying (3.5) with T^n in place of T we conclude that $\sup_x \varphi(\|T^n x\|^2)/\varphi(\|x\|^2) \leq \kappa_\sigma$. This and (1.1) imply

$$\|C_T^n\|^2 \leq \frac{\kappa_\sigma}{|\det A|^n}.$$

Using the Gelfand formula we get $r(C_T) \leq 1/\sqrt{|\det A|}$.

Assume now that $\|A\| = 1$ and $a \in \mathcal{R}(I - AA^*)\setminus\{0\}$. Then, by Theorem 2.2, $\overline{L}_\tau := \sup\{\varphi'(t)/\varphi(t) : t \geq \tau\} < +\infty$. It follows from Lemma 3.2 (ii) that the sequence $\{((I - A^n A^{*n})^{-1} a_n, a_n)\}_{n=1}^\infty$ is bounded by some $\beta \geq 0$. Applying Lemma 3.2 (ii) and Lemma 2.1 (the latter to T^n in place of T), we get $\|T^n x\|^2 \leq \|x\|^2 + ((I - A^n A^{*n})^{-1} a_n, a_n) \leq \|x\|^2 + \beta$ for $n \geq 1$. This and Lemma 1.1 imply

$$\frac{\varphi(\|T^n x\|^2)}{\varphi(\|x\|^2)} \leq \exp(\overline{L}_\tau\,(\|T^n x\|^2 - \|x\|^2)) \leq \exp(\overline{L}_\tau\,\beta),$$

provided $\|T^n x\|^2 \geq \|x\|^2 \geq \tau$ and $n \geq 1$. By (3.5) we have $|\det A|^n\,\|C_T^n\|^2 \leq \kappa_\tau \exp(\overline{L}_\tau\,\beta)$ for $n \geq 1$. The Gelfand formula gives us $\sqrt{|\det A|}\,r(C_T) \leq 1$.

(ii) Take an arbitrary $\sigma \geq \tau$. Let $\underline{M}_\sigma$ and $\overline{M}_\sigma$ be as in Lemma 1.2. Then, by Theorem 2.2, $\overline{M}_\sigma$ is finite. It follows from Lemma 1.2 that

$$\frac{\varphi(\|T^n x\|^2)}{\varphi(\|x\|^2)} \leq \exp(2\overline{M}_\sigma\,(\|T^n x\| - \|x\|)) \leq \exp(2\overline{M}_\sigma\,\|a_n\|),$$

provided $\|T^n x\|^2 \geq \|x\|^2 \geq \sigma$ and $n \geq 1$. Applying (3.5) and the Gelfand formula we obtain

$$|\det A|\,r(C_T)^2 \leq \exp(2\overline{M}_\sigma \lim_{n\to+\infty} \|a_n\|/n),$$

which, by Lemma 3.2 (iii), yields

$$\sqrt{|\det A|}\,r(C_T) \leq \exp(\overline{M}_\sigma \|Pa\|).$$

Letting $\sigma \to +\infty$ we get one of the inequalities in (ii).

To prove the other one, it is sufficient to consider the case $Pa \neq 0$ (because of (3.6)). It follows from Lemma 3.2 (iii) that for any $\sigma \geq \tau$, there exists $n_0 \geq 1$ such that $\|a_n\|^2 \geq \sigma$ for $n \geq n_0$. Consequently, by Lemma 1.2, we have

$$|\det A|^n \|C_T^n\|^2 \geq \frac{\varphi(\|a_n\|^2)}{\varphi(0)}$$

$$\geq \frac{1}{\kappa_\sigma} \frac{\varphi(\|a_n\|^2)}{\varphi(\sigma)}$$
$$\geq \frac{1}{\kappa_\sigma} \exp(2\underline{M}_\sigma (\|a_n\| - \sqrt{\sigma})), \quad n \geq n_0.$$

Arguing similarly to the previous paragraph we infer the desired inequality.

(iii) Take an arbitrary $\sigma \geq \tau$. Let $\underline{N}_\sigma$ and $\overline{N}_\sigma$ be as in Lemma 1.3. Due to Theorem 2.2 and Proposition 2.3, $\overline{N}_\sigma$ is finite. Without loss of generality we may assume that $\|A^n\| > 1$ for all $n \geq 1$ (consequently $r(A) \geq 1$). Otherwise there exists $n \geq 1$ such that $\|A^n\| < 1$, so we can apply (i), or $\|A^n\| = 1$, so we can apply either (i) or (ii) (because $\underline{M} = \overline{M} = 0$).

Since $\|A\| > 1$, one can deduce from Lemma 2.1 (i) that the set $\{x : \|Tx\|^2 \geq \|x\|^2 \geq \sigma\}$ is nonempty. It follows from Lemma 1.3 and (3.5) that

$$\left(\sup\left\{\frac{\|Tx\|^2}{\|x\|^2} : \|Tx\|^2 \geq \|x\|^2 \geq \sigma\right\}\right)^{\underline{N}_\sigma}$$
$$\leq |\det A| \|C_T\|^2$$
$$\leq \kappa_\sigma \left(\sup\left\{\frac{\|Tx\|^2}{\|x\|^2} : \|Tx\|^2 \geq \|x\|^2 \geq \sigma\right\}\right)^{\overline{N}_\sigma}. \tag{3.7}$$

Lemma 2.1 (i) gives us $\sup\{\|Tx\|^2/\|x\|^2 : \|Tx\|^2 \geq \|x\|^2 \geq \sigma\} \geq \|A\|^2$. Thus $\|A\|^{2\underline{N}_\sigma} \leq |\det A| \|C_T\|^2$. Letting $\sigma \to +\infty$ we get $\|A\|^{\underline{N}} \leq \sqrt{|\det A|}\, \|C_T\|$. Replacing T by T^n in the last inequality and using the Gelfand formula we obtain $r(A)^{\underline{N}} \leq \sqrt{|\det A|}\, r(C_T)$.

