Hokkaido Mathematical Journal Vol. 26 (1997) p. 297–322

Some applications of pseudo-differential operators to elasticity

Alexander KOZHEVNIKOV and Tatiana SKUBACHEVSKAYA

(Received July 17, 1995; Revised December 17, 1996)

Abstract. The paper deals with four basic boundary value problem of static elasticity (BPET). It was calculated the principal symbol of a pseudo-differential operator on the boundary whose eigenvalues are the Cosserat eigenvalues of the original BPET. This principal symbol is presented in terms of the principal curvatures and the coefficients of the first quadratic form of the boundary. It was found the principal term in the asympotics of the Cosserat eigenvalues.

Key words: elasticity, isotropic and homogeneous elastic body, Lamé equation, boundary value problems, Poisson constant, pseudo-differential operators, principal symbol, Cosserat spectrum, asymptotics.

Introduction

This paper deals with four basic boundary value problems of static elasticity theory; from now on, we shall call them BPETs. The main tool in our investigation is the calculus of pseudo-differential operators (ΨDO). These methods have been used over the last few years by various authors for investigation both BPET and Stokes problems [3], [6], [7], [11], [12], [13].

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with an infinitely smooth boundary Γ and let an isotropic, homogeneous elastic body fill Ω. It is well-known that the vector of displacement $u = u(z) = (u_1, u_2, u_3)^t$ satisfies the following Lamé equation (or the Navier equation according to Gurtin [8, p.90]):

$$L_\omega u := \Delta u + \omega \operatorname{grad}\operatorname{div} u = 0, \qquad z \in \Omega \tag{0.1}$$

where Δ is the Laplace operator in $\mathbb{R}^3$, $\omega = (1-2\sigma)^{-1}$ and σ is the Poisson constant. The upper index t denotes the transposition.

Let $g(z) = (g_1, g_2, g_3)^t$ be a given vector-function on Γ, i.e. at $z \in \Gamma$. Let also $N = (N_1, N_2, N_3)^t$ be the inner unit normal vector, τ_1 and τ_2 be orthogonal tangent vectors at each point of boundary Γ (τ_1, τ_2, N form a basis). We shall consider four BPETs following [8], [16, §40] and [14, Chap.3].

1991 *Mathematics Subject Classification* : 35J55, 47A56, 73C02, 76D07.

The first and second BPETs are defined by the following boundary conditions, respectively:

$$\gamma u = g(z) \tag{0.2}$$

$$\gamma\left[(\omega-1)N_l \operatorname{div} u + \sum_{m=1}^{3}\left(\frac{\partial u_l}{\partial z_m}+\frac{\partial u_m}{\partial z_l}\right)N_m\right] = g_l(z), \qquad l=1,2,3 \tag{0.3}$$

where γ is the operator of the restriction to the boundary Γ of a function with the domain $\overline{\Omega} = \Omega \cup \Gamma$.

Denote by T_ω the matrix differential operator which assigns the stress vector $T_\omega u$ to the displacement vector u. Its components are the expressions in square brackets on the left-hand sides of (0.3).

Let $\gamma_1 = \gamma$ and $\gamma_2 = \gamma T_\omega$. Then the boundary conditions (0.2) and (0.3) can be written as

$$\gamma_1 u = g \tag{0.4}$$

and

$$\gamma_2 u = g \tag{0.5}$$

The third and fourth BPETs are defined by the following conditions respectively:

$$\begin{cases} \langle \gamma_2 u - N\langle N, \gamma_2 u\rangle, \tau_k\rangle = g_k(z), & k=1,2 \\ \langle \gamma u, N\rangle = g_3(z) \end{cases} \tag{0.6}$$

and

$$\begin{cases} \langle N, \gamma_2 u\rangle = g_1(z) \\ \langle \gamma u - N\langle N, \gamma u\rangle, \tau_k\rangle = g_{k+1}(z), & k=1,2 \end{cases} \tag{0.7}$$

where $\langle\ ,\ \rangle$ means the usual inner product in $\mathbb{R}^3$.

Denote by γ_3 the matrix operator generated by the left-hand side of the equation (0.6) and by γ_4 the matrix operator generated by the left-hand side of the equation (0.7). Then the boundary conditions (0.6) and (0.7) can be rewritten as

$$\gamma_3 u = g \tag{0.8}$$

and

$$\gamma_4 u = g \tag{0.9}$$

We observe that the conditions (0.6) and (0.7) are equivalent respectively to the following equalities:

$$\gamma_2 u - N\langle N, \gamma_2 u\rangle = h, \qquad \langle N, \gamma_1 u\rangle = h_4$$

and

$$\gamma_1 u - N\langle N, \gamma_1 u\rangle = h, \qquad \langle N, \gamma_2 u\rangle = h_4,$$

where h and h_4 are respectively vector and scalar functions given on Γ.

We observe also that the first BPET is usually called the problem with given displacements. The second is called the problem with given stresses, the third is called the problem of a hard contact and the fourth is called the problem of a soft contact.

Consider the operator P_j which is generated by the j-th BPET for $\omega = \varkappa$ where $\varkappa > 1/3$ is a fixed number (for the first BPET $\varkappa > -1$ is possible).

$$P_j u := (\Delta u + \varkappa \operatorname{graddiv} u, \gamma_j u)^t, \qquad j = 1, 2, 3, 4 \tag{0.10}$$

In the boundary operators γ_j we also put $\omega = \varkappa$. According to the general theory of elliptic boundary value problems, the operator P_j is invertible on the space of pairs (F, g). Here F and g are vector-valued functions defined respectively in Ω and on Γ and orthogonal in the space $[L_2(\Omega)]^3 \oplus [L_2(\Gamma)]^3$ to the co-kernel of the j-th BPET. Denote by A_j the operator which is inverse to P_j. Put $f = (\alpha_j + \varkappa)\gamma \operatorname{div} A_j(0, g)^t$, $j = 1, 2, 3, 4$, where α_j are the following constants: $\alpha_1 = 2,\ \alpha_2 = 0,\ \alpha_3 = 1,\ \alpha_4 = 1$.

It has been proved in [13], that the j-th BPET, $j = 1, 2, 3, 4$ for each $\omega \neq -1, \infty$ can be reduced to the equivalent Fredholm regular integral equation on Γ relative to unknown function θ:

$$(\alpha_j + \varkappa)\theta + (\omega - \varkappa)K_j\theta = f \tag{0.11}$$

where K_j are ΨDO on Γ of order -1, i.e. compact in $L_2(\Gamma)$ integral operators with a weak singularity of the first order.

Let ω be a spectral parameter. A spectrum of the operator pencil induced by the first and the second boundary value problems for the Lamé equation $\Delta u + \omega \operatorname{graddiv} u = 0$ was studied by E. and F. Cosserat and later by S.G. Mikhlin and V.G. Mazya. A bibliography can be found in [15]. In particular, the limit points for finite-multiple eigenvalues (the Cosserat spectrum) of the first and the second problems have been obtained in this paper.

