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Lattice homomorphism — Korovkin type inequalities for vector valued functions

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Abstract. Considered here is the space of continuous functions from a compact and convex subset of a normed vector space into an abstract Banach lattice. Also considered are lattice homomorphisms from the above space into itself or into the associated space of vector valued bounded functions. The uniform convergence of such operators to the unit operator with rates is mainly studied in this article. The produced quantitative results are inequalities which engage the modulus of continuity of the involved continuous function or of its higher order Fréchet derivative.

Key words: Lattice homomorphism, positive operator, Banach lattice, Banach space, modulus of continuity, Fréchet derivatives, unit operator, rate of convergence, Korovkin type inequalities, uniform convergence, continuous function, bounded function.

1. Introduction

The study of the convergence of positive linear operators became more intense and attracted a lot of attention when P. Korovkin (1953) proved his famous theorem (see [8], p. 14).

Korovkin's First Theorem *Let $[a,b]$ be a compact interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators L_n mapping $C([a,b])$ into itself. Suppose that (L_nf) converges uniformly to f for the three test functions $f=1,x,x^2$. Then (L_nf) converges uniformly to f on $[a,b]$ for all functions of $f\in C([a,b])$.*

So a lot of authors since then are working on the theoretical aspects of above convergence. But R.A. Mamedov (1959) (see [9]) was the first to put Korovkin's theorem in a quantitative form.

Mamedov's Theorem *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators in the space $C([a,b])$, for which $L_n1=1$, $L_n(t,x)=x+\alpha_n(x)$,*

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$L_n(t^2, x) = x^2 + \beta_x(x)$. *Then*

$$\|L_n(f, x) - f(x)\|_\infty \leq 3\omega_1(f, \sqrt{d_n}),$$

where ω_1 *is the first modulus of continuity and* $d_n := \|\beta_n(x) - 2x\alpha_n(x)\|_\infty$.

An improvement of the last theorem was the following

Shisha and Mond's Theorem (1968, see [12]). *Let* $[a, b] \subset \mathbb{R}$ *be a compact interval. Let* $\{L_n\}_{n\in\mathbb{N}}$ *be a sequence of positive linear operators acting on* $C([a, b])$. *For* $n = 1, 2, \ldots$, *suppose* $L_n(1)$ *is bounded. Let* $f \in C([a, b])$. *Then for* $n = 1, 2, \ldots$, *we have*

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \cdot \|L_n 1 - 1\|_\infty + \|L_n(1) + 1\|_\infty \cdot \omega_1(f, \mu_n),$$

where

$$\mu_n := \|(L_n((t - x)^2))(x)\|_\infty^{1/2}.$$

Shisha-Mond inequality generated and inspired a lot of research done by many authors worldwide on the rate of *convergence of a sequence of positive linear operators to the unit operator*, always producing similar inequalities however in many different directions, e.g., see the important work of H. Gonska of 1983 in [5], etc.

In his 1993 research monograph, the author (see [4]), establishes in many directions best upper bounds for $|(L_n f)(x_0) - f(x_0)|$, $x_0 \in Q \subseteq \mathbb{R}^n$, $n \geq 1$, compact and convex, which lead for the first time to sharp/attained inequalities of Shisha-Mond type. The method of proving is probabilistic from the theory of moments. His pointwise approach is closely related to the study of the weak convergence with rates of a sequence of finite measures to the unit measure at a specific point.

All of the above have inspired and motivated the work in this article. Here is what we do: Let X be a normed vector space, Y be a Banach lattice, $M \subset X$ is a compact and convex subset. Consider the space of continuous functions from M into Y, denoted by $C(M, Y)$, also consider the space of bounded functions $B(M, Y)$. Here we study the rate of the uniform convergence of lattice homomorphisms $T : C(M, Y) \to C(M, Y)$ or $T : C(M, Y) \to B(M, Y)$ to the unit operator I. For that see Theorems 10, 12 and Corollary 14. In the last two results we assume that X is a Banach space. The produced inequalities (19), (23) and (25), respectively, are of

Shisha-Mond type, i.e., Korovkin type.

In there we find upper bounds to $|||Tf - f|||_\infty$, $f \in C(M,Y)$, and $|||TP-P|||_\infty$, $P \in C^n(M,Y)$, $n \in \mathbb{N}$, (space of n-times continuously Fréchet differentiable functions), where $|||\cdot|||_\infty$ is the supremum norm in $C(M,Y)$ or $B(M,Y)$. These inequalities involve the modulus of continuity of f or $P^{(n)}$.

The rest of the material of this article makes sure that the right-hand sides of the main inequalities (19), (23) and (25) are finite. At the end we give several examples. To the best of our knowledge this is the first treatise for Korovkin type inequalities for lattice homomorphisms over vector valued functions. Since $C(M,Y)$ is a Banach lattice, the lattice homomorphism T as described above, is also a positive operator and thus a continuous linear operator.

2. Background

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be real normed vector spaces, and let $M \subset X$ be a set. Assume that $(Y, \|\cdot\|, <)$ is a Banach lattice, see [1], p. 197. Denote by $C(M,Y)$ the vector space of continuous functions from M into Y.

Lemma 1 *$C(M,Y)$ is a vector lattice.*

Proof. Let $f, g \in C(M,Y)$ and x_n, $x \in M$, such that $x_n \to x$, then $f(x_n) \to f(x)$, and $g(x_n) \to g(x)$, as $n \to +\infty$. I.e., $\|f(x_n) - f(x)\| < \varepsilon$, $\|g(x_n) - g(x)\| < \varepsilon$ for $\varepsilon > 0$ arbitrarily small, iff $|f(x_n) - f(x)| < \varepsilon \cdot i$, $|g(x_n) - g(x)| < \varepsilon \cdot i$, $i \in Y^+$, positive cone of Y, such that $\|i\| = 1$. Here $|f| := f \vee (-f)$, where $\vee$, $\wedge$ stand for the supremum and infimum, respectively. Denote by $\circ$ either of $\vee$, $\wedge$.

We introduce the order $f < g$ iff $f(x) < g(x)$, all $x \in M$. It holds that

$$\begin{aligned} (f \vee g)(x) &= f(x) \vee g(x), \\ (f \wedge g)(x) &= f(x) \wedge g(x), \\ |f|(x) &= |f(x)|, \quad \text{all } x \in M. \end{aligned} \tag{1}$$

Obviously here $|f(x)| < |g(x)|$ iff

$$\|f(x)\| < \|g(x)\|, \quad \text{all } x \in M.$$

We observe that (cf. Theorem 24.1, p. 194, [1])

$$|f(x_n) \circ g(x_n) - f(x) \circ g(x)|$$

$$\leq |f(x_n) \circ g(x_n) - f(x) \circ g(x_n)| + |f(x) \circ g(x_n) - f(x) \circ g(x)|$$
$$\leq |f(x_n) - f(x)| + |g(x_n) - g(x)| \leq 2\varepsilon \cdot i.$$

Therefore

$$|f(x_n) \circ g(x_n) - f(x) \circ g(x)| \leq 2\varepsilon \cdot i,$$

iff

$$\|f(x_n) \circ g(x_n) - f(x) \circ g(x)\| \leq 2\varepsilon.$$

That is,

$$\|(f \circ g)(x_n) - (f \circ g)(x)\| \leq 2\varepsilon,$$

for any $\varepsilon > 0$ small.

Hence $(f \circ g)(x_n) \to (f \circ g)(x)$, i.e., $f \vee g$, $f \wedge g$ are continuous functions. Thus $C(M, Y)$ is a vector lattice. □

From now on we assume that M is compact. Define for $f \in C(M, Y)$

$$|||f|||_\infty := \sup\{\|f(x)\| : x \in M\}. \tag{2}$$

One can easily see that $|||\cdot|||_\infty$ defines a norm on $C(M, Y)$. For $f, g \in C(M, Y)$ we have that, $|f| \leq |g|$ iff $|f|(x) \leq |g|(x)$ iff $|f(x)| \leq |g(x)|$ iff $\|f(x)\| \leq \|g(x)\|$, all $x \in M$. The last implies $|||f|||_\infty \leq |||g|||_\infty$. I.e., if

$$|f| \leq |g| \Longrightarrow |||f|||_\infty \leq |||g|||_\infty.$$

Hence $|||\cdot|||_\infty$ is a lattice norm, and $C(M, Y)$ is a normed vector lattice, where M is a compact subset of X.