Take $\theta > r(A)$. Then the sequence $\{\|A^n\|/\theta^n\}_{n=1}^\infty$ tends to 0 and consequently so does the sequence $\{n^{-1}(1 + \|A\|\theta^{-1} + \cdots + \|A^n\|\theta^{-n})\}_{n=1}^\infty$. This in turn implies that $1 + \|A\| + \cdots + \|A^n\| \leq n\theta^n$ for n large enough (because $\theta > r(A) \geq 1$). Using (3.1) we get the following estimate for n large enough and $\sigma \geq \|a\|^2$

$$\frac{\|T^n x\|}{\|x\|} \leq \|A^n\| + \frac{\|a_n\|}{\sqrt{\sigma}} \leq 1 + \|A\| + \cdots + \|A^n\| \leq n\theta^n,$$

provided $\|x\|^2 \geq \sigma$. Applying (3.7) to T^n in place of T we get

$$|\det A|^n \, \|C_T^n\|^2 \leq \kappa_\sigma (n\theta^n)^{2\overline{N}_\sigma},$$

for n large enough. Using once more Gelfand's formula, then letting $\sigma \to$

$+\infty$ and finally letting $\theta \to r(A)$ we get the conclusion. □

Corollary 3.5 *Assume that $\varphi \in \mathcal{E}$ and C_T is bounded. If A is a contraction and $a \in \mathcal{D}(A_c)$, then*

$$r(C_T)\sqrt{|\det A|} = 1.$$

Proof. Note that $a \in \mathcal{D}(A_c) \subseteq \mathcal{N}(I-A)^{\perp}$, so the conclusion follows from parts (i) and (ii) of Theorem 3.4. □

Theorem 3.4 enables us to calculate spectral radius of C_T in case the appropriate limits exist.

Theorem 3.6 *Assume that $\varphi \in \mathcal{E}$ and C_T is bounded.*

(i) *If $r(A) \le 1$ and there exists $M = \lim_{t\to+\infty} \sqrt{t}\,\varphi'(t)/\varphi(t) \le +\infty$, then*

$$r(C_T)\sqrt{|\det A|} = \exp(M\,\|Pa\|), \tag{3.8}$$

with the usual convention $\infty \cdot 0 = 0$; P is the orthogonal projection of $\mathbb{R}^d$ onto $\mathcal{N}(I-A)$.

(ii) *If $r(A) > 1$ and there exists $N = \lim_{t\to+\infty} t\varphi'(t)/\varphi(t) \le +\infty$, then*

$$r(C_T)\sqrt{|\det A|} = r(A)^N. \tag{3.9}$$

Proof. We first consider the case $\|A\| > 1$. Then, by Theorem 2.2, Proposition 2.3 and Theorem 3.4 (iii), we have $\overline{N} < +\infty$ and

$$\max\{1, r(A)\}^{\underline{N}} \le r(C_T)\sqrt{|\det A|} \le \max\{1, r(A)\}^{\overline{N}}. \tag{3.10}$$

It is clear that (ii) is a consequence of (3.10). On the other hand if $r(A) \le 1$, then (3.10) implies (3.8) as $M = 0$ (the latter follows from $\overline{N} < +\infty$).

Consider now the case $\|A\| \le 1$. If $a \in \mathcal{R}(I - AA^*)$, then $Pa = 0$, so (3.8) follows from Theorem 3.4 (i). If $\|A\| = 1$ and $a \notin \mathcal{R}(I - AA^*)$, then we can apply Theorem 3.4 (ii). This completes the proof. □

In general, the quantities $\underline{M}$ and $\overline{M}$ (resp. $\underline{N}$ and $\overline{N}$) appearing in Theorem 3.4 do not coincide. It may happen that $\underline{M} \ne \overline{M}$ even for entire functions $\varphi \in \mathcal{H}_0$ (see [2] for a general method of constructing such examples). On the other hand, if $\varphi \in \mathcal{H}_0$ and $\|A\| > 1$, then $\underline{N} = \overline{N}$. Indeed, due to Corollary 2.4, φ is a polynomial, so $\underline{N} = \overline{N} = \deg \varphi$. This observation, Corollary 2.4 and Theorem 3.4 lead to the following

Corollary 3.7 *Assume that $\varphi \in \mathcal{H}_0$ and C_A is bounded. Then*

$$r(C_A) = \begin{cases} \dfrac{1}{\sqrt{|\det A|}} & \text{if } \varphi \text{ is not a polynomial} \\ \dfrac{1}{\sqrt{|\det A|}} \max\{1, r(A)\}^{\deg \varphi} & \text{if } \varphi \text{ is a polynomial.} \end{cases}$$

Note that Corollary 3.7 can be deduced from [16, Lemma 2.1] and [16, Prop. 2.2]. Applying these last two results one can also show that

$$r(C_A) < \|C_A\| \iff (\|A\| > 1 \text{ and } r(A) < \|A\|)$$

(then obviously φ has to be a polynomial). The case $\varphi(0) = 0$ is a little bit more complicated; it can be described with help of [16, Lemma 2.1] and [16, Prop. 2.2].