The asymptotics of the Cosserat spectrum has been investigated in [13]. It has been proved that for the j-th BPET, $j = 1, 2, 3, 4$, outside the ε neighborhood of the limit point there are $C_j \varepsilon^{-2} + o(\varepsilon^{-2})$ finite-multiple eigenvalues as $\varepsilon \to 0$. The coefficients C_j have not been found.

It should be noted that the point $\omega = -1$ is an isolated infinite-multiple eigenvalue of the j-th BPET, $j = 1, 2$. This fact has been proved in [15].

In this paper we obtain the principal symbols of ΨDOs K_j (see (0.11)). Using the formulae for the principal symbols we establish the kernels of the corresponding integral operators. We also find the coefficients C_j in the asymptotics of the Cosserat spectrum.

Let us now formulate the basic results. Denote by k_1 and k_2 the principal curvatures of the surface Γ and by E_1 and E_2 the coefficients of the first quadratic form of Γ.

Theorem 1 *ΨDOs K_j $j = 1, 2, 3, 4$ (see (0.11)) have the following principal symbols*:

$$
\begin{aligned}
\sigma_0(K_1) &= (\varkappa + 2)^{-1} \sum_{l=1}^{2} (k_l E_l^{-1} \xi_l^2 \|\xi'\|^{-2} - k_l) \|\xi'\|^{-1}, \\
\sigma_0(K_2) &= \varkappa^{-1} \sum_{l=1}^{2} (k_l - k_l E_l^{-1} \xi_l^2 \|\xi'\|^{-2}) \|\xi'\|^{-1}, \\
\sigma_0(K_3) &= (\varkappa + 1)^{-1} \sum_{l=1}^{2} k_l E_l^{-1} \xi_l^2 \|\xi'\|^{-3}, \\
\sigma_0(K_4) &= (\varkappa + 1)^{-1} \sum_{l=1}^{2} k_l \|\xi'\|^{-1},
\end{aligned}
$$

where $\xi_1 \in \mathbb{R}$, $\xi_2 \in \mathbb{R}$, $\xi' = (\xi_1, \xi_2)^t$, $\|\xi'\|^2 = E_1^{-1}\xi_1^2 + E_2^{-1}\xi_2^2$.

Corollary *The principal symbols of the ΨDOs K_j, $j = 1, 2, 3, 4$ (see (0.11)) induce the following integral operators B_j* :

$$
B_j \varphi(x') = \frac{1}{2\pi} \int_{\Gamma} b_j(x', x' - y') \varphi(y') \sqrt{E_1 E_2} dy'
$$

with the kernels:

$$\begin{aligned}
b_1(x', x'-y') &= -(k_1h_1^2+k_2h_2^2)|h|^{-3}(\varkappa+2)^{-1},\\
b_2(x', x'-y') &= (k_1h_1^2+k_2h_2^2)|h|^{-3}\varkappa^{-1},\\
b_3(x', x'-y') &= (k_1h_1^2+k_2h_2^2)|h|^{-3}(\varkappa+1)^{-1},\\
b_4(x', x'-y') &= (k_1+k_2)|h|^{-1}(\varkappa+1)^{-1},
\end{aligned}$$

where $h=(h_1,h_2)=(\sqrt{E_1}(x_1-y_1),\sqrt{E_2}(x_2-y_2))$, $\mid h\mid^2=h_1^2+h_2^2$, k_1, k_2, E_1, E_2 *are the functions in the local coordinates* x_1, x_2; φ *is a function in the local coordinates* y_1, y_2; $x'=(x_1,x_2)$, $y'=(y_1,y_2)$.

It has been proved in [13] that the Cosserat eigenvalues of the j-th BPET has the unique limit point $-\alpha_j$.

Theorem 2 *There are* $C_j\varepsilon^{-2}+o(\varepsilon^{-2})$ *as* $\varepsilon\to 0$ *finite-multiple Cosserat eigenvalues of the* j*-th BPET,* $j=1,2,3,4$ *outside of the* ε*-neghborhood of the limit point and*

$$\begin{aligned}
C_1=C_2=C_3 &= \frac{1}{32\pi}\int_\Gamma\sqrt{E_1E_2}(3k_1^2+3k_2^2+2k_1k_2)d\Gamma,\\
C_4 &= \frac{1}{4\pi}\int_\Gamma\sqrt{E_1E_2}(k_1+k_2)^2d\Gamma.
\end{aligned}$$

At the end of the paper we study some spectral properties of operator $\mathrm{div}\Delta_0^{-1}\mathrm{grad}$ where Δ_0^{-1} is an operator solving the Dirichlet problem for the Poisson equation:

$$\Delta_0^{-1}: f\to v,\quad \text{where } \Delta v=f \quad \text{in } \Omega,\quad v=0\quad \text{on } \Gamma.$$

The operator acts in the Sobolev spaces and it is important for the Stockes problem from hydromechanics (see [12], [13]).

The plan of the paper is as follows:

In Section 1 we introduce all the necessary notations and obtain the auxiliary results.

In Section 2 we find the principal symbols of the ΨDO K_j (see (0.11)) and the kernels of the respective integral operators.

In Section 3 the asymptotics of the Cosserat spectrum are studied.

Section 4 deals with the operator $\mathrm{div}\Delta_0^{-1}\mathrm{grad}$.

1. The Auxiliary Results

Let us take an arbitrary point $z \in \Gamma$ and introduce the local coordinate system in its neighborhood. Let the boundary Γ of the domain Ω be given locally by infinitely differentiable functions $z_l = z_l(x_1, x_2)$ $l = 1, 2, 3$ in the variables x_1, x_2. These variables are chosen so that the coordinate lines $x_1 = \text{const}$, $x_2 = \text{const}$ are the curvature lines. We enumerate x_1, x_2 so that the direction of the vector product $(\frac{\partial z}{\partial x_1}) \times (\frac{\partial z}{\partial x_2})$ coincides with the inner unit normal $N(x')$ to Γ, where $x' = (x_1, x_2)$.

Introduce in the neighborhood of Γ the coordinates x_1, x_2, x_3, where x_3 is the distance from the point $z = (z_1, z_2, z_3) \in \Omega$ to Γ. Then

$$z = z(x') + x_3 N(x') \equiv f(x),$$

where $z(x') \in \Gamma$, $x_3 \in (-\varepsilon, \varepsilon)$. Here $\varepsilon > 0$ is taken so small that the representation of z in terms of $z(x') \in \Gamma$ and $x_3 \in (-\varepsilon, \varepsilon)$ is unique and smooth, i.e., f is bijective and is C^∞ with C^∞ inverse, from $\Gamma \times (-\varepsilon, \varepsilon)$ to the set $f(\Gamma \times (-\varepsilon, \varepsilon)) \subset \mathbb{R}^3$.