Proposition 2 *$C(M, Y)$ is a Banach lattice.*

Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $C(M, Y)$. Then given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|||f_n - f_m|||_\infty < \varepsilon$, all $n, m > n_0$. Therefore for any $x \in M$ we have

$$\|f_n(x) - f_m(x)\| \leq |||f_n - f_m|||_\infty < \varepsilon,$$

which implies that $\{f_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach lattice Y. Thus, by completeness of Y, $\{f_n(x)\}$ converges in Y for every $x \in M$; let $f(x) := \lim_{n \to +\infty} f_n(x)$, all $x \in M$. Since $\|f_n(x) - f_m(x)\| < \varepsilon$, all

$n, m > n_0$, for a fixed $n > n_0$ we get that

$$\|f_n(x) - f_m(x)\| < \varepsilon, \quad \text{all} \ \ m > n_0.$$

By continuity of $\|\cdot\|$ and taking the limit in the last inequality as $m \to +\infty$ we obtain $\|f_n(x) - f(x)\| < \varepsilon$, true for all $n > n_0$ and all $x \in M$. That is,

$$|||f_n - f|||_\infty < \varepsilon, \quad \text{all} \ \ n > n_0,$$

i.e., $\lim_{n\to+\infty} f_n = f$ in $|||\cdot|||_\infty$.

Let x_N, $x \in M$ be such that $x_N \to x$, then $f_n(x_N) \to f_n(x)$, by $f_n \in C(M, Y)$. Next see that

$$\begin{aligned}\|f(x_N) - f(x)\| &\leq \|f(x_N) - f_n(x_N)\| + \|f_n(x_N) - f_n(x)\| \\ + \|f_n(x) - f(x)\| &\leq \varepsilon + \varepsilon + \varepsilon,\end{aligned}$$

by $f_n \to f$ and f_n-continuity. I.e., $\|f(x_N) - f(x)\| \leq 3\varepsilon$, $\varepsilon > 0$ small. Hence $f(x_N) \to f(x)$, as $N \to +\infty$, i.e., $f \in C(M, Y)$. That is, $C(M, Y)$ is complete, proving that it is a Banach lattice. □

Definition 3 Let $T : C(M, Y) \to C(M, Y)$ be a linear operator. T is called a *lattice homomorphism* iff it fulfills one of the following equivalent statements:

(i) $T(f \vee g) = T(f) \vee T(g)$, all $f, g \in C(M, Y)$,
(ii) $T(f \wedge g) = T(f) \wedge T(g)$, all $f, g \in C(M, Y)$,
(iii) $T(f) \wedge T(g) = 0$ holds whenever $f \wedge g = 0$,
(iv) $|T(f)| = T(|f|)$, all $f \in C(M, Y)$, see [1], p. 202.

Obviously a lattice homomorphism is a *positive* one, i.e., whenever $f \geq g$ we get that $T(f) \geq T(g)$, $f, g \in C(M, Y)$. Since $C(M, Y)$ is a Banach lattice, then a positive operator T from $C(M, Y)$ into itself is a *continuous* one, see [1], p. 200.

In this paper, we will be dealing mainly with lattice homomorphisms $T : C(M, Y) \to C(M, Y)$. We need the following auxiliary results.

Proposition 4 *Let $f \in C(M, Y)$ and $T : C(M, Y) \to C(M, Y)$ be a continuous linear operator. Then $(T(f(x_0)))(x_0)$ is a continuous function of $x_0 \in M$.*

Proof. Let x_n, $x_0 \in M$ be such that $x_n \to x_0$, then $\rho_n(x) := f(x_n) \to f(x_0) =: \rho(x)$, all $x \in M$, i.e., $\rho_n \to \rho$ uniformly. By continuity of T we get that $T(\rho_n) \to T(\rho)$ uniformly, i.e., $T(f(x_n)) \to T(f(x_0))$ uniformly. (Here

$T(f(x_n))$, $T(f(x_0)) \in C(M,Y)$.) That is, $T(f(x_n)) - T(f(x_0)) \xrightarrow{u} 0$, i.e., given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|||T(f(x_n)) - T(f(x_0))|||_\infty < \varepsilon$, all $n > n_0$.

Hence given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|(T(f(x_n)))(x_n) - (T(f(x_0)))(x_n)\| \\ \leq |||T(f(x_n)) - T(f(x_0))|||_\infty < \varepsilon,$$

all $n > n_0$. Notice that (as $x_n \to x_0$)

$$\|(T(f(x_n)))(x_n) - (T(f(x_0)))(x_0)\| \\ \leq \|(T(f(x_n)))(x_n) - (T(f(x_0)))(x_n)\| \\ + \|(T(f(x_0)))(x_n) - (T(f(x_0)))(x_0)\| < 2\varepsilon.$$

The last establishes the claim of the proposition. □

Remark 5 From Proposition 4 we get that $(T(f - f(x_0)))(x_0)$ is a continuous function of $x_0 \in M$ with values in Y. Also let $i \in Y^+$ be such that $\|i\| = 1$, then $T(i) \in C(M,Y)$.

Proposition 6 *Let $T : C(M,Y) \to C(M,Y)$ be a continuous linear operator and $r > 0$. Then $(T(\|x - x_0\|^r \cdot i))(x_0)$ is a continuous function of $x_0 \in M$ with values in Y, where $i \in Y^+$ is such that $\|i\| = 1$.*

Proof. Let x_n, $x_0 \in M$ be such that $x_n \to x_0$, as $n \to +\infty$. Observe that

$$\|(T(\|x - x_n\|^r \cdot i))(x_n) - (T(\|x - x_0\|^r \cdot i))(x_0)\| \\ \leq \|(T(\|x - x_n\|^r \cdot i))(x_n) - (T(\|x - x_0\|^r \cdot i))(x_n)\| \\ + \|(T(\|x - x_0\|^r \cdot i))(x_n) - (T(\|x - x_0\|^r \cdot i))(x_0)\| =: (*).$$

Notice that $\|x - x_0\|^r \cdot i$ is a continuous function of x ans so is $(T(\|x - x_0\|^r \cdot i))(x)$. That is,

$$(T(\|x - x_0\|^r \cdot i))(x_n) \to (T(\|x - x_0\|^r \cdot i))(x_0),$$

i.e.,

$$\|(T(\|x - x_0\|^r \cdot i))(x_n) - (T(\|x - x_0\|^r \cdot i))(x_0)\| < \varepsilon_1,$$

where $\varepsilon_1 > 0$ is small.

Therefore

$$(*) < |||T(\|x - x_n\|^r \cdot i) - T(\|x - x_0\|^r \cdot i)|||_\infty + \varepsilon_1 =: (**).$$

Notice that

$$\sup_{x\in M} |\|x-x_n\| - \|x-x_0\|| \le \|x_n - x_0\| \to 0,$$

thus

$$\||\|x-x_n\| - \|x-x_0\||\|_{\infty,x} \to 0, \quad (\|\cdot\|_{\infty,x} \text{ supremum in } x)$$

i.e.,

$$\|x-x_n\| \xrightarrow{u} \|x-x_0\|, \quad \text{uniformly.}$$

Here $\|x-y\| \le \Delta$ — the diameter of M, all $x, y \in M$, i.e., $\|x-x_n\|$, $\|x-x_0\| \le \Delta$.

We have proved in [3], pp. 20–21 that $\|x-x_n\|^r \xrightarrow{u} \|x-x_0\|^r$, uniformly, i.e., for $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\||\|x-x_n\|^r - \|x-x_0\|^r\|_{\infty,x} < \varepsilon,$$

for all $n > n_0$.