4. Lack of Seminormality

In this section we investigate the question: do there exist bounded seminormal composition operators with nontrivial translation part ? Roughly speaking, the answer is in the negative in all but one case where $\|A\| = 1$, $a \notin \mathcal{R}(I - AA^*)$ and $\limsup_{t \to +\infty} \sqrt{t}\, \varphi'(t)/\varphi(t) < +\infty$.

To begin with we state some more or less known characterizations of normal matrices.

Lemma 4.1 (i) *If $r(A) = \|A\|$, then there is a nonzero linear subspace H reducing A to a multiple of a unitary operator such that $\|A|_H\| = \|A\|$.*
(ii) *A is normal if and only if*

$$\|x\|^2 \leq \|Ax\|\|A^{-1}x\|, \quad x \in \mathbb{R}^d. \tag{4.1}$$

(iii) *A is a multiple of a unitary operator if and only if*

$$\|x\|^2 \geq \|Ax\|\|A^{-1}x\|, \quad x \in \mathbb{R}^d. \tag{4.2}$$

Proof. (i) Without loss of generality we can assume that $\|A\| = 1$. It follows from Lemma 3.1 (ii) that $\mathcal{D}(A_u) = \mathcal{N}(I - A^s A^{*s})$ for some $s \geq 1$. However $\|A^s\| = 1$, because $r(A) = \|A\| = 1$, so $\mathcal{D}(A_u) \neq \{0\}$.

(ii) If (4.1) holds, then $\|Ax\|^2 \leq \|A^2x\|\|x\|$ (i.e. A is paranormal), so $r(A) = \|A\|$ (cf. [9, Th. 7.1.7]). Repeated application of (i) leads to the conclusion.

(iii) If (4.2) holds, then

$$\|Ax\|\|A^{-1}x\| \leq \|x\|^2 = (A^*x, A^{-1}x) \leq \|A^*x\|\|A^{-1}x\|,$$

which implies that A^* is hyponormal and consequently normal. This in turn yields $\|Ax\|\|A^{-1}x\| = \|x\|^2$. Due to the proof of step 3 of [16, Prop. 2.3], A is a multiple of a unitary operator. □

It is worth while to note that in general, the part (ii) of Lemma 4.1 is no longer true in infinite dimensional Hilbert spaces (cf. [9, Th. 8.3.29] for some generalizations). On the other hand the part (iii) of Lemma 4.1 is always true (this will be proved in a separate paper).

The following estimate from below on the norm of C_T will be used in the sequel.

Lemma 4.2 *Assume that $\varphi \in \mathcal{E}$ is strictly increasing on $[\tau, +\infty)$ $(\tau \geq 0)$, $a \neq 0$ and C_T is bounded. If $\tau = 0$ or $\|A\| \geq 1$, then*

$$\|C_T\|\sqrt{|\det A|} > 1.$$

Proof. If $\tau = 0$, then $\|C_T\|^2|\det A| \geq \varphi(\|a\|^2)/\varphi(0) > 1$. Assume that $\|A\| \geq 1$. Take x_0 such that $\|x_0\| = 1$, $\|A\| = \|Ax_0\|$ and $(x_0, A^*a) \geq 0$. Since $\|A\| \geq 1$ we have

$$\|T(\sqrt{\tau}\, x_0)\|^2 = \tau\|A\|^2 + 2\sqrt{\tau}(x_0, A^*a) + \|a\|^2 > \tau.$$

This and the monotonicity of φ imply

$$|\det A|\|C_T\|^2 \geq \frac{\varphi(\|T(\sqrt{\tau}\, x_0)\|^2)}{\varphi(\|\sqrt{\tau}\, x_0\|^2)} > \frac{\varphi(\tau)}{\varphi(\tau)} = 1,$$

which completes the proof. □

Remark 4.3 Note that a function $\varphi \in \mathcal{E}$ is strictly increasing in some neighbourhood of $+\infty$ if and only if φ is not constant in any neighbourhood of $+\infty$, which in turn is equivalent to $\lim_{t\to+\infty} \varphi(t) = +\infty$. We can strengthen a part of Lemma 4.2 as follows:

- *Let φ be a positive Borel function such that $\lim_{t\to+\infty} \varphi(t) = +\infty$. If C_T is bounded, $a \neq 0$ and $\|A\| \geq 1$, then $\|C_T\|\sqrt{|\det A|} > 1$.*

Indeed, since $\varphi(\|T(\sqrt{t}\,x_0)\|^2)/\varphi(\|\sqrt{t}\,x_0\|^2) = \varphi(t+\delta(t))/\varphi(t)$, where x_0 is as in the proof of Lemma 4.2 and $\delta(t) := t(\|A\|^2 - 1) + 2\sqrt{t}(x_0, A^*a) + \|a\|^2$, it is sufficient to show that there exists $t \geq 0$ such that $\varphi(t+\delta(t)) > \varphi(t)$. Suppose, contrary to our claim, that $\varphi(t+\delta(t)) \leq \varphi(t)$ for every $t \geq 0$. Define the sequence $\{t_n\}_{n=0}^{\infty}$ by $t_0 = 0$ and $t_{n+1} := t_n + \delta(t_n)$ for $n \geq 0$. Then $\{\varphi(t_n)\}_{n=0}^{\infty}$ is monotonically decreasing and $\lim_{n\to+\infty} t_n = +\infty$ (because $t_{n+1} - t_n \geq \|a\|^2 > 0$). Thus $\liminf_{t\to+\infty} \varphi(t) < +\infty$, which contradicts $\lim_{t\to+\infty} \varphi(t) = +\infty$.