Near Γ there is defined a normal vector field $N(x) = (N_1(x), N_2(x), N_3(x))$, as follows:

$$N(x) = N(x') \text{ for } z \text{ of the form } z = z(x') + x_3 N(x'),$$

where $z(x') \in \Gamma$, $x_3 \in (-\varepsilon, \varepsilon)$. The derivative along N is denoted D_N (the normal derivative): $D_N g = -i \sum_{k=1}^{3} N_k(x)(\partial / \partial x_k) g$ defined for $x \in f(\Gamma \times (-\varepsilon, \varepsilon)) \subset \mathbb{R}^3$.

Let the first and the second quadratic forms of the surface Γ be

$$\begin{aligned} I(x', dx') &= E_1(x')(dx_1)^2 + E_2(x')(dx_2)^2, \\ II(x', dx') &= L_1(x')(dx_1)^2 + L_2(x')(dx_2)^2. \end{aligned}$$

The following orthogonality relations are valid:

$$\begin{cases} \left\langle \dfrac{\partial z}{\partial x_l}, N \right\rangle = 0 \\ \left\langle \dfrac{\partial z}{\partial x_l}, \dfrac{\partial z}{\partial x_m} \right\rangle = E_l \delta_{lm} \end{cases}, \tag{1.1}$$

where δ_{lm} is the Kronecker delta, $l, m = 1, 2$.

We shall also need the Rodrigues relations:

$$\frac{\partial N}{\partial x_l} = -k_l \frac{\partial z}{\partial x_l} \quad l = 1, 2, \tag{1.2}$$

where k_l are the principal normal curvatures.

The following equality holds for k_l:

$$k_l = L_l(x')/E_l(x') \tag{1.3}$$

We have thus introduced the local coordinate system (x_1, x_2, x_3). The coordinate line x_3 is directed as the normal N. We assume that Γ has no umbilical points.

In the constructed local coordinate system the operators $\partial/\partial z_m$ have the form

$$\frac{\partial}{\partial z_m} = \sum_{l=1}^{2} (1 - x_3 k_l)^{-1} E_l^{-1} \frac{\partial z_m}{\partial x_l} \frac{\partial}{\partial x_l} + N_m \frac{\partial}{\partial x_3}, \quad m = 1, 2, 3$$

We denote by τ_1 and τ_2 the unit vectors tangent to Γ, which are collinear to $\partial z/\partial x_1$, $\partial z/\partial x_2$. Then $\tau_k = E_k^{-1/2}(\partial z/\partial x_k)$, $k = 1, 2$.

All the vectors N, τ_k are considered as column-vectors.

Let E be the unit matrix of dimensions 3×3 and β be a column-vector $\beta = (\beta_1, \beta_2, \beta_3)^t$ such that

$$\beta = \sum_{j=1}^{2} E_j^{-1/2} \xi_j \tau_j (1 - x_3 k_j)^{-1}, \quad \text{where} \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Let $i = \sqrt{-1}$ and

$$|||\, \xi' \,|||^2 = (1 - x_3 k_1)^{-2} E_1^{-1} \xi_1^2 + (1 - x_3 k_2)^{-2} E_2^{-1} \xi_2^2;$$

Denote by $T_\varkappa$ the matrix of differential operators corresponding to the second BPET (see (0.3)).

Hence in the introduced local coordinate system the following formulae hold for the principal symbols $\sigma_0(B)$ of various operators B connected with BPET.

$$\sigma_0(\partial/\partial z_m) = i\beta_m + i\xi_3 N_m, \tag{1.4}$$

$$\sigma_0(\text{grad}) = i\beta + i\xi_3 N, \tag{1.5}$$

$$\sigma_0(\mathrm{div}) = i\beta^t + i\xi_3 N^t, \tag{1.6}$$

$$\sigma_0(T_\varkappa) = i\beta N^t + i\xi_3 E + i\varkappa\xi_3 NN^t + i(\varkappa - 1)N\beta^t, \tag{1.7}$$

$$\sigma_0(T_\varkappa^t) = i(\varkappa - 1)\beta N^t + i\xi_3 E + i\varkappa\xi_3 NN^t + iN\beta^t, \tag{1.8}$$

$$\sigma_0(\Delta) = -(|||\,\xi'\,|||^2 + \xi_3^2)E,$$

$$\begin{aligned}\sigma_0(\Delta + \varkappa\,\mathrm{graddiv}) \;=\; & -(|||\,\xi'\,|||^2 + \xi_3^2)E \\ & -\varkappa\Big(\beta\beta^t + \xi_3(\beta N^t + N\beta^t) + \xi_3^2 NN^t\Big),\end{aligned} \tag{1.9}$$

Let $\Pi_\varkappa$ be a parametrix of the Lamé operator $L_\varkappa = \Delta + \varkappa\,\mathrm{graddiv}$. This means that the product $\Pi_\varkappa L_\varkappa$ is equal to the identity operator up to infinitely smoothing operator. Then

$$\begin{aligned}&\sigma_0(\Pi_\varkappa) \\ &= \frac{\varkappa\beta\beta^t + \varkappa\xi_3(N\beta^t + \beta N^t) + \varkappa\xi_3^2 NN^t - (1+\varkappa)(|||\,\xi'\,|||^2 + \xi_3^2)E}{(1+\varkappa)(|||\,\xi'\,|||^2 + \xi_3^2)^2}\end{aligned} \tag{1.10}$$

Let $\mu_1 = E_1^{-1/2}\xi_1\tau_1, \quad \mu_2 = E_2^{-1/2}\xi_2\tau_2.$

Lemma 1 *In the introduced local coordinate system at $x_3 = 0$ the following formulae hold for next to the principal symbols $\sigma_1(L_\varkappa)$ and $\sigma_1(\Pi_\varkappa)$ of ΨDO $L_\varkappa$ and $\Pi_\varkappa$:*

$$\begin{aligned}&\sigma_1(L_\varkappa) \\ &= \sum_{l=1}^{2}\Big\{-E_l^{-2}\frac{\partial E_l}{\partial x_l}i\xi_l - k_l i\xi_3\Big\}E + \frac{1}{2}\sum_{l,m=1}^{2} E_l^{-1}E_m^{-1}\frac{\partial E_m}{\partial x_l}i\xi_l E \\ &\quad + \varkappa\sum_{l=1}^{2} ik_l(N\mu_l^t - \tau_l\tau_l^t\xi_3) + \varkappa\sum_{l,m=1}^{2} i\mu_l\Big(\frac{\partial \mu_m}{\partial x_l}\Big)^t\xi_l^{-1},\end{aligned} \tag{1.11}$$