Observe that

$$\begin{aligned}
&\||\,\|x-x_n\|^r \cdot i - \|x-x_0\|^r \cdot i\||_\infty \\
&\quad = \sup_{x\in M} \|\|x-x_n\|^r \cdot i - \|x-x_0\|^r \cdot i\| \\
&\quad = \sup_{x\in M} |\|x-x_n\|^r - \|x-x_0\|^r| \cdot \|i\| \\
&\quad = \||\|x-x_n\|^r - \|x-x_0\|^r\|_{\infty,x} < \varepsilon.
\end{aligned}$$

That is,

$$\||\,\|x-x_n\|^r \cdot i - \|x-x_0\|^r \cdot i\||_\infty < \varepsilon, \quad \text{all } \ n > n_0,$$

i.e.,

$$\|id - x_n\|^r \cdot i \xrightarrow{u} \|id - x_0\|^r \cdot i,$$
$$\text{uniformly } (id \text{ is the identity map}).$$

By the continuity of operator T we get that

$$T(\|x-x_n\|^r \cdot i) \xrightarrow{u} T(\|x-x_0\|^r \cdot), \quad \text{uniformly.}$$

I.e.,

$$\||T(\|x-x_n\|^r \cdot i) - T(\|x-x_0\|^r \cdot i)\||_\infty < \varepsilon_1,$$

where $\varepsilon_1 > 0$ is small as above. Consequently

$$(**) < 2\varepsilon_1.$$

We have established that

$$(T(\|x - x_n\|^r \cdot i))(x_n) \to (T(\|x - x_0\|^r \cdot i))(x_0),$$

as $x_n \to x_0$ and $n \to +\infty$. That is we have proved the claim of the proposition. □

From now on we assume that M is a compact and convex subset of X.

Definition 7 Let $f \in C(M, Y)$, its (first) modulus of continuity is defined by

$$\omega_1(f, h) := \sup\{\|f(x) - f(y)\| : \text{all } x, y \in M \text{ such that } \|x - y\| \leq h\}, \tag{3}$$

$h > 0$. Here Y can be just a normed vector space.

Lemma 8 (From Lemma 7.11, p. 208, [4]). *Let $f \in C(M, Y)$, $h > 0$ and fixed $x_0 \in M$. Then*

$$\|f(x) - f(x_0)\| \leq \omega_1(f, h) \cdot \left\lceil \frac{\|x - x_0\|}{h} \right\rceil \leq \omega_1(f, h) \cdot \left(1 + \frac{\|x - x_0\|}{h}\right), \quad \text{all } x \in M. \tag{4}$$

Here $\lceil \cdot \rceil$ stands for the ceiling of the number. Here Y can be just a normed vector space. Obviously one can have that

$$\|f(x) - f(x_0)\| \leq \left\| \omega_1(f, h) \cdot \left(1 + \frac{\|x - x_0\|}{h}\right) \cdot i \right\|, \tag{5}$$

where $i \in Y^+$ such that $\|i\| = 1$. Thus, for $f \in C(M, Y)$ and $h > 0$ we get that

$$\||f - f(x_0)\||_\infty \leq \||\omega_1(f, h) \cdot \left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\||_\infty, \tag{6}$$

$x_0 \in M$ be fixed.

From now on in this section we would assume that X is a Banach space and that P maps X into the Banach lattice Y. Here $M \subset X$ still is a compact and convex subset. Furthermore we would assume that P is n-times continuously Fréchet differentiable on M, i.e., $P|_M \in C^n(M,Y)$. Obviously $P|_M \in C(M,Y)$ which is a Banach lattice. That is, here $(P|_M)^{(k)}$ is a continuous map from M into the space of all k-linear bounded operators from X into Y, all $k = 1,\dots,n$. It is clear that all $(P|_M)^{(k)}$, $k = 0,1,\dots,n$ are norm bounded by continuity, and thus they are integrable. Here for any x_0, $x_1 \in M$ we form $x(\theta) := \theta x_1 + (1-\theta)x_0 \in M$, $0 \le \theta \le 1$, and we identify $P(\theta) \equiv P(x(\theta))$. That is, $P^{(k)}(\theta) = P^{(k)}(x(\theta))(x_1 - x_0)^k$, where $(x_1 - x_0)^k := (x_1 - x_0, \dots, x_1 - x_0)$ is a k-tuple, $k = 1,\dots,n$. We are following [10], pp. 87–127, Chapter 3.

In particular from [10], p. 124, Theorem 20.2 (Taylor's Theorem) for any x_0, $x_1 \in M$ we get that

$$P(x_1) = P(x_0) + \sum_{k=1}^{n} \frac{1}{k!} \cdot P^{(k)}(x_0)(x_1 - x_0)^k + \mathcal{R}_n(x_0, x_1), \tag{7}$$

where

$$\begin{aligned}\mathcal{R}_n(x_0,x_1) \;:=\; &\int_0^1 (P^{(n)}(\theta x_1 + (1-\theta)x_0)(x_1 - x_0)^n \\ &- P^{(n)}(x_0)(x_1 - x_0)^n) \cdot \frac{(1-\theta)^{n-1}}{(n-1)!} \cdot d\theta,\end{aligned} \tag{8}$$

is a vector valued abstract Riemann integral. Call

$$\Delta(x_1, x_0) := P(x_1) - P(x_0) - \sum_{k=1}^{n} \frac{1}{k!} P^{(k)}(x_0)(x_1 - x_0)^k. \tag{9}$$

Notice that

$$\Delta(x_1, x_0) = \mathcal{R}_n(x_0, x_1). \tag{10}$$

It is clear that

$$\Delta(\bullet, x_0) = \mathcal{R}_n(x_0, \bullet) \in C(M,Y).$$

Consider also

$$\phi_n(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \cdot \frac{(|t| - s)^{n-1}}{(n-1)!} \cdot ds, \quad n \in \mathbb{N}, \quad t \in \mathbb{R}, \tag{11}$$

which is a continuous function in t. From [4], p. 210 we see that

$$\phi_n(\|x_1 - x_0\|)$$

$$\leq \left(\frac{\|x_1 - x_0\|^{n+1}}{(n+1)! \cdot h} + \frac{\|x_1 - x_0\|^n}{2 \cdot n!} + \frac{h \cdot \|x_1 - x_0\|^{n-1}}{8 \cdot (n-1)!} \right). \tag{12}$$

Obviously, $i \cdot \phi_n \cdot (\|x - x_0\|) \in C(M, Y)$, where $i \in Y^+$ such that $\|i\| = 1$.

We observe the following

$$\begin{aligned} \|\mathcal{R}_n(x_0, x_1)\| &= \Big\| \int_0^1 (P^{(n)}(\theta x_1 + (1-\theta)x_0) \\ &\quad - P^{(n)}(x_0))(x_1 - x_0)^n \cdot \frac{(1-\theta)^{n-1}}{(n-1)!} \cdot d\theta \Big\| \\ &\leq \int_0^1 \|(P^{(n)}(\theta x_1 + (1-\theta)x_0) \\ &\quad - P^{(n)}(x_0))(x_1 - x_0)^n\| \cdot \frac{(1-\theta)^{n-1}}{(n-1)!} \cdot d\theta \\ &\leq \int_0^1 \|(P^{(n)}(\theta x_1 + (1-\theta)x_0) \\ &\quad - P^{(n)}(x_0)\| \cdot \|x_1 - x_0\|^n \cdot \frac{(1-\theta)^{n-1}}{(n-1)!} \cdot d\theta \leq (*). \end{aligned}$$

Remember that $P^{(n)}(x)$, $x \in M$, is an n-linear bounded operator from X into Y.