One can show (similarly to [16]) that a bounded C_T is hyponormal (resp. cohyponormal) if and only if

$$\begin{aligned} &\varphi(\|Tx\|^2)\,\varphi(\|T^{-1}x\|^2) \leq \varphi(\|x\|^2)^2, \quad x \in \mathbb{R}^d \\ \text{(resp.}\quad &\varphi(\|Tx\|^2)\,\varphi(\|T^{-1}x\|^2) \geq \varphi(\|x\|^2)^2, \quad x \in \mathbb{R}^d). \end{aligned} \tag{4.3}$$

We are now in a position to state the first result excluding the existence of seminormal composition operators C_T with $a \neq 0$ in case $\|A\| \leq 1$ and $a \in \mathcal{R}(I - AA^*)$.

Theorem 4.4 *Assume that $\varphi \in \mathcal{E}$ is strictly increasing on $[\tau, +\infty)$ ($\tau \geq 0$), $a \neq 0$ and C_T is bounded. If any of the following three conditions holds*
(i) $r(A) < 1$ *and* $\tau = 0$,
(ii) $\|A\| = 1$ *and* $Pa = 0$ (P *is the orthogonal projection onto* $\mathcal{N}(I - A)$),
(iii) $r(A) \leq 1$ *and* $\|A\| > 1$,
then C_T is not seminormal.

Proof. (i) Suppose that C_T is seminormal. Then (cf. [1]) $r(C_T) = \|C_T\|$ and consequently (cf. [15]) $\|C_T^n\| = \|C_T\|^n$ for $n \geq 1$, which in turn implies

$$r(C_T^n) = \|C_T^n\| \quad n \geq 1. \tag{4.4}$$

Since $r(A) < 1$, there exists $k \geq 1$ such that $\|A^k\| < 1$. Hence $\mathcal{N}(I - A) = \{0\}$ and, consequently, $a_k = (I - A)^{-1}(I - A^k)a \neq 0$ (compare with (3.4)), so by Lemma 4.2 and Theorem 3.4 (i) we have

$$\|C_T^k\|\sqrt{|\det A^k|} > 1 = r(C_T^k)\,\sqrt{|\det A^k|},$$

which contradicts (4.4).

(ii) & (iii) It follows from Theorem 3.4 that $r(C_T)\sqrt{|\det A|} = 1$. Since, by Lemma 4.2, $\|C_T\|\sqrt{|\det A|} > 1$, we conclude that C_T is not seminormal. □

An inspection of the proof of Theorem 4.4 shows that under its assumptions C_T is not even normaloid, i.e. $r(C_T) \neq \|C_T\|$.

Example 4.5 It may happen that C_T is cohyponormal, while $\|A\| < 1$, $a \neq 0$ and $\varphi \in \mathcal{E}$ is monotonically increasing but not strictly increasing. Consequently, we have $\|C_T\| \sqrt{|\det A|} = 1$.

Let $0 < \alpha < 1$, $a \neq 0$, $Ax = \alpha x$. Then there exists $t_0 \geq 0$ such that $\|Ax + a\|^2 + \|A^{-1}(x - a)\|^2 \geq 2\|x\|^2$ for $\|x\|^2 \geq t_0$. Define φ as follows

$$\varphi(t) = \begin{cases} \exp(t_0) & \text{if} \quad t < t_0 \\ \exp(t) & \text{if} \quad t \geq t_0. \end{cases}$$

To prove the cohyponormality of C_T it is enough to show (see (4.3)) that

$$\varphi(\|Tx\|^2)\,\varphi(\|T^{-1}x\|^2) \geq \varphi(\|x\|^2)^2.$$

Consider two cases. If $\|x\|^2 < t_0$, then

$$\varphi(\|Tx\|^2)\,\varphi(\|T^{-1}x\|^2) \geq \exp(2t_0) = \varphi(\|x\|^2)^2.$$

On the other hand, if $\|x\|^2 \geq t_0$, then

$$\begin{aligned} \varphi(\|Tx\|^2)\,\varphi(\|T^{-1}x\|^2) &\geq \exp(\|Tx\|^2)\exp(\|T^{-1}x\|^2) \\ &\geq \exp(2\|x\|^2) \\ &= \varphi(\|x\|^2)^2, \end{aligned}$$

which proves the cohyponormality of C_T.

The case $r(A) > 1$ is investigated below.

Theorem 4.6 *Assume that $\varphi \in \mathcal{E}$ and $\lim_{t\to+\infty} \sqrt{t}\,(N - \frac{t\varphi'(t)}{\varphi(t)}) = 0$ for some $N > 0$. If C_T is seminormal, then A is normal and $a = 0$.*

Proof. Let $\tau \geq 1$ be such that $\varphi \in \mathcal{E}_\tau$. Since $N = \lim_{t\to+\infty} t\varphi'(t)/\varphi(t)$, the operator C_T is bounded. First we show that φ fulfills the condition

$$\lim_{t\to+\infty} \sqrt{t}\,(N - \underline{N}_t) = 0. \tag{4.5}$$

Indeed, since for every $t \geq \tau$ there exists $s \geq t$ such that $s\varphi'(s)/\varphi(s) - \underline{N}_t < t^{-1}$, we have

$$\begin{aligned} |\sqrt{t}\,(N - \underline{N}_t)| &\leq \sqrt{t}\left|N - \frac{s\varphi'(s)}{\varphi(s)}\right| + \sqrt{t}\left(\frac{s\varphi'(s)}{\varphi(s)} - \underline{N}_t\right) \\ &\leq \sqrt{s}\left|N - \frac{s\varphi'(s)}{\varphi(s)}\right| + \frac{1}{\sqrt{t}}, \end{aligned}$$

which implies (4.5).