$$\begin{aligned}&\sigma_1(\Pi_\varkappa) \\ &= \Bigg\{\sum_{l=1}^{2} i\Big(E_l^{-2}\frac{\partial E_l}{\partial x_l}\xi_l + k_l\xi_3\Big)E - \frac{1}{2}\sum_{l,m=1}^{2} E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\xi_m E \\ &\quad + \frac{\varkappa}{\varkappa+1}\sum_{l=1}^{2} ik_l\Big(\tau_l\tau_l^t\xi_3 - N\mu_l^t\Big)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\varkappa}{\varkappa+1} \sum_{l,m=1}^{2} i\mu_l \Big(\frac{\partial \mu_m}{\partial x_l} \xi_l^{-1} \Big)^t \Big\} \|\xi\|^{-4} \\
&+ \Big\{ \sum_{l=1}^{2} i4k_l E_l^{-1} \xi_l^2 \xi_3 E - 2 \sum_{l,m=1}^{2} E_l^{-2} \frac{\partial E_l}{\partial x_m} E_m^{-1} \xi_l^2 \xi_m E \\
&+ \frac{\varkappa}{\varkappa+1} \sum_{l=1}^{2} 2NN^t i \Big(2k_l E_l^{-1} \xi_l^2 \xi_3 - k_l \xi_3^3 - E_l^{-2} \frac{\partial E_l}{\partial x_l} \xi_l \xi_3^2 \Big) \\
&+ \frac{\varkappa}{\varkappa+1} \sum_{l,m=1}^{2} i \Big[\Big(N\mu_l^t + \mu_l N^t \Big) \Big(k_m \Big(E_m^{-1} \xi_m^2 - 2\xi_3^2 \Big) \\
&- E_m^{-2} \xi_m^2 \xi_3 \Big(\frac{\partial E_m}{\partial x_l} \xi_l^{-1} + 2 \frac{\partial E_m}{\partial x_m} \xi_m^{-1} \Big) \Big) \\
&+ 2 \Big(\frac{\partial \mu_l}{\partial x_m} N^t + N \Big(\frac{\partial \mu_l}{\partial x_m} \Big)^t \Big) E_m^{-1} \xi_m \xi_3 \Big] \\
&+ \frac{\varkappa}{\varkappa+1} \sum_{l,m,p=1}^{2} \Big[\Big(N\mu_p^t \xi_3 + \mu_p N^t \xi_3 \\
&+ \sum_{s=1}^{2} \mu_p \mu_s^t + \frac{1}{2} NN^t \xi_3^2 \Big) E_l^{-1} E_m^{-1} \frac{\partial E_l}{\partial x_m} \xi_m \\
&+ 2 \Big(\mu_l \Big(\frac{\partial \mu_p}{\partial x_m} \Big)^t + \frac{\partial \mu_p}{\partial x_m} \mu_l^t \Big) E_m^{-1} \xi_m - \mu_p \mu_l^t \xi_m \Big(2k_m \xi_p^{-1} \xi_3 \\
&+ E_m^{-2} \Big(\frac{\partial E_m}{\partial x_l} \xi_l^{-1} \xi_m + \frac{\partial E_m}{\partial x_p} \xi_p^{-1} \xi_m + 2 \frac{\partial E_m}{\partial x_m} \Big) \Big) \Big] \Big\} \|\xi\|^{-6} \\
&+ 6 \frac{\varkappa}{\varkappa+1} \Big\{ \sum_{l=1}^{2} i \Big[NN^t \xi_3^2 + \sum_{m=1}^{2} (N\mu_m^t \xi_3 + \mu_m N^t \xi_3) \\
&+ \sum_{m,p=1}^{2} \mu_p \mu_m^t \Big] \Big(-2k_l E_l^{-1} \xi_l^2 \xi_3 + \sum_{s=1}^{2} E_l^{-2} \frac{\partial E_l}{\partial x_s} E_s^{-1} \xi_l^2 \xi_s \Big) \Big\} \|\xi\|^{-8},
\end{aligned}
\tag{1.12}
$$

where $\|\xi\|^2 = \xi_3^2 + \|\xi'\|^2$, $\|\xi'\|^2 = E_1^{-1}\xi_1^2 + E_2^{-1}\xi_2^2$.

Proof. The following formulae hold for the principal and next to the principal symbols of the product of two ΨDOs A and B [9]:

$$\sigma_0(AB) = \sigma_0(A)\sigma_0(B)$$

$$\sigma_1(AB) = \sigma_1(A)\sigma_0(B) + \sigma_0(A)\sigma_1(B) + \sum_{|\nu|=1} (-i)\partial_\xi^\nu \sigma_0(A)\partial_x^\nu \sigma_0(B) \tag{1.13}$$

Here $\nu = (\nu_1, \nu_2, \nu_3)$, ν_l are non-negative integers $l = 1, 2, 3$, $|\nu| = \nu_1 + \nu_2 + \nu_3$, $\partial_x^\nu = (\partial^{\nu_1}/\partial x_1{}^{\nu_1})(\partial^{\nu_2}/\partial x_2{}^{\nu_2})(\partial^{\nu_3}/\partial x_3{}^{\nu_3})$.

It is obvious that $\sigma_1(\partial/\partial z_l) = 0$, $l = 1, 2, 3$, $\sigma_1(\mathrm{grad}) = 0$, $\sigma_1(\mathrm{div}) = 0$.

We obtain (1.11) applying (1.13) to (1.4), (1.5), (1.6). We also use orthogonality relations (1.1) and Rodrigues relations (1.2) here.

Since $\Pi_\varkappa$ is a parametrix of $L_\varkappa$, then

$$\sigma_1(\Pi_\varkappa) = -\Big(\sigma_0(\Pi_\varkappa)\sigma_1(L_\varkappa) + \sum_{|\nu|=1} (-i)\partial_\xi^\nu \sigma_0(\Pi_\varkappa)\partial_x^\nu \sigma_0(L_\varkappa)\Big)\sigma_0(\Pi_\varkappa) \tag{1.14}$$

Because of (1.14), (1.9), (1.10) and (1.11), we obtain (1.12). Lemma 1 is proved. □

The following Lemma 2 presents some particular case of a result proved in [5, p.499].

Lemma 2 *Let Q be a differential operator of order d in $\mathbb{R}^3$, written in the form*

$$Qf = \sum_{l=0}^{d} S_l D_N^l f,$$

where S_l is a differential operator of order $d - l$, which does not contain derivatives with respect to x_3, $f \in C_0^\infty(\overline{\Omega})$. Let P be a parametrix of an elliptic differential operator in $\mathbb{R}^3$. Then

$$r^+PQe^+f - r^+Pe^+r^+Qe^+f = \sum_{m=0}^{d-l}(-i)\sum_{l=m+1}^{d} r^+(PS_lD_N^{l-1-m})((\gamma D_N^m f)\delta_\Gamma),$$

where δ_Γ is a distribution in $\mathbb{R}^3$ such that $\delta_\Gamma(\varphi) = \int_\Gamma \varphi d\Gamma$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$; e^+ denotes the "extension by zero" operator mapping a function u into a function e^+u, equal to u in Ω and equal to 0 in $\mathbb{R}^3 \backslash \Omega$; r^+ denotes the restriction operator into Ω.