We define $(h > 0)$

$$\begin{aligned} \omega_1(P^{(n)}, h) &:= \sup\{\|P^{(n)}(x) - P^{(n)}(y)\| : \\ &\qquad \text{all } x, y \in M \text{ such that } \|x - y\| \leq h\}. \end{aligned} \tag{13}$$

By Lemma 8 we obtain

$$\begin{aligned} (*) &\leq \int_0^1 \omega_1(P^{(n)}, h) \cdot \left\lceil \frac{\theta \cdot \|x_1 - x_0\|}{h} \right\rceil \cdot \|x_1 - x_0\|^n \cdot \frac{(1-\theta)^{n-1}}{(n-1)!} \cdot d\theta \\ &= \omega_1(P^{(n)}, h) \cdot \|x_1 - x_0\|^n \cdot \int_0^1 \left\lceil \frac{\theta \cdot \|x_1 - x_0\|}{h} \right\rceil \cdot \frac{(1-\theta)^{n-1}}{(n-1)!} \cdot d\theta \\ &= \omega_1(P^{(n)}, h) \cdot \phi_n(\|x_1 - x_0\|). \end{aligned}$$

I.e., we have proved that

$$\|\mathcal{R}_n(x_0, x_1)\| \leq \omega_1(P^{(n)}, h) \cdot \phi_n(\|x_1 - x_0\|). \tag{14}$$

Now by (12) we have

$$\|\mathcal{R}_n(x_0,x_1)\| \le \omega_1(P^{(n)},h)\cdot\left\{\frac{\|x_1-x_0\|^{n+1}}{(n+1)!\cdot h}+\frac{\|x_1-x_0\|^{n}}{2\cdot n!}+\frac{h\cdot\|x_1-x_0\|^{n-1}}{8\cdot(n-1)!}\right\}<+\infty, \tag{15}$$

for all x_0, $x_1 \in M$. Here $0<\omega_1(P^{(n)},h)<+\infty$, by M being compact and $P^{(n)}$ being continuous. Notice also that R.H.S. (15) is continuous in $x_1 \in M$. Thus

$$\|\Delta(x,x_0)\| \le \left\|\omega_1(P^{(n)},h)\cdot\left\{\frac{\|x-x_0\|^{n+1}}{(n+1)!\cdot h}+\frac{\|x-x_0\|^{n}}{2\cdot n!}+\frac{h\cdot\|x-x_0\|^{n-1}}{8\cdot(n-1)!}\right\}\cdot i\right\|, \tag{16}$$

all $x\in M$, $i\in Y^+$ such that $\|i\|=1$. Obviously the function within the long $\|\cdot\|$ in the R.H.S. (16) belongs to $C(M,Y)$. Hence

$$|||\Delta(\cdot,x_0)|||_\infty \le |||\omega_1(P^{(n)},h)\cdot\left\{\frac{\|id-x_0\|^{n+1}}{(n+1)!\cdot h}+\frac{\|id-x_0\|^{n}}{2\cdot n!}+\frac{h\cdot\|id-x_0\|^{n-1}}{8\cdot(n-1)!}\right\}\cdot i|||_\infty<+\infty. \tag{17}$$

Obviously $C^n(M,Y)\subset C(M,Y)$.

To remind, we are going to be dealing with lattice homomorphisms T from $C(M,Y)$ into itself. We need the following.

Proposition 9 *Let $T: C(M,Y)\to C(M,Y)$ be a continuous linear operator, $P: X\to Y$ and $P|_M \in C^n(M,Y)$ (in the Fréchet sense). Then $(T(P^{(k)}(x_0)(x-x_0)^k))(x_0)$ is a continuous function for any $x_0\in M$, where $(x-x_0)^k := (x-x_0,\dots,x-x_0)$, k-tuple; $1\le k\le n$, $x\in M$.*

Proof. Let x_n, $x_0\in M$ such that $x_n\to x_0$ as $n\to+\infty$. We need to prove that

$$(T(P^{(k)}(x_n)(x-x_n)^k))(x_n)\to(T(P^{(k)}(x_0)(x-x_0)^k))(x_0).$$

For that we observe

$$\begin{aligned}&\|(T(P^{(k)}(x_n)(x-x_n)^k))(x_n)-(T(P^{(k)}(x_0)(x-x_0)^k))(x_0)\|\\ &\quad\le\|(T(P^{(k)}(x_n)(x-x_n)^k))(x_n)-(T(P^{(k)}(x_0)(x-x_0)^k))(x_n)\|\end{aligned}$$

$$+ \|(T(P^{(k)}(x_0)(x-x_0)^k))(x_n) - (T(P^{(k)}(x_0)(x-x_0)^k))(x_0)\| =: (*).$$

Here $j(x) := (x-x_0)^k$ is continuous in $x \in X$ and the continuous linear operator $P^{(k)}(x_0)$ maps X^k into Y, i.e., $P^{(k)}(x_0)(x-x_0)^k \in Y$. Notice that

$$(P^{(k)}(x_0) \circ j)(x) = P^{(k)}(x_0)(x-x_0)^k \text{ is continuous in } x \in X.$$

I.e., $P^{(k)}(x_0)(x-x_0)^k := P^{(k)}(x_0)(x-x_0)^k|_M \in C(M,Y)$, therefore

$$T(P^{(k)}(x_0)(x-x_0)^k) \in C(M,Y).$$

Thus for arbitrarily small $\varepsilon_1 > 0$ we get that

$$\|(T(P^{(k)}(x_0)(x-x_0)^k))(x_n) - (T(P^{(k)}(x_0)(x-x_0)^k))(x_0)\| < \varepsilon_1.$$

Hence

$$(*) < \||(T(P^{(k)}(x_n)(x-x_n)^k)) - (T(P^{(k)}(x_0)(x-x_0)^k))\||_\infty + \varepsilon_1. \tag{18}$$

We observe that

$$\begin{aligned}&\||P^{(k)}(x_n)(x-x_n)^k - P^{(k)}(x_0)(x-x_0)^k\||_\infty \\ &\quad \le \||P^{(k)}(x_n)(x-x_n)^k - P^{(k)}(x_n)(x-x_0)^k\||_\infty \\ &\qquad + \||P^{(k)}(x_n)(x-x_0)^k - P^{(k)}(x_0)(x-x_0)^k\||_\infty =: A + B.\end{aligned}$$

By assumption $P^{(k)}$ maps M into the space of k-linear bounded operators from X^k into Y and $P^{(k)}$ is assumed to be continuous. I.e., $P^{(k)}(x_n) \to P^{(k)}(x_0)$ as $x_n \to x_0$, that is, $\|P^{(k)}(x_n) - P^{(k)}(x_0)\| \to 0$. Also it holds $\|x - x_0\| < \text{diameter}(M) =: d(M) < +\infty$. Consequently we obtain that

$$\begin{aligned}&\|P^{(k)}(x_n)(x-x_0)^k - P^{(k)}(x_0)(x-x_0)^k\| \\ &\quad = \|(P^{(k)}(x_n) - P^{(k)}(x_0))(x-x_0)^k\| \\ &\quad \le \|P^{(k)}(x_n) - P^{(k)}(x_0)\| \cdot \|x-x_0\|^k \\ &\quad \le \|P^{(k)}(x_n) - P^{(k)}(x_0)\| \cdot (d(M))^k \to 0.\end{aligned}$$

That is for arbitrarily small $\varepsilon > 0$ we have

$$B := \||P^{(k)}(x_n)(x-x_0)^k - P^{(k)}(x_0)(x-x_0)^k\||_\infty < \varepsilon,$$

for all $n > n_0 \in \mathbb{N}$. By assumption we have that $\|P^{(k)}(x)\| < \gamma < +\infty$, for all $x \in M$. Therefore,

$$\begin{aligned}
&\|P^{(k)}(x_n)(x-x_n)^k - P^{(k)}(x_n)(x-x_0)^k\| \\
&\quad = \|P^{(k)}(x_n)((x-x_n)^k - (x-x_0)^k)\| = \|P^{(k)}(x_n)(x_0-x_n)^k\| \\
&\quad \le \|P^{(k)}(x_n)\| \cdot \|x_n - x_0\|^k < \gamma \cdot \|x_n - x_0\|^k \to 0.
\end{aligned}$$

Finally, one can get that

$$A := |||P^{(k)}(x_n)(x-x_n)^k - P^{(k)}(x_n)(x-x_0)^k|||_\infty \le \varepsilon,$$

for all $n > n_1 \in \mathbb{N}$. And

$$|||P^{(k)}(x_n)(x-x_n)^k - P^{(k)}(x_0)(x-x_0)^k|||_\infty \le 2\varepsilon,$$

for all $n > \max(n_0, n_1)$, i.e.,

$$P^{(k)}(x_n)(x-x_n)^k \xrightarrow{u} P^{(k)}(x_0)(x-x_0)^k,$$

uniformly. Since T is continuous, we get that

$$T(P^{(k)}(x_n)(x-x_n)^k) \xrightarrow{u} T(P^{(k)}(x_0)(x-x_0)^k),$$

uniformly. So for sufficiently large n we obtain

$$|||T(P^{(k)}(x_n)(x-x_n)^k) - T(P^{(k)}(x_0)(x-x_0)^k)|||_\infty \le \varepsilon_1.$$