We split the proof into a few steps.

Step 1. If $r(A) \geq 1$ and $r(C_T) = \|C_T\|$, then $r(A) = \|A\|$. Moreover, $(Ax, a) = 0$ for every $x \in \mathbb{R}^d$ such that $\|Ax\| = \|A\|\|x\|$.

Take x such that $\|Ax\| = \|A\|\|x\|$. Without loss of generality we can assume that $\|x\| = 1$ and $(Ax, a) \geq 0$. Then $\|T(\sqrt{t}\,x)\|^2 \geq \|\sqrt{t}\,x\|^2$ for $t \geq \tau$. Applying Theorem 3.4 and Lemma 1.3 we obtain

$$\begin{aligned} r(A)^{2N} &\geq |\det A|\, r(C_T)^2 = |\det A|\, \|C_T\|^2 \\ &\geq \frac{\varphi(\|T(\sqrt{t}\,x)\|^2)}{\varphi(\|\sqrt{t}\,x\|^2)} \\ &\geq \left(\frac{\|T(\sqrt{t}\,x)\|^2}{t}\right)^{\underline{N}_t} \\ &\geq \left(\|A\|^2 + \frac{(Ax, a)}{\sqrt{t}}\right)^{\underline{N}_t} \\ &\geq \|A\|^{2\underline{N}_t}\left(1 + \frac{(Ax, a)}{\sqrt{t}\,\|A\|^2}\right)^{\underline{N}_t}, \quad t \geq \tau. \end{aligned} \tag{4.6}$$

In particular, we have $r(A)^N \geq \|A\|^{\underline{N}_t}$ for $t \geq \tau$. Letting $t \to +\infty$ we get $r(A) = \|A\|$. The last equality and (4.6) imply

$$r(A)^{2\sqrt{t}\,(N - \underline{N}_t)} \geq \left(1 + \frac{(Ax, a)}{\sqrt{t}\,\|A\|^2}\right)^{\sqrt{t}\,\underline{N}_t} \geq 1 + \underline{N}_t\,\frac{(Ax, a)}{\|A\|^2}$$

for t large enough. Hence, by (4.5), $(Ax, a) = 0$.

Step 2. If $r(A) \geq 1$ and C_T is seminormal, then there exists $H \neq \{0\}$ which reduces A to a normal operator and $a \perp H$.

Since C_T is seminormal , we have $r(C_T) = \|C_T\|$. Due to Step 1, $r(A) = \|A\|$. Let H be as in Lemma 4.1 (i). If $x \in H$, then $\|Ax\| = \|A\|\|x\|$.

Hence, by Step 1, $(Ax, a) = 0$ for every $x \in H$ or equivalently $a \perp H$.

Step 3. If C_T is seminormal, $K \neq \mathbb{R}^d$ is a linear space which reduces A to a normal operator and $a \perp K$, then there exists a linear space $\tilde{K}$, essentially larger than K, which reduces A to a normal operator and $a \perp \tilde{K}$.

The transformation T decomposes into the orthogonal sum $T = T_1 \oplus T_2$, where $T_1 = T|_K$ is a normal linear operator in K and $T_2 = A_2 + a$ with $A_2 := A|_{K^\perp}$. It follows from (4.3) that C_{T_2} is seminormal. If $r(A_2) \geq 1$, then, by Step 2, there exists $H \neq \{0\}$ which reduces A_2 to a normal operator and $a \perp H$. It is clear that the space $\tilde{K} := K \oplus H$ has the required properties.

Suppose that $r(A_2) < 1$. Then $r(A_2^{-1}) > 1$, $C_{T_2^{-1}} = C_{T_2}^{-1}$ is bounded and seminormal (use Theorem 2.2 and Proposition 2.3). By Step 2, there exists $H \neq \{0\}$ which reduces A_2^{-1} to a normal operator and $A_2^{-1}a \perp H$. Consequently, H reduces A to a normal operator and $a \perp (A_2^*)^{-1}(H) = H$. Therefore the space $\tilde{K} := K \oplus H$ has the required properties.

The conclusion of the theorem follows from Step 3. $\square$

It may happen that the limit $\lim_{t\to+\infty} \sqrt{t}\,(N - \frac{t\varphi'(t)}{\varphi(t)})$ does not exist. This case is treated in the following theorem.

Theorem 4.7 *Let $\varphi \in \mathcal{E}$ be such that $\limsup_{t\to+\infty} t\varphi'(t)/\varphi(t) < +\infty$. Assume that $\varphi(t) = \alpha t^N + \mathcal{O}(t^{N-\epsilon})$ for some positive reals α, N, ϵ. If C_T is seminormal, then A is normal. If moreover $\epsilon > 1/2$, then $a = 0$.*

Proof. Put $\rho(t) := \varphi(t) - \alpha t^N$. Without loss of generality we may assume that N is a positive integer. Otherwise we can consider φ^θ instead of φ with appropriate $\theta > 1$ still preserving seminormality of C_T (use (4.3)). For simplicity we assume also that $\alpha = 1, \epsilon < N$.