Lemma 3 *Let $u \in C^\infty(\overline{\Omega})$ satisfy the homogeneous Lamé equation: $L_\varkappa u =$*

0 in Ω and let $\Pi_\varkappa$ be a parametrix of the Lamé operator $L_\varkappa$ in $\mathbb{R}^3$. Then

$$u = \Pi_\varkappa T^t_\varkappa((\gamma u)\delta_\Gamma) + \Pi_\varkappa((\gamma_2 u)\delta_\Gamma) + \Pi_\varkappa V((\gamma u)\delta_\Gamma) + \cdots \tag{1.15}$$

Here dots denote an integral operator with infinitely smooth kernel and V is a ΨDO.

The symbol of V at $x_3 = 0$ has the form:

$$\sigma_0(V) = -(k_1(1 + \varkappa\tau_1\tau_1^t) + k_2(1 + \varkappa\tau_2\tau_2^t)).$$

Proof. We put $Q = L_\varkappa$ and $P = \Pi_\varkappa$. Applying Lemma 2 and using also (1.9) and (1.11), we obtain:

$$\begin{aligned} r^+\Pi_\varkappa L_\varkappa e^+ u \;=\; & r^+\Pi_\varkappa e^+ r^+ L_\varkappa e^+ u \\ & + (-i)r^+(\Pi_\varkappa S_1((\gamma u)\delta_\Gamma) + \Pi_\varkappa S_2 D_N((\gamma u)\delta_\Gamma) \\ & + \Pi_\varkappa S_2((\gamma D_N u)\delta_\Gamma), \end{aligned} \tag{1.16}$$

where

$$\begin{aligned} S_1 \;=\; & \sum_{m=1}^{2} i(1 - x_3 k_m)^{-1}\Big\{-k_m(1 + \varkappa\tau_m\tau_m^t) \\ & + \varkappa E_m^{-1}\Big(\frac{\partial z}{\partial x_m}N^t + N\Big(\frac{\partial z}{\partial x_m}\Big)^t\Big)\frac{\partial}{\partial x_m}\Big\}, \\ S_2 \;=\; & -(1 + \varkappa N N^t) \end{aligned}$$

One can easily to see, that the expression in the left-hand part of (1.16) is equal to $(u + T_{-\infty}u)$, where $T_{-\infty}$ is the integral operator with infinitely smooth kernel. The first term in the right-hand part of (1.16) is equal to zero, since $L_\varkappa u = 0$. The other terms can be reduced to the form:

$$\Pi_\varkappa T^t_\varkappa((\gamma u)\delta_\Gamma) + \Pi_\varkappa((\gamma_2 u)\delta_\Gamma) + \Pi_\varkappa V((\gamma u)\delta_\Gamma).$$

Thus we obtain (1.15). □

Lemma 4 *Let $\varphi \in C^\infty(\Gamma)$ and Q be a ΨDO in the domain Ω and let each term of its symbol $\sum_{l=1}^\infty q_l(x,\xi)$ be a rational function of ξ in the given local coordinate system. Then the operator $Q_m\varphi = \gamma D_N^m Q(\varphi\delta_\Gamma)$ is a ΨDO on Γ with the symbol $\sum_{l=0}^\infty q_l^m(x',\xi')$ and*

$$q_l^m(x',\xi') = \frac{1}{2\pi}\int_{\Gamma_+}(D_N + \xi_3)^m q_l(x',0,\xi)d\xi_3,$$

$l = 0, \cdots \infty$, $x' = (x_1, x_2)$, $\xi' = (\xi_1, \xi_2)$. Here the contour "Γ_+" is a circle in the semiplane $Im\,\xi_3 > 0$ with all the singularities of the symbol $q_l(x', 0, \xi)$ within it.

Proof of Lemma 4 can be found in [10].

Lemma 5 *Let Q be a ΨDO in Ω and let each term of its symbol $\sum_{l=0}^{\infty} q_l(x, \xi)$ be a rational function of ξ and Λ be the operator of the harmonic continuation in Ω, i.e. the function $u = \Lambda g$ is a solution of the boundary value problem: $\Delta u = 0$ in Ω, $u = g$ on Γ. Then the operator $A = \gamma Q \Lambda$ is a ΨDO on Γ with the symbol $\sum_{l=0}^{\infty} a_l(x', \xi')$ and*

$$\begin{aligned} a_0(x', \xi') &= (2\pi i)^{-1} \int_{\Gamma_+} (\xi_3 - i\|\xi'\|)^{-1} q_0(x', 0, \xi) d\xi_3, \\ a_1(x', \xi') &= (2\pi i)^{-1} \int_{\Gamma_+} \Big\{ (\xi_3 - i\|\xi'\|)^{-1} q_1(x', 0, \xi) \\ &\quad + (\xi_3 + i\|\xi'\|) \Big(-q_0(x', 0, \xi) \varphi_1(x', 0, \xi) \\ &\quad + \sum_{|\nu|=1} \Big(i\partial_\xi^\nu q_0(x', 0, \xi) \partial_x^\nu \varphi_0(x', 0, \xi) \\ &\quad - \partial_\xi^\nu (q_0(x', 0, \xi)\varphi_0(x', 0, \xi)) \partial_x^\nu (\|\xi'\|) \Big) \\ &\quad - q_0(x', 0, \xi)\varphi_0(x', 0, \xi)\varphi_2(x', \xi') \Big) \Big\} d\xi_3. \end{aligned}$$

Here contour Γ_+ is the same as in Lemma 4;

$$\begin{aligned} \varphi_0(x, \xi) &= \sigma_0(\Pi_\varkappa) \qquad at \quad \varkappa = 0; \\ \varphi_1(x', 0, \xi) &= \sigma_1(\Pi_\varkappa) \qquad at \quad \varkappa = 0, \quad x_3 = 0; \\ \varphi_2(x', \xi') &= \frac{i}{2} \sum_{l=1}^{2} k_l \Big(E_l^{-1} \frac{\xi_l^2}{\|\xi'\|^2} + 1 \Big) - \frac{1}{2} \sum_{l=1}^{2} E_l^{-2} \frac{\partial E_l}{\partial x_l} \frac{\xi_l}{\|\xi'\|} \\ &\quad + \frac{1}{4} \sum_{l,m=1}^{2} E_l^{-1} E_m^{-1} \frac{\partial E_l}{\partial x_m} \Big(1 + E_l^{-1} \frac{\xi_l^2}{\|\xi'\|^2} \Big) \frac{\xi_m}{\|\xi'\|}. \end{aligned}$$