Consequently from (18) we get $(*) \le 2\varepsilon_1$. Thus

$$\|(T(P^{(k)}(x_n)(x-x_n)^k))(x_n) - (T(P^{(k)}(x_0)(x-x_0)^k))(x_0)\| \to 0,$$

as $x_n \to x_0$. We have established the claim of the proposition. □

3. Main Results

Next comes our first main result

Theorem 10 *Let M be a compact and convex subset of $(X, \|\cdot\|)$ and $(Y, \|\cdot\|, <)$ is a Banach lattice. Let T be a lattice homomorphism from $C(M,Y)$ into itself, and $f \in C(M,Y)$. Then*

$$\begin{aligned}
|||Tf - f|||_\infty \le\ & |||((T(f(x_0))) - f)(x_0)|||_{\infty,x_0} \\
& + \omega_1(f, |||(T(\|x-x_0\| \cdot i))(x_0)|||_{\infty,x_0}) \\
& \cdot (1 + |||T(i)|||_\infty),
\end{aligned} \tag{19}$$

where $|||\cdot|||_\infty$, $|||\cdot|||_{\infty,x_0}$ *are the supremum norms taken over* M *and over all* $x_0 \in M$*, respectively, and* $i \in Y^+$ *is such that* $\|i\| = 1$.

Remark 11 Notice that R.H.S. (19) is finite. This comes from the definition of T, M being compact, $f \in C(M, Y)$, Proposition 4 and Proposition 6.

Proof of Theorem 10. We observe that

$$\begin{aligned}(Tf)(x_0) - f(x_0) &\\ &= (Tf)(x_0) - (T(f(x_0))(x_0) + (T(f(x_0)))(x_0) - f(x_0)\\ &= [(T(f - f(x_0)))(x_0)] + [(T(f(x_0)))(x_0) - f(x_0)].\end{aligned}$$

Thus

$$\begin{aligned}\|(Tf)(x_0) - f(x_0)\| &\\ &\le \|(T(f - f(x_0)))(x_0)\| + \|(T(f(x_0)))(x_0) - f(x_0)\|.\end{aligned}$$

Hence

$$\begin{aligned}|||Tf - f|||_\infty \le\ & |||(T(f - f(x_0)))(x_0)|||_{\infty,x_0}\\ &+ |||(T(f(x_0)))(x_0) - f(x_0)|||_{\infty,x_0}.\end{aligned} \tag{20}$$

From Remark 5 we have that $(T(f - f(x_0)))(x_0)$ is a continuous function for any $x_0 \in M$ with values in Y, therefore its supremum norm is finite.

We notice also that $(h > 0)$

$$\begin{aligned}\left(T\left(\left(1 + \frac{\|x - x_0\|}{h}\right)\cdot i\right)\right)(x_0) &\\ &= (T(i))(x_0) + \frac{1}{h}\cdot(T(\|x - x_0\|\cdot i))(x_0))\end{aligned}$$

is a continuous function for any $x_0 \in M$. By Lemma 8 we get that

$$\|f(x) - f(x_0)\| \le \omega_1(f, h)\cdot\left(1 + \frac{\|x - x_0\|}{h}\right), \quad \text{all } x, x_0 \in M.$$

I.e.,

$$\|f(x) - f(x_0)\| \le \left\|\omega_1(f, h)\cdot\left(1 + \frac{\|x - x_0\|}{h}\right)\cdot i\right\|,$$

iff

$$|f(x) - f(x_0)| \le \left|\omega_1(f, h)\cdot\left(1 + \frac{\|x - x_0\|}{h}\right)\cdot i\right|$$

iff

$$|f - f(x_0)|(x) \le \left|\omega_1(f,h) \cdot \left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\right|(x), \quad \text{all } x \in M.$$

Hence by the positivity of T and being a lattice homomorphism we obtain

$$T|f - f(x_0)| \le T\left|\omega_1(f,h) \cdot \left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\right|$$

and

$$|T(f - f(x_0))| \le \left|T\left(\omega_1(f,h) \cdot \left(1 + \frac{\|id - x_0\|}{h}\right) i\right)\right|,$$

iff

$$|T(f - f(x_0))|(x) \le \left|T\left(\omega_1(f,h) \cdot \left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\right)\right|(x), \quad \text{all } x \in M.$$

Furthermore we have

$$|(T(f - f(x_0)))(x)| \le \left|\left(T\left(\omega_1(f,h) \cdot \left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\right)\right)(x)\right|, \quad \text{all } x \in M.$$

Since Lemma 8 is true for any $x_0 \in M$, we get that

$$|(T(f - f(x_0)))(x_0)| \le \left|\left(T\left(\omega_1(f,h) \cdot \left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\right)\right)(x_0)\right|,$$

true for any $x_0 \in M$. Ans since Y is a normed vector lattice, the last inequality implies

$$\|(T(f - f(x_0)))(x_0)\| \le \omega_1(f,h) \cdot \left\|\left(T\left(\left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\right)\right)(x_0)\right\|,$$

which in turn implies

$$|||(T(f - f(x_0)))(x_0)|||_{\infty,x_0} \le \omega_1(f,h) \cdot |||\left(T\left(\left(1 + \frac{\|id - x_0\|}{h}\right) \cdot i\right)\right)(x_0)|||_{\infty,x_0}$$

$$= \omega_1(f,h) \cdot |||(T(i))(x_0) + \frac{1}{h} \cdot (T(\|id - x_0\| \cdot i))(x_0)|||_{\infty,x_0}$$
$$\leq \omega_1(f,h) \cdot \left(|||T(i)|||_\infty + \frac{1}{h} \cdot |||(T(\|id - x_0\| \cdot i))(x_0)|||_{\infty,x_0} \right).$$

Picking

$$h := |||(T(\|id - x_0\| \cdot i))(x_0)|||_{\infty,x_0}, \tag{21}$$

we find that

$$|||(T(f - f(x_0)))(x_0)|||_{\infty,x_0}$$
$$\leq \omega_1(f, |||(T(\|id - x_0\| \cdot i))(x_0)|||_{\infty,x_0}) \cdot (1 + |||T(i)|||_\infty). \tag{22}$$

It is clear now that inequalities (20) and (22) imply inequality (19). The theorem is now proved. □

In the following we give our second main result.

Theorem 12 *Let X be a Banach space, Y be a Banach lattice, M be a compact and convex subset of X, $h > 0$, and T is a lattice homomorphism from $C(M,Y)$ into itself. Consider a function P from X into Y such that $P|_M \in C^n(M,Y)$, $n \in \mathbb{N}$. Then*

$$|||TP - P|||_\infty$$
$$\leq |||(T(P(x_0)))(x_0) - P(x_0)|||_{\infty,x_0}$$
$$+ \sum_{k=1}^{n} \frac{1}{k!} \cdot |||(T(P^{(k)}(x_0)(x - x_0)^k))(x_0)|||_{\infty,x_0}$$
$$+ \omega_1(P^{(n)}, h) \cdot \left[\frac{1}{(n+1)! \cdot h} \cdot |||(T(\|x - x_0\|^{n+1} i))(x_0)|||_{\infty,x_0} \right.$$
$$+ \frac{1}{2 \cdot n!} \cdot |||(T(\|x - x_0\|^n \cdot i))(x_0)|||_{\infty,x_0}$$
$$\left. + \frac{h}{8 \cdot (n-1)!} \cdot |||(T(\|x - x_0\|^{n-1}, i))(x_0)|||_{\infty,x_0} \right]. \tag{23}$$

Here $i \in Y^+$ is such that $\|i\| = 1$, and $||| \cdot |||_\infty$, $||| \cdot |||_{\infty,x_0}$ are the supremum norms taken over M and over all $x_0 \in M$, respectively.

Remark 13 Observe that R.H.S. (23) is finite. This comes from the definition of T, M being compact, $P|_M \in C^n(M,Y)$, Proposition 4, Proposition 6 and Proposition 9.