First we show that A is normal. If C_T is cohyponormal, then, by (4.3), we have

$$\begin{aligned}
1 &\leq \lim_{t\to+\infty} \frac{\varphi(\|T(tx)\|^2)\,\varphi(\|T^{-1}(tx)\|^2)}{\varphi(\|tx\|^2)^2} \\
&= \lim_{t\to+\infty} \frac{\varphi(\|T(tx)\|^2)}{\|T(tx)\|^{2N}}\,\frac{\varphi(\|T^{-1}(tx)\|^2)}{\|T^{-1}(tx)\|^{2N}}\,\frac{\|T(tx)\|^{2N}\,\|T^{-1}(tx)\|^{2N}}{\|tx\|^{4N}} \\
&= \lim_{t\to+\infty} \left(\frac{\|T(tx)\|\,\|T^{-1}(tx)\|}{\|tx\|^2}\right)^{2N}
\end{aligned}$$

$$= \left(\frac{\|Ax\| \|A^{-1}x\|}{\|x\|^2} \right)^{2N}, \quad x \neq 0. \tag{4.7}$$

It follows from Lemma 4.1 (ii) that A is normal. Similar arguments can be applied to deduce the normality of A from the hyponormality of C_T.

To prove the other part of the conclusion, assume that C_T is seminormal and $\epsilon > 1/2$. Since A is normal, A can be decomposed as

$$A = \sum_{j=1}^{m} \oplus \kappa_j U_j,$$

where $\kappa_1 > \kappa_2 > \ldots > \kappa_m > 0$ and U_j is a unitary operator on $H_j \neq \{0\}$. Take any j such that $\kappa_j \neq 1$. We show that the orthogonal projection a_j of a onto H_j vanishes. Take $x \in H_j$ such that $\|x\| = 1$. First notice that $\rho(\|T(tx)\|^2) = \rho(\mathcal{O}(t^2)) = \mathcal{O}((t^2)^{N-\epsilon}) = o(|t|^{2N-1})$ and $\rho(\|T^{-1}(tx)\|^2) = o(|t|^{2N-1})$. This in turn implies

$$\begin{aligned} \varphi(\|T(tx)\|^2) &= t^{2N}\|Ax\|^{2N} \\ &\quad + 2Nt^{2N-1}\|Ax\|^{2N-2}(Ax, a) + o(|t|^{2N-1}), \\ \varphi(\|T^{-1}(tx)\|^2) &= t^{2N}\|A^{-1}x\|^{2N} \\ &\quad - 2Nt^{2N-1}\|A^{-1}x\|^{2N-2}(A^{-1}x, A^{-1}a) \\ &\quad + o(|t|^{2N-1}), \\ \varphi(\|tx\|^2) &= t^{2N} + o(|t|^{2N-1}). \end{aligned}$$

Consequently

$$\begin{aligned} &\varphi(\|T(tx)\|^2)\,\varphi(\|T^{-1}(tx)\|^2) - \varphi(\|tx\|^2)^2 \\ &\quad = t^{4N}((\|Ax\|\|A^{-1}x\|)^{2N} - 1) \\ &\qquad + 2Nt^{4N-1}(\|Ax\|\|A^{-1}x\|)^{2N-2}(\|A^{-1}x\|^2(Ax, a) \\ &\qquad - \|Ax\|^2(A^{-1}x, A^{-1}a)) + o(|t|^{4N-1}) \\ &\quad = 2Nt^{4N-1}(\kappa_j^{-1}(U_j x, a_j) - (x, a_j)) + o(|t|^{4N-1}). \end{aligned}$$

Since $4N - 1$ is odd and the first term of the above chain of equalities is either globally nonnegative or globally nonpositive we get

$$\kappa_j^{-1}(U_j x, a_j) - (x, a_j) = 0, \quad x \in H_j,$$

which in turn implies that $U_j^* a_j = \kappa_j a_j$. However $\kappa_j \neq 1$, so $a_j = 0$.

If there exists n such that $\kappa_n = 1$ (n is unique), then by what has been proved in the previous paragraph, $a = a_n \in H_n$. We show that $a_n = 0$. Set $T_n = U_n + a_n$ and note that C_{T_n} is seminormal. Suppose, contrary to our claim, that $a_n \neq 0$. Since $\lim_{t\to+\infty} \varphi(t) = +\infty$, Remark 4.3 implies that $\|C_{T_n}\|\sqrt{|\det U_n|} > 1$. On the other hand $a_n \notin \mathcal{R}(I - U_nU_n^*)$ and $\underline{M} = \overline{M} = 0$ (as $\limsup_{t\to+\infty} t\varphi'(t)/\varphi(t) < +\infty$), so, by Theorem 3.4 (ii), $r(C_{T_n})\sqrt{|\det U_n|} = 1$, which contradicts the seminormality of C_{T_n}. This completes the proof. □

Remark 4.8 It is worth while to notice that Theorem 4.6 does not imply Theorem 4.7 and vice verse. Indeed, the function

$$\varphi(t) = t^N \left(1 - \frac{1}{\sqrt{t}\,\ln t}\right),$$

where $N > 1$, satisfies all the assumptions of Theorem 4.6 but not those of Theorem 4.7. More precisely φ is not of the form $\varphi(t) = \alpha t^N + \mathcal{O}(t^{N-\epsilon})$ for any $\epsilon > 1/2$.