Proof. Denote by Φ a parametrix of the operator Δ in $\mathbb{R}^3$. Lemma 1 and Lemma 3 (at $\varkappa = 0$) induce the following formula for the harmonic function f in Ω:

$$f = i\Phi((\gamma D_N f)\delta_\Gamma) + i\Phi D_N((\gamma f)\delta_\Gamma) + \Phi V_1((\gamma f)\delta_\Gamma) + \cdots, \quad (1.17)$$

where $\sigma_0(V_1)(x',0,\xi)=-(k_1+k_2)$; $\sigma_0(\Phi)(x,\xi)=\sigma_0(\Pi_\varkappa)$ and $\sigma_1(\Phi)(x',0,\xi)=\sigma_1(\Pi_\varkappa)(x',0,\xi)$ at $\varkappa=0$.

Applying the operator γ to equality (1.17), we get:

$$\gamma f - (i\gamma\Phi D_N + \gamma\Phi V_1)((\gamma f)\delta_\Gamma) = i\gamma\Phi((\gamma D_N f)\delta_\Gamma) + \cdots .$$

We denote $f_0 = \gamma f$ and $f_1 = \gamma D_N f$. We also denote by A_1 and A_0 the following operators:

$$A_1 f_1 := i\gamma\Phi(f_1\delta_\Gamma), \quad A_0 f_0 := (i\gamma(\Phi D_N) + \gamma\Phi V_1)(f_0\delta_\Gamma).$$

By Lemma 4, these operators are ΨDOs.

Since $f_1 = \gamma D_N \Lambda f_0$ then

$$(I - A_0)f_0 = A_1(\gamma D_N\Lambda)f_0 + \cdots,$$

where dots denote the operator of order $-\infty$.

Hence the principal and next to the principal symbols of the operator $I - A_0$ are equal to the principal and next to the principal symbols of the operator $A_1(\gamma D_N\Lambda)$, respectively.

Using the calculus of ΨDO and Lemma 4, we obtain the principal and next to the principal symbols of the operator $\gamma D_N\Lambda$:

$$\sigma_0(\gamma D_N\Lambda) = i\|\xi'\|, \tag{1.18}$$

$$\begin{aligned}\sigma_1&(\gamma D_N\Lambda)\\ &= \frac{i}{2}\sum_{l=1}^{2} k_l\Big(E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2} - 1\Big) - \frac{1}{2}\sum_{l=1}^{2} E_l^{-2}\frac{\partial E_l}{\partial x_l}\frac{\xi_l}{\|\xi'\|}\\ &\quad + \frac{1}{4}\sum_{l,m=1}^{2} E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\Big(1 + E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\Big)\frac{\xi_m}{\|\xi'\|}.\end{aligned} \tag{1.19}$$

Applying to (1.17) first ΨDO Q and then operator γ, we get

$$\begin{aligned}\gamma Q\Lambda f_0 &= i\gamma Q\Phi((\gamma D_N\Lambda f_0)\delta_\Gamma)\\ &\quad + i\gamma Q\Phi D_N(f_0\delta_\Gamma) + i\gamma Q\Phi V_1(f_0\delta_\Gamma) + \cdots\end{aligned}$$

Using the calculus of ΨDO, Lemma 4 and also the relations (1.18) and (1.19), we obtain the principal and next to the principal symbols of the operator A. Lemma 5 is proved. □

2. Calculation of the Principal Symbols of ΨDO K_j

First we consider a scheme of the reducing of the j-th BPET to an equivalent Fredholm integral equation. (see [13]).

Let us rewrite the j-th BPET using a fixed number $\varkappa > 1/3$ (possibly, $\varkappa > -1$ for the first BPET). We have

$$\Delta u + \varkappa \operatorname{graddiv} u = (\varkappa - \omega) \operatorname{graddiv} u$$

$$\gamma_j u = g + \delta_{2j}(\varkappa - \omega) N \gamma \operatorname{div} u + \delta_{4j}(\varkappa - \omega)\bar{i}\gamma \operatorname{div} u$$

where $\bar{i} = (1,0,0)^t$, δ_{lj} is the Kronecker delta ($j = 1,2,3,4$). We put

$$P_j u := (\Delta u + \varkappa \operatorname{graddiv} u,\ \gamma_j u)^t$$

It is known from the general theory of elliptic boundary value problems that the operator P_j is invertible on a set of pairs (F, g) so that F and g are vector-valued functions defined respectively in Ω and on Γ and orthogonal to co-kernel of the j-th BPET. These co-kernels are described in [8], [13], [14], [16].

Denote by A_j an operator inverse to P_j. Then

$$\begin{aligned} u \;=\; & A_j((\varkappa - \omega)\operatorname{graddiv} u,\ g + \delta_{2j}(\varkappa - \omega) N \gamma \operatorname{div} u \\ & + \delta_{4j}(\varkappa - \omega)\bar{i}\gamma \operatorname{div} u)^t. \end{aligned}$$

It follows from this

$$u + (\omega - \varkappa) A_j(\operatorname{graddiv} u, \delta_{2j} N\gamma \operatorname{div} u + \delta_{4j}\bar{i}\gamma \operatorname{div} u)^t = A_j(0,g)^t.$$

Applying successively to the latter equality the operators div and γ and replacing $\gamma \operatorname{div} u$ by θ, we get

$$\theta + (\omega - \varkappa) L_j \theta = f, \quad j = 1,2,3,4, \tag{2.1}$$

where $L_j\theta = \gamma \operatorname{div} A_j(\operatorname{grad} \Lambda\theta, \delta_{2j} N\theta + \delta_{4j}\bar{i}\theta)^t$, $f = \gamma \operatorname{div} A_j(0,g)^t$.

Thus it is proved that if the vector-function u satisfies the j-th BPET then the function $\theta = \gamma \operatorname{div} u$ satisfies the integral equation (2.1).

Vice versa, if $\theta \in C^\infty(\Gamma)$ satisfies the integral equation (2.1), then we put

$$u = (\varkappa - \omega) A_j(\operatorname{grad} \Lambda\theta,\ \delta_{2j} N\theta + \delta_{4j}\bar{i}\theta)^t + A_j(0,g)^t.$$

It can be verified that this vector-function u satisfies the j-th BPET.