Corollary 14 (Same setting and assumptions as in Theorem 12.) *Choose*

$$h := h^* := \frac{1}{(n+1)!} \cdot \max\Big\{ |||(T(\|x-x_0\|^{n+1} \cdot i))(x_0)|||_{\infty,x_0}, \\ |||(T(\|x-x_0\|^{n} \cdot i))(x_0)|||_{\infty,x_0}, \\ |||(T(\|x-x_0\|^{n-1} \cdot i))(x_0)|||_{\infty,x_0} \Big\}. \tag{24}$$

Then

$$\begin{aligned} &|||TP - P|||_\infty \\ &\quad \le |||(T(P(x_0)))(x_0) - P(x_0)|||_{\infty,x_0} \\ &\quad + \sum_{k=1}^{n} \frac{1}{k!} \cdot |||(T(P^{(k)}(x_0)(x-x_0)^k))(x_0)|||_{\infty,x_0} \\ &\quad + \omega_1(P^{(n)}, h^*) \cdot \left[1 + \left(\frac{n+1}{2} \right) \cdot h^* + \frac{n \cdot (n+1)}{8} \cdot h^{*2} \right]. \end{aligned} \tag{25}$$

Proof. Notice that

$$\psi := \frac{1}{(n+1)!} \cdot |||(T(\|x-x_0\|^{n+1} \cdot i))(x_0)|||_{\infty,x_0} \le h^*,$$

i.e., $\frac{\psi}{h^*} \le 1$. Also we see that

$$\frac{1}{2 \cdot n!} \cdot |||(T(\|x-x_0\|^{n} \cdot i))(x_0)|||_{\infty,x_0} \le \frac{(n+1)}{2} \cdot h^*,$$

and

$$\begin{aligned} &\frac{h^*}{8 \cdot (n-1)!} \cdot |||(T(\|x-x_0\|^{n-1} \cdot i))(x_0)|||_{\infty,x_0} \\ &\qquad \le \frac{(h^*)^2}{8} \cdot n \cdot (n+1). \end{aligned}$$

Therefore

$$\begin{aligned} \text{Remainder (23)} \le\ &\omega_1(P^{(n)}, h^*) \\ &\cdot \left[1 + \left(\frac{n+1}{2} \right) \cdot h^* + \frac{n \cdot (n+1)}{8} \cdot (h^*)^2 \right]. \end{aligned}$$

□

Proof of Theorem 12. From (7) we get on M that

$$P(\bullet) = P(x_0) + \sum_{k=1}^{n} \frac{1}{k!} \cdot P^{(k)}(x_0)(\bullet - x_0)^k + \mathcal{R}_n(x_0, \bullet), \tag{26}$$

where $x_0 \in M$. Thus

$$\begin{aligned} &(T(P))(x_0) - (T(P(x_0)))(x_0) - \sum_{k=1}^{n} \frac{1}{k!} \cdot (T(P^{(k)}(x_0)(id - x_0)^k))(x_0) \\ &\quad = (T(\mathcal{R}_n(x_0, \cdot)))(x_0) \in C(M, Y), \end{aligned} \tag{27}$$

by Proposition 4, 9. Also we find that

$$\begin{aligned} &(TP)(x_0) - P(x_0) \\ &\quad = (TP)(x_0) - (T(P(x_0)))(x_0) + (T(P(x_0)))(x_0) - P(x_0) \\ &\quad = [(T(P - P(x_0)))(x_0)] + [(T(P(x_0)))(x_0) - P(x_0)]. \end{aligned}$$

Thus

$$\begin{aligned} \|(TP)(x_0) - P(x_0)\| \le\ & \|(T(P - P(x_0)))(x_0)\| \\ & + \|(T(P(x_0)))(x_0) - P(x_0)\|, \end{aligned} \tag{28}$$

and so we find that

$$\begin{aligned} |||TP - P|||_\infty \le\ & |||(T(P - P(x_0)))(x_0)|||_{\infty, x_0} \\ & + |||(T(P(x_0)))(x_0) - P(x_0)|||_{\infty, x_0}. \end{aligned} \tag{29}$$

Notice that all functions involved in (26) belong to $C(M, Y)$, which is a normed vector lattice.

Consequently,

$$\begin{aligned} &|(T(P - P(x_0)))(x)| \\ &\quad = \left| \left(T \left(\sum_{k=1}^{n} \frac{1}{k!} \cdot P^{(k)}(x_0)(id - x_0)^k + \mathcal{R}_n(x_0, x) \right) \right) (x) \right|, \end{aligned} \tag{30}$$

true for all $x \in M$, for arbitrary $x_0 \in M$. I.e., we get that

$$\begin{aligned} &|(T(P - P(x_0)))(x_0)| \\ &\quad = \left| \left(T \left(\sum_{k=1}^{n} \frac{1}{k!} \cdot P^{(k)}(x_0)(id - x_0)^k + \mathcal{R}_n(x_0, x) \right) \right) (x_0) \right|, \end{aligned} \tag{31}$$

is true for any $x_0 \in M$. Since Y is a normed vector lattice the last implies that

$$\begin{aligned}
&\|(T(P - P(x_0)))(x_0)\| \\
&= \left\| \left(T \left(\sum_{k=1}^{n} \frac{1}{k!} \cdot P^{(k)}(x_0)(id - z_0)^k + \mathcal{R}_n(x_0, x) \right) \right) (x_0) \right\|, \qquad (32)
\end{aligned}$$

And

$$\begin{aligned}
&|||(T(P - P(x_0)))(x_0)|||_{\infty,x_0} \\
&= ||| \left(T \left(\sum_{k=1}^{n} \frac{1}{k!} \cdot P^{(k)}(x_0)(id - x_0)^k + \mathcal{R}_n(x_0, x) \right) \right) (x_0)|||_{\infty,x_0} \\
&\leq \sum_{k=1}^{n} \frac{1}{k!} \cdot |||(T(P^{(k)}(x_0)(id - x_0)^k))(x_0)|||_{\infty,x_0} \\
&\quad + |||(T(\mathcal{R}_n(x_0, x)))(x_0)|||_{\infty,x_0}. \qquad (33)
\end{aligned}$$

From (15) we obtain

$$\begin{aligned}
&|||\mathcal{R}_n(x_0, x)|||_\infty \\
&\leq |||i \cdot \omega_1(P^{(n)}, h) \cdot \left\{ \frac{\|x - x_0\|^{n+1}}{(n+1)! \cdot h} + \frac{\|x - x_0\|^n}{2 \cdot n!} \right. \\
&\quad \left. + \frac{h \cdot \|x - x_0\|^{n-1}}{8 \cdot (n-1)!} \right\} |||_\infty < +\infty, \qquad (34)
\end{aligned}$$

true for arbitrary $x_0 \in M$. Since $C(M, Y)$ is a Banach lattice by (15) again we have

$$\begin{aligned}
|\mathcal{R}_n(x_0, x)| \leq & \left| \left\{ \frac{\|x - x_0\|^{n+1}}{(n+1)! \cdot h} + \frac{\|x - x_0\|^n}{2 \cdot n!} \right. \right. \\
& \left. \left. + \frac{h \cdot \|x - x_0\|^{n-1}}{8 \cdot (n-1)!} \right\} \cdot \omega_1(P^{(n)}, h) \cdot i \right| =: |\varphi|. \qquad (35)
\end{aligned}$$

That is, by positivity of T we have

$$T|\mathcal{R}_n(x_0, x)| \leq T|\varphi|, \qquad (36)$$

i.e.,

$$(T|\mathcal{R}_n(x_0, x)|)(x) \leq (T|\varphi|)(x), \qquad (37)$$

for all $x \in M$, for any $x_0 \in M$.