On the other hand, the function

$$\varphi(t) = t^N + t^{N-\epsilon}\cos(t^{1-\epsilon}),$$

where $N > 1$ and $2/3 < \epsilon \leq 3/4$, satisfies all the assumptions of Theorem 4.7, but $\lim_{t\to+\infty} \sqrt{t}\,(N - \frac{t\varphi'(t)}{\varphi(t)})$ does not exist. In fact, we have

$$-\liminf_{t\to+\infty} \sqrt{t}\left(N - \frac{t\varphi'(t)}{\varphi(t)}\right)$$

$$= \limsup_{t\to+\infty} \sqrt{t}\left(N - \frac{t\varphi'(t)}{\varphi(t)}\right) = \begin{cases} +\infty & \epsilon < \frac{3}{4} \\ \frac{1}{4} & \epsilon = \frac{3}{4}. \end{cases}$$

Details are left to the reader.

Among functions φ satisfying the assumptions of Theorems 4.6 and 4.7, there are polynomials of degree $n \geq 1$ with nonnegative coefficients.

Proposition 4.9 *If φ is a nonconstant polynomial with nonnegative coefficients, then the following conditions are equivalent*

(i) *C_T is bounded and cohyponormal,*

(ii) *A is normal and $a = 0$,*

(iii) *C_T is bounded and cosubnormal.*

Proof. Note that if C_T is bounded, then $\varphi(0) > 0$ or $a = 0$. Indeed, otherwise $\lim_{x\to 0} \varphi(\|Tx\|^2)/\varphi(\|x\|^2) = +\infty$, which contradicts (1.1).

(i) $\Rightarrow$ (ii) That $a = 0$ follows from the above observation and Theorem 4.7. Applying (4.7) with $N = \deg \varphi$ and Lemma 4.1 we conclude that A is normal.

(ii) $\Rightarrow$ (iii) This is a consequence of [16, Th. 2.5]. □

Proposition 4.10 *Let φ be a nonconstant polynomial with nonnegative coefficients. If φ is a monomial (resp. φ is not a monomial), then the following conditions are equivalent*
(i) *C_T is bounded and hyponormal,*
(ii) *A is a multiple of a unitary operator (resp. A is unitary) and $a = 0$,*
(iii) *C_T is bounded and normal (resp. C_T is unitary).*

Proof. (i) $\Rightarrow$ (ii) Analysis similar to that in the proof of Proposition 4.9 shows that $a = 0$ and A is a multiple of a unitary operator. The remaining part of (ii) follows from [16, Prop. 2.3].

The implication (ii) $\Rightarrow$ (iii) can be verified directly (see also [16, (NO) and (UN)]). □

5. Cohyponormality.

In this section we distinguish a class of cohyponormal composition operators with nontrivial translation part. According to Section 4., such operators may exist only in case $\|A\| = 1$ and $a \notin \mathcal{R}(I - AA^*)$. We show that the convexity of the function $t \longmapsto \ln \varphi(t^2)$ characterizes cohyponormal composition operators induced by pure translations. First we formulate an elementary fact concerning convex functions.

Lemma 5.1 *If $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ is an even function, then ω is convex if and only if $\omega|_{[0,+\infty)}$ is convex and monotonically increasing.*

Proposition 5.2 *Assume that φ is continuous and C_{A+a} is bounded for all a in a linear subspace H of $\mathcal{N}(I - A)$.*

(i) *If $H \neq \{0\}$ and C_{A+a} is cohyponormal for all $a \in H$, then $t \longmapsto \ln \varphi(t^2)$ is convex on $\mathbb{R}$.*

(ii) *If $A = A^*$ and $t \longmapsto \ln \varphi(t^2)$ is convex on $\mathbb{R}$, then C_{A+a} is cohyponormal for all $a \in H$.*

Proof. Set $\omega(t) = \ln \varphi(t^2)$, $t \in \mathbb{R}$.

(i) It follows from (4.3) that

$$\omega(\|x\|) \leq \frac{\omega(\|x+a\|)+\omega(\|x-a\|)}{2}, \quad x, a \in H. \tag{5.1}$$

Take $s, t \in \mathbb{R}$ and a normalized vector $v \in H$. Setting $x = \frac{1}{2}(s+t)v$ and $a = \frac{1}{2}(s-t)v$ in (5.1), we get

$$\omega\left(\frac{s+t}{2}\right) \leq \frac{\omega(s)+\omega(t)}{2}. \tag{5.2}$$

Since φ is continuous, (5.2) implies that ω is convex.

(ii) Take $a \in H$ and set, as usual, $T = A + a$. If $A = A^*$, then

$$\begin{aligned} \|(A^{-1}+A)x\|^2 &= ((A^{-2}+2+A^2)x, x) \\ &= 4\|x\|^2 + ((A^{-1}-A)^2 x, x) \\ &\geq 4\|x\|^2, \end{aligned}$$

so

$$\|Tx\| + \|T^{-1}x\| \geq \|Tx + T^{-1}x\| = \|(A^{-1}+A)x\| \geq 2\|x\|.$$

Since, by Lemma 5.1, the function $\omega|_{[0,+\infty)}$ is monotonically increasing and convex, we have

$$\omega(\|x\|) \leq \omega\left(\frac{\|Tx\|+\|T^{-1}x\|}{2}\right) \leq \frac{\omega(\|Tx\|)+\omega(\|T^{-1}x\|)}{2}.$$

This in turn implies

$$\varphi(\|x\|^2)^2 \leq \varphi(\|Tx\|^2)\,\varphi(\|T^{-1}x\|^2),$$

which is equivalent to the cohyponormality of C_T (use (4.3)). $\square$

Corollary 5.3 *If φ is continuous and C_{I+a} is bounded for all a, then C_{I+a} is cohyponormal for all a if and only if $t \longmapsto \ln\varphi(t^2)$ is convex on $\mathbb{R}$.*

Corollary 5.3 is related to Lemma 3.3 in [5] via the discussion carried in [7, Example 4.2].