The investigation of the ΨDO L_j allows us to prove that the equation (2.1) is regular. More precisely, let Λ_j be the operator which solves for $\omega = \varkappa$ the j-th BPET. Then it holds

$$\begin{aligned} L_j &= \gamma \operatorname{div} \Pi_\varkappa \operatorname{grad} \Lambda - \gamma \operatorname{div} \Lambda_j \gamma_j \Pi_\varkappa \operatorname{grad} \Lambda \\ &\quad + \delta_{2j} \gamma \operatorname{div} \Lambda_2 N + \delta_{4j} \gamma \operatorname{div} \Lambda_4 \bar{\imath}. \end{aligned} \tag{2.2}$$

Each term in the right-hand part of (2.2) is a ΨDO of zero order on Γ.

By virtue of (2.2) we obtain the principal symbols of the operators L_j. It has been proved in [13] that $\sigma_0(L_j) = 1/(\alpha_j + \varkappa)$, where $\alpha_1 = 2, \alpha_2 = 0, \alpha_3 = 1, \alpha_4 = 1$. This allowes us rewrite (2.1) in the following form:

$$(\alpha_j + \omega)\theta + (\omega - \varkappa)K_j\theta = (\alpha_j + \varkappa)f, \quad j = 1,2,3,4,$$

where K_j are different ΨDOs on Γ of order -1, i.e. compact integral operators with a weak singularity of the first order.

It is obvious that

$$\sigma_0(K_j) = (\alpha_j + \varkappa)\sigma_1(L_j). \tag{2.3}$$

To obtain the principal symbol of the operator K_j one should therefore calculate the next to the principal symbol of the operator L_j, which is actually its subprincipal symbol.

Proof of Theorem 1 from Introduction. Our aim is to calculate the principal and next to the principal symbols of ΨDOs contained in the right-hand part of (2.2).

First we consider the operators $\gamma \operatorname{div} \Lambda_j$, $j = 1,2,3,4$. It has been shown in [13] that the principal symbols of these ΨDO are the following:

$$\sigma_0(\gamma \operatorname{div} \Lambda_1) = \frac{2}{(\varkappa + 2)} \Big(\sum_{l=1}^{2} i\mu_l^t - N^t \|\xi'\| \Big) \tag{2.4}$$

$$\sigma_0(\gamma \operatorname{div} \Lambda_2) = -\frac{1}{\varkappa} \Big(\sum_{l=1}^{2} i\mu_l^t \|\xi'\|^{-1} - N^t \Big), \tag{2.5}$$

$$\sigma_0(\gamma \operatorname{div} \Lambda_3) = \frac{1}{(\varkappa + 1)} \Big(-E_1^{-1/2} \frac{i\xi_1}{\|\xi'\|}, -E_2^{-1/2} \frac{i\xi_2}{\|\xi'\|}, \ -2\|\xi'\| \Big), \tag{2.6}$$

$$\sigma_0(\gamma \operatorname{div} \Lambda_4) = \frac{1}{(\varkappa + 1)} \Big(1, \ 2E_1^{-1/2} i\xi_1, \ 2E_2^{-1/2} i\xi_2 \Big). \tag{2.7}$$

We now calculate the next to the principal symbols of these operators.

If a function u satisfies the Lamé equation $L_\varkappa u = 0$, then according to Lemma 3, it satisfies equality (1.15). Applying to both parts of (1.15) the operator γ, we get

$$\gamma u - \gamma\Pi_\varkappa T^t_\varkappa((\gamma u)\delta_\Gamma) - \gamma\Pi_\varkappa V((\gamma u)\delta_\Gamma) = \gamma\Pi_\varkappa((\gamma_2 u)\delta_\Gamma) + \cdots,$$

where dots denote an operator of order $-\infty$.

Let $\{\gamma Q\}(\gamma u) := \gamma Q\Big((\gamma u)\delta_\Gamma\Big)$ where Q is a ΨDO in Ω.

Hence we have the following equality for the operator $\gamma_2\Lambda_1$, which maps γu into $\gamma_2 u$:

$$\gamma_2\Lambda_1 = \{\gamma\Pi_\varkappa\}^{-1}(I - \{\gamma(\Pi_\varkappa T^t_\varkappa)\} - \{\gamma(\Pi_\varkappa V)\}) + \cdots, \tag{2.8}$$

where $\{\gamma\Pi_\varkappa\}^{-1}$ is a parametrix of the operator $\gamma\Pi_\varkappa((\cdot)\delta_\Gamma)$ and I is the identity operator.

Applying to the both parts of (1.15) first the operator div and then the operator γ, we get

$$\begin{aligned}\gamma\operatorname{div} u \;=\; & \gamma\operatorname{div}\Pi_\varkappa T^t_\varkappa((\gamma u)\delta_\Gamma) + \gamma\operatorname{div}\Pi_\varkappa\Big((\gamma_2\Lambda_1(\gamma u))\delta_\Gamma\Big) \\ & + \gamma\operatorname{div}\Pi_\varkappa V((\gamma u)\delta_\Gamma) + \cdots .\end{aligned}$$

Hence

$$\gamma\operatorname{div}\Lambda_1 = \{\gamma\operatorname{div}\Pi_\varkappa T^t_\varkappa\} + \{\gamma\operatorname{div}\Pi_\varkappa\}\gamma_2\Lambda_1 + \{\gamma\operatorname{div}\Pi_\varkappa V\} + \cdots .$$

By virtue of (2.8), we have

$$\begin{aligned}\gamma\operatorname{div}\Lambda_1 = & \{\gamma\operatorname{div}\Pi_\varkappa T^t_\varkappa\} \\ & + \{\gamma\operatorname{div}\ \Pi_\varkappa\}(\{\gamma\Pi_\varkappa\}^{-1}(I - \{\gamma(\Pi_\varkappa T^t_\varkappa)\}\{\gamma\Pi_\varkappa V\}) + \cdots) \\ & + \{\gamma\operatorname{div}\Pi_\varkappa V\} + \cdots .\end{aligned}$$

Using calculus of ΨDO, Lemmas 1 and 4, we obtain the next to the principal symbol of the operator $\gamma\operatorname{div}\Lambda_1$:

$$\begin{aligned}\sigma_1(\gamma\operatorname{div}\Lambda_1) \;=\; & \frac{1}{(\varkappa+2)^2}\sum_{l=1}^{2} k_l\Big(1 - E_l^{-1}\frac{\xi_l^2}{\|\xi'\|}\Big)\Big(2N^t + \varkappa\sum_{m=1}^{2}\frac{i\mu^t_m}{\|\xi'\|}\Big) \\ & + \frac{1}{2(\varkappa+2)}\sum_{l=1}^{2}\Big\{-2E_l^{-2}\frac{\partial E_l}{\partial x_l}\frac{i\xi_l}{\|\xi'\|}\end{aligned} \tag{2.9}$$

$$+\sum_{m=1}^{2} E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\left(1+E_l^{-1}\frac{\xi_l^2}{\|\xi'\|}\right)\frac{i\xi_m}{\|\xi'\|^2}\Big\}N^t.$$

The boundary conditions (0.6), (0.7) and also the equality (2.8) imply formulae for the principal and next to the principal symbols of the operators $\gamma_3\Lambda_1$ and $\gamma_4\Lambda_1$.