Since T is a lattice homomorphism we get

$$|T(\mathcal{R}_n(x_0,x))|(x) \le |T(\varphi)|(x), \tag{38}$$

true for all $x \in M$, for any $x_0 \in M$. That is,

$$|T(\mathcal{R}_n(x_0,x))|(x_0) \le |T(\varphi)|(x_0), \tag{39}$$

for any $x_0 \in M$. And

$$|(T(\mathcal{R}_n(x_0,x)))(x_0)| \le |(T(\varphi))(x_0)|, \tag{40}$$

for any $x_0 \in M$. Since $C(M,Y)$ is a normed vector lattice, and both sides of inequality (40) in $|\bullet|$'s belong to $C(M,Y)$ (for the last statement see Propositions 4, 6, 9 and the definition of T) we obtain

$$|||(T(\mathcal{R}_n(x_0,x)))(x_0)|||_{\infty,x_0} \le |||(T(\varphi))(x_0)|||_{\infty,x_0}. \tag{41}$$

I.e.,

$$\begin{aligned}&|||(T(\mathcal{R}_n(x_0,x)))(x_0)|||_{\infty,x_0}\\ &\quad\le |||\Big(T\Big(\Big\{\frac{\|x-x_0\|^{n+1}}{(n+1)!\cdot h}+\frac{\|x-x_0\|^{n}}{2\cdot n!}+\frac{h\cdot\|x-x_0\|^{n-1}}{8\cdot(n-1)!}\Big\}\\ &\qquad\cdot\omega_1(P^{(n)},h)\cdot i\Big)\Big)(x_0)|||_{\infty,x_0}.\end{aligned} \tag{42}$$

That is,

$$\begin{aligned}&|||(T(\mathcal{R}_n(x_0,x)))(x_0)|||_{\infty,x_0}\\ &\quad\le \omega_1(P^{(n)},h)\cdot\Big[\frac{1}{(n+1)!\cdot h}\cdot|||(T((\|x-x_0\|^{n+1})\cdot i))(x_0)|||_{\infty,x_0}\\ &\qquad+\frac{1}{2\cdot n!}\cdot|||(T((\|x-x_0\|^{n})\cdot i))(x_0)|||_{\infty,x_0}\\ &\qquad+\frac{h}{8\cdot(n-1)!}\cdot|||(T((\|x-x_0\|^{n-1})\cdot i))(x_0)|||_{\infty,x_0}\Big]\\ &\quad=:\lambda<+\infty.\end{aligned} \tag{43}$$

The last quantity λ is finite by Proposition 6.

Next, from (33) and (43), we find that

$$\begin{aligned}&|||(T(P-P(x_0)))(x_0)|||_{\infty,x_0}\\ &\quad\le\sum_{k=1}^{n}\frac{1}{k!}\cdot|||(T(P^{(k)}(x_0)(x-x_0)^k))(x_0)|||_{\infty,x_0}+\lambda.\end{aligned} \tag{44}$$

Finally, inequality (23) follows from inequalities (29) and (44). The proof of Theorem 12 now has been completed. □

4. Further Discussion

Here the whole setting will be the same as introduced in §2. We would like to consider further the vector space

$$\begin{aligned} B(M,Y) := \{F : M \to Y \mid \exists \theta_F > 0 : \\ \|F(x)\| \leq \theta_F,\ \forall x \in M\}, \end{aligned} \tag{45}$$

which is the space of normed bounded functions on M. Obviously here the associated norm $|||\cdot|||_\infty$ is again a lattice norm. The completeness of $B(M,Y)$ is established in a similar manner as in Proposition 2. That is $B(M;Y)$ is a Banach lattice.

For the last we still need to prove that

Lemma 15 *$B(M,Y)$ is a vector lattice.*

Proof. Let $f, g \in B(M,Y)$, it is enough to prove that $f \vee g$, $f \wedge g \in B(M,Y)$. There exists $M^* > 0$ such that $\|f(x)\| < M^*$, $\|g(x)\| < M^*$, for all $x \in M$. That is

$$\begin{aligned} \pm f(x) &< |f(x)| < M^* \cdot i, \\ \pm g(x) &< |g(x)| < M^* \cdot i, \end{aligned}$$

where $i \in Y^+$ such that $\|i\| = 1$. Here Y^+ is the positive cone of Y.

i) Hence

$$f(x) \vee g(x) < M^* \cdot i$$

and

$$-(f(x) \vee g(x)) = (-f(x)) \wedge (-g(x)) < M^* \cdot i.$$

Hence

$$(f \vee g)(x) < M^* \cdot i$$

and

$$-(f \vee g)(x) < M^* \cdot i,$$

imply

$$|(f \vee g)(x)| < M^* \cdot i.$$

And since Y is a Banach lattice we get that

$$\|(f \vee g)(x)\| < M^*, \quad \text{all} \ \ x \in M.$$

That is,

$$|||f \vee g|||_\infty < M^*,$$

i.e., $f \vee g \in B(M,Y)$.

ii) Obviously

$$f(x) \wedge g(x) < M^* \cdot i$$

and

$$-(f(x) \wedge g(x)) = (-f(x)) \vee (-g(x)) < M^* \cdot i.$$

Hence

$$(f \wedge g)(x) < M^* \cdot i$$

and

$$-(f \wedge g)(x) < M^* \cdot i,$$

imply that

$$|(f \wedge g)(x)| < M^* \cdot i.$$

That is,

$$\|(f \wedge g)(x)\| < M^*, \quad \text{all} \ \ x \in M,$$

and so $|||f \wedge g|||_\infty < M^*$. I.e., $f \wedge g \in B(M,Y)$. □

Here we would like to consider *lattice homomorphisms* T from $C(M,Y)$ into $B(M,Y)$ and produce similar results as in §3. Obviously such a T is a *positive operator*, and since $C(M,Y)$ is a Banach lattice we get that T is *continuous*.

To proceed we need the following auxiliary results.

Lemma 16 *Let $f \in C(M,Y)$ and the lattice homomorphism T :*

$C(M,Y) \to B(M,Y)$. Then

$$(T(f(x_0)))(x_0) \in B(M,Y),$$

as a function of $x_0 \subset M$.

Proof. Since $f \in C(M,Y)$ and M is a compact subset of X we get for any $x_0 \in M$ that

$$\|f(x_0)\| \leq \theta_f = \|\theta_f \cdot i\| < +\infty,$$

where $i \in Y^+$ such that $\|i\| = 1$.

Equivalently we have

$$|f(x_0)| \leq |\theta_f \cdot i|.$$

Call $\varphi(x) := f(x_0)$, $\rho(x) := \theta_f \cdot i$, all $x \in M$. I.e., $|\varphi(x)| \leq |\rho(x)|$, that is,

$$|\varphi|(x) \leq |\rho|(x), \quad \text{all} \;\; x \in M,$$

iff

$$|\varphi| \leq |\rho|.$$

Since T is a positive operator we get

$$T|\varphi| \leq T|\rho|.$$

And because T is a lattice homomorphism, we have

$$|T(\varphi)| \leq |T(\rho)|,$$

and

$$|T(\varphi)|(x) \leq |T(\rho)|(x), \quad \text{all} \;\; x \in M.$$

In particular, it holds

$$|(T(\varphi))(x_0)| \leq |(T(\rho))(x_0)|.$$

Since Y is a Banach lattice we get that

$$\|(T(\varphi))(x_0)\| \leq \|(T(\rho))(x_0)\|.$$

I.e.,

$$\|(T(f(x_0)))(x_0)\| \;\leq\; \|(T(\theta_f \cdot i))(x_0)\|$$

$$= \theta_f \cdot \|(T(i))(x_0)\|$$
$$\leq \theta_f \cdot \||T(i)\||_\infty =: \Omega_{f,T} < +\infty.$$

That is,

$$\|(T(f(x_0)))(x_0)\| \leq \Omega_{f,T} < +\infty,$$

for any arbitrary $x_0 \in M$. We have established that $(T(f(\cdot)))(\cdot) \in B(M,Y)$. □

Obviously $(T(f - f(x_0)))(x_0) \in B(M,Y)$.