Theorem 5.4 *Let $\varphi \in \mathcal{E}_0$ be such that $M := \sup_{t\geq 0} \sqrt{t}\,\varphi'(t)/\varphi(t) < +\infty$. Assume that $t \longmapsto \ln\varphi(t^2)$ is convex on $[0,+\infty)$. If $\|A\| = 1$ and $Aa = a$,* *then C_T is bounded and*

$$r(C_T) = \|C_T\| = \frac{1}{\sqrt{|\det A|}} \exp(M\,\|a\|).$$

Moreover, if $A = A^*$, then C_T is cohyponormal.

Proof. Set $\psi(t) := \varphi(t^2)$, $t \geq 0$. The function ψ is monotonically increasing because $\varphi \in \mathcal{E}_0$. Since $\ln\psi$ is convex, the function $(\ln\psi)'$ is monotonically increasing. Consequently $\lim_{t\to+\infty} \sqrt{t}\,\varphi'(t)/\varphi(t) = \frac{1}{2}\sup\psi'/\psi = M$. Applying Theorem 3.4 we obtain

$$r(C_T)^2 = \frac{1}{|\det A|} \exp(2M\,\|a\|). \tag{5.3}$$

Since $\psi \in \mathcal{E}_0$, the inequality (1.2) of Lemma 1.1 yields

$$\begin{aligned} |\det A|\,\|C_T\|^2 &= \sup_x \frac{\psi(\|Tx\|)}{\psi(\|x\|)} \\ &\leq \sup_x \frac{\psi(\|x\| + \|a\|)}{\psi(\|x\|)} \\ &\leq \exp\left(\|a\|\,\sup \frac{\psi'}{\psi}\right). \end{aligned}$$

This and (5.3) imply the first part of the conclusion. The other one follows from Lemma 5.1 and Proposition 5.2 (ii) with $H = \mathcal{N}(I - A)$. $\square$

Note that the functions $\varphi_1(t) = \cosh(\sqrt{t})$ and $\varphi_2(t) = \sinh(\sqrt{t})/\sqrt{t}$ $(t > 0)$ satisfy all the assumptions of Theorem 5.4 and both belong to $\mathcal{H}_0$. It turns out that if $d = 1$, then C_{I+a} is a cosubnormal operator on $L^2(\varphi_j(\|x\|^2)^{-1}\mathrm{d}x)$ for $j = 1, 2$. This can be proved by repeating some reasonings from [6] via analysis carried in [7, Example 4.2].

6. Concluding Remarks

1^0. In this paper we have investigated composition operators C_T on the Hilbert space $L^2(\varphi(\|x\|^2)^{-1}\mathrm{d}x)$, where the (continuous) function φ has been assumed to be convex, monotonically increasing and continuously differentiable in some neighbourhood of infinity. It is easily seen that the implications (i) $\Rightarrow$ (ii) of Lemmata 1.1, 1.2 and 1.3, which play the essential role in all the estimates of $\|C_T\|$ and $r(C_T)$, hold for φ which is *monotonically increasing and continuously differentiable in some neighbourhood of* *infinity.* Consequently, for such φ all Theorems of the paper, except the "only if" part of Theorem 2.2, remain true provided we add, everywhere where it is necessary, one of the conditions (i) $\div$ (iv) of Theorem 2.2 (which of course implies the boundedness of C_T). The details are left to the reader.

2^0. All the results of the paper remain true if we replace the canonical norm $\|\cdot\|$ involved in the definition of $L^2(\varphi(\|x\|^2)^{-1}\mathrm{d}x)$ by an arbitrary one coming from an inner product on $\mathbb{R}^d$.

3^0. As in 2^0 all the results of the paper remain true for composition operators induced by complex affine isomorphisms T of $\mathbb{C}^d$; we only have to replace the quantity $|\det A|$ by the new one $|\det A|^2$, where in the latter case $\det A$ stands for the determinant of a complex matrix associated with A.

4^0. Set $\rho(x) = \varphi(\|x\|^2)^{-1}$. It is a matter of direct verification to show that

$$C_{T,1/\rho} = U_\rho(|\det A|^{-1} C^*_{T^{-1},\rho}) U_\rho^{-1}, \tag{6.1}$$

where $C_{S,\omega}$ stands for the composition operator induced by S on $L^2(\omega(x)\mathrm{d}x)$ and $U_\rho : L^2(\rho(x)\mathrm{d}x) \longrightarrow L^2(\rho(x)^{-1}\mathrm{d}x)$ is the unitary operator defined by $U_\rho f = f\rho$ for $f \in L^2(\rho(x)\mathrm{d}x)$ (compare with (AD) and (UE) in [16]). The equality (6.1) should be understood as follows: $C_{T,1/\rho}$ is bounded if and only if so is $C_{T^{-1},\rho}$; if this is the case, then (6.1) holds. Basing on (6.1) one can easily formulate appropriate versions of all the results of the paper for composition operators induced by T on $L^2(\varphi(\|x\|^2)\mathrm{d}x)$.

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