Lemma 17 *Let $r > 0$, $i \in Y^+$ such that $\|i\| = 1$, $T : C(M,Y) \to B(M,Y)$ a lattice homomorphism and $x_0 \in M$ be arbitrary. Then $(T(\|x - x_0\|^r \cdot i))(x_0) \in B(M,Y)$ as a function of x_0.*

Proof. For any $x, x_0 \in M$, by the compactness of M we get that

$$\|x - x_0\|^r \leq \ell, \quad \text{for some } \ \ell > 0.$$

Here $\|x - x_0\|^r \cdot i \in C(M,Y)$, thus

$$\||\,\|x - x_0\|^r \cdot i\|_{\infty,x} \leq \ell,$$

and

$$|\|x - x_0\|^r \cdot i| \leq |\ell \cdot i|.$$

Hence

$$T|\|x - x_0\|^r \cdot i| \leq T|\ell \cdot i|$$

and

$$|(T(\|x - x_0\|^r \cdot i))| \leq |(T(\ell \cdot i))|.$$

Furthermore,

$$|(T(\|x - x_0\|^r \cdot i))|(x) \leq |T(\ell i)|(x), \quad \text{all } \ x \in M.$$

In particular

$$|(T(\|x - x_0\|^r \cdot i))|(x_0) \leq |T(\ell i)|(x_0),$$

for any arbitrary $x_0 \in M$. That is,

$$|(T(\|x - x_0\|^r \cdot i))(x_0)| \leq |(T(\ell i))(x_0)|,$$

for any $x_0 \in M$. The last implies that

$$\begin{aligned} \|(T(\|x-x_0\|^r \cdot i))(x_0)\| &\le \|(T(\ell i))(x_0)\| \\ = \ell \cdot \|(T(i))(x_0)\| &\le \ell \cdot |||T(i)|||_\infty < \theta_{\ell,T} < +\infty, \end{aligned}$$

for some constant $\theta_{\ell,T} > 0$. We have proved that

$$|||(T(\|x-x_0\|^r \cdot i))(x_0)|||_{\infty,x_0} < +\infty,$$

i.e.,

$$(T(\|x-x_0\|^r \cdot i))(x_0) \in B(M,Y)$$

as a function of $x_0 \in M$. □

The last result we need here is

Lemma 18 *Let X be a Banach space, Y a Banach lattice, $P : X \to Y$ such that $P|_M \in C^n(M,Y)$, $n \in \mathbb{N}$, where M is a convex and compact subset of X. Let T be a lattice homomorphism from $C(M,Y)$ into $B(M,Y)$. Here $k = 1,\dots,n$. Then*

$$(T(P^{(k)}(x_0)(x-x_0)^k))(x_0) \in B(M,Y)$$

as a function of $x_0 \in M$.

Proof. We have that

$$P^{(k)}(x_0)(x-x_0)^k \in Y, \quad \text{for all } x \in X,$$

and $P^{(k)}(x_0)(x-x_0)^k$ is continuous in $x \in X$.

Since $P^{(k)}(x_0)$ is bounded k-linear operator and $P|_M \in C^n(M,Y)$ we get that

$$\|P^{(k)}(x_0)(x-x_0)^k\| \le \|P^{(k)}(x_0)\| \cdot \|x-x_0\|^k < +\infty,$$

i.e., there exists $\mathcal{D}_P > 0$ such that

$$\|P^{(k)}(x_0)(x-x_0)^k\| \le \mathcal{D}_P < +\infty,$$

for all $x \in M$ and any $x_0 \in M$. That is,

$$\|P^{(k)}(x_0)(x-x_0)^k\| \le \|\mathcal{D}_P \cdot i\|,$$

where $i \in Y^+$ such that $\|i\| = 1$. Equivalently

$$|P^{(k)}(x_0)(x-x_0)^k| \le |\mathcal{D}_P \cdot i|.$$

Since T is a lattice homomorphism we get

$$T|P^{(k)}(x_0)(x-x_0)^k| \le T|\mathcal{D}_P \cdot i|,$$

i.e.,

$$(T|P^{(k)}(x_0)(x-x_0)^k|)(x) \le (T|\mathcal{D}_P \cdot i|)(x), \quad \text{all } \ x \in M.$$

Hence

$$|T(P^{(k)}(x_0)(x-x_0)^k)|(x) \le |T(\mathcal{D}_P \cdot i)|(x),$$

and

$$|(T(P^{(k)}(x_0)(x-x_0)^k))(x)| \le |(T(\mathcal{D}_P \cdot i))(x)|,$$

all $x \in M$, and for any $x_0 \in M$.

In particular it holds

$$|(T(P^{(k)}(x_0)(x-x_0)^k))(x_0)| \le |(T(\mathcal{D}_P \cdot i))(x_0)|,$$

for any arbitrary $x_0 \in M$. Therefore,

$$\|(T(P^{(k)}(x_0)(x-x_0)^k))(x_0)\| \le \|(T(\mathcal{D}_P \cdot i))(x_0)\|,$$

for any $x_0 \in M$. Consequently,

$$|||(T(P^{(k)}(x_0)(x-x_0)^k))(x_0)|||_{\infty,x_0} \\ \le \zeta_{P,T} := \mathcal{D}_P \cdot |||T(i)|||_\infty < +\infty.$$

We have established that $(T(P^{(k)}(x_0)(x-x_0)^k))(x_0) \in B(M,Y)$ as a function of $x_0 \in M$, all $k = 1, \ldots, n$. □

Next one can prove again in exactly the same way inequalities (19), (23) and (25) of Theorem 10, 12 and of Corollary 14, respectively, within the same settings — *except that now T is a lattice homomorphism from $C(M,Y)$ into $B(M,Y)$.*

The valid inequalities (19), (23), (25), under the extended T as above, have again finite right-hand sides. The last is justified by the use of Lemmas 16, 17, 18.

5. Examples

Next we prove that the set of lattice homomorphisms where our theory can be applied is not an empty one.

1) Let $\tau : Y \to Y$ be a lattice homomorphism and $f \in C(M,Y)$, then $F : C(M,Y) \to C(M,Y)$ defined by $F(f) := \tau \circ f$ is a lattice homomorphism.

2) Let $\varphi : \mathbb{N} \to \mathbb{R}_+ - \{0\}$ be such that $\varphi(n) \to 1$ as $n \to +\infty$ (e.g., $\varphi(n) = 1 + \frac{1}{n}$). Let $f \in C(M,Y)$, then $T_n : C(M,Y) \to C(M,Y)$ defined by $(T_n f)(x) := \varphi(n) \cdot f(x)$, all $x \in M$ is a lattice homomorphism. Furthermore $T_n \to I$-unit operator, as $n \to +\infty$.

3) Let $\gamma_n : M \to \mathbb{R}_+ - \{0\}$ be such that $\|\gamma_n\|_\infty \leq \alpha_n$, $\alpha_n > 0$, $\forall n \in \mathbb{N}$ and $\lim_{n\to+\infty} \gamma_n = 1$, uniformly (e.g., $\gamma_n(x) := e^{-\|x\|/n}$). Let $f \in C(M,Y)$, define $(T_n f)(x) := \gamma_n(x) \cdot f(x)$, all $x \in M$. Then T_n determines a lattice homomorphism from $C(M,Y)$ into $B(M,Y)$ such that $T_n \to I$.

4) Let T_n, $n \in \mathbb{N}$, be a positive linear operator from $C(M,Y)$ into itself such that $T_n \to 0$. Assume that T_n is an *orthomorphism* (see [2], p. 109). Then (by Exercise 2, p. 124 of [2]) $E_n := I + T_n$ *is a lattice homomorphism*, where I is the unit operator. Our theory (§3) when applied to E_n gives the convergence of $T_n \to 0$ with rates.

5) Let $\alpha_n > 0$ be such that $\alpha_n \to 1$ as $n \to +\infty$. Let $f \in C(X,Y)$ and define $(T_n f)(x) := f(\alpha_n x)$, all $x \in X$. Then T_n is a lattice homomorphism from $C(X,Y)$ into itself such that $T_n f \to f$ pointwise, all $f \in C(X,Y)$.

6) Let $0 < \alpha_n \to 0$ as $n \to +\infty$ and $j \in X$ such that $\|j\| = 1$. Let $f \in C(X,Y)$ and define

$$(T_n f)(x) := f(x + \alpha_n \cdot j) \quad (\to f(x)),$$

all $x \in X$. Then T_n is a lattice homomorphism from $C(X,Y)$ into itself such that $T_n f \to f$ pointwise, all $f \in C(X,Y)$. E.T.C.

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