The next to the principal symbols of the operators $\gamma\mathrm{div}\,\Lambda_j$, $j=2,3,4$ are obtained from the following equations:

$$\begin{aligned}(\gamma\mathrm{div}\,\Lambda_2)(\gamma_2\Lambda_1) &= \gamma\mathrm{div}\,\Lambda_1,\\ (\gamma\mathrm{div}\,\Lambda_3)(\gamma_3\Lambda_1) &= \gamma\mathrm{div}\,\Lambda_1,\\ (\gamma\mathrm{div}\,\Lambda_4)(\gamma_4\Lambda_1) &= \gamma\mathrm{div}\,\Lambda_1.\end{aligned}$$

Thus we have

$$\begin{aligned}&\sigma_1(\gamma\mathrm{div}\,\Lambda_2)\\ &=\frac{1}{\varkappa^2}\sum_{l=1}^{2}k_l\left(E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}-1\right)\left(\frac{1}{2}N^t(\varkappa-2)\frac{1}{\|\xi'\|}+\sum_{m=1}^{2}\frac{i\mu_m^t}{\|\xi'\|^2}\right)\\ &\quad-\frac{1}{2\varkappa}\sum_{l,m=1}^{2}E_l^{-2}\frac{\xi_l}{\|\xi'\|^3}\left(\frac{\partial E_l}{\partial x_l}\xi_m+\frac{\partial E_l}{\partial x_m}\xi_l\right)\mu_m^t\xi_m^{-1}\\ &\quad+\frac{1}{4\varkappa}\sum_{l,m,p=1}^{2}E_l^{-1}E_p^{-1}\frac{\partial E_l}{\partial x_p}\left(1+3E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\frac{\xi_p}{\|\xi'\|^3}\mu_m^t. \qquad (2.10)\end{aligned}$$

$$\begin{aligned}&\sigma_1(\gamma\mathrm{div}\,\Lambda_3)\\ &=\Bigg\{\Bigg(-\frac{1}{2(\varkappa+1)}E_1^{-1/2}\frac{i\xi_1}{\|\xi'\|^2}\left[4k_1+\sum_{l=1}^{2}k_l\left(1-\frac{(3\varkappa+1)}{(\varkappa+1)}E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\right]\\ &\quad+\frac{1}{2(\varkappa+2)}E_1^{-1/2}E_2^{-1}\left[\frac{\partial E_2}{\partial x_1}+\frac{1}{(\varkappa+1)}\frac{\partial E_1}{\partial x_2}\right]\Bigg),\\ &\quad\Bigg(-\frac{1}{2(\varkappa+1)}E_2^{-1/2}\frac{i\xi_2}{\|\xi'\|^2}\left[4k_2+\sum_{l=1}^{2}k_l\left(1-\frac{(3\varkappa+1)}{(\varkappa+1)}E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\right]\\ &\quad+\frac{1}{2(\varkappa+1)}E_2^{-1/2}E_1^{-1}\left[\frac{\partial E_1}{\partial x_2}+\frac{1}{(\varkappa+1)}\frac{\partial E_2}{\partial x_1}\right]\Bigg),\\ &\quad\left(-\frac{1}{(\varkappa+1)}\sum_{l=1}^{2}k_l\frac{1}{\|\xi'\|}\left(1-\frac{(\varkappa-1)}{(\varkappa+1)}E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\right)\Bigg\}. \qquad (2.11)\end{aligned}$$

$$
\begin{aligned}
&\sigma_1(\gamma \operatorname{div} \Lambda_4)\\
&= \left\{ \frac{(\varkappa+2)}{(\varkappa+1)^2} \sum_{l=1}^{2} k_l \frac{1}{\|\xi'\|}, \frac{E_1^{-1/2} E_2^{-1}}{(\varkappa+2)} \left[\frac{\partial E_2}{\partial x_1} + \frac{1}{(\varkappa+1)} \frac{\partial E_1}{\partial x_2} \right] \right.\\
&\quad + \frac{2E_1^{-1/2}}{(\varkappa+1)^2} \frac{i\xi_1}{\|\xi'\|} \sum_{l=1}^{2} k_l, \frac{E_2^{-1/2} E_1^{-1}}{(\varkappa+2)} \left[\frac{\partial E_1}{\partial x_2} + \frac{1}{(\varkappa+1)} \frac{\partial E_2}{\partial x_1} \right]\\
&\quad \left. + \frac{2E_2^{-1/2}}{(\varkappa+1)^2} \frac{i\xi_2}{\|\xi'\|} \sum_{l=1}^{2} k_l \right\}. \qquad (2.12)
\end{aligned}
$$

Now we consider the operators $\gamma \operatorname{div} \Pi_\varkappa \operatorname{grad} \Lambda$ and $\gamma_j \Pi_\varkappa \operatorname{grad} \Lambda$, $j = 1, 2, 3, 4$.

The principal symbols of these ΨDOs are as follows:

$$
\sigma_0(\gamma \operatorname{div} \Pi_\varkappa \operatorname{grad} \Lambda) = (\varkappa+1)^{-1}, \qquad (2.13)
$$

$$
\sigma_0(\gamma_1 \Pi_\varkappa \operatorname{grad} \Lambda) = -\frac{1}{4(\varkappa+1)} \left(\sum_{l=1}^{2} \frac{i\mu_l}{\|\xi'\|^2} + N \frac{1}{\|\xi'\|} \right), \qquad (2.14)
$$

$$
\sigma_0(\gamma_2 \Pi_\varkappa \operatorname{grad} \Lambda) = \frac{(2\varkappa+1)}{2(\varkappa+1)} N - \frac{1}{2(\varkappa+1)} \sum_{l=1}^{2} \frac{i\mu_l}{\|\xi'\|}, \qquad (2.15)
$$

$$
\begin{aligned}
&\sigma_0(\gamma_3 \Pi_\varkappa \operatorname{grad} \Lambda)\\
&= \frac{1}{2(\varkappa+1)} \left\{ E_1^{-1/2} \frac{i\xi_1}{\|\xi'\|}, E_2^{-1/2} \frac{i\xi_2}{\|\xi'\|}, \frac{1}{2\|\xi'\|} \right\}^t \qquad (2.16)
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_0(\gamma_4 \Pi_\varkappa \operatorname{grad} \Lambda)\\
&= \left\{ \frac{(2\varkappa+1)}{2(\varkappa+1)}, -\frac{1}{4(\varkappa+1)} E_1^{-1/2} \frac{i\xi_1}{\|\xi'\|^2}, -\frac{1}{4(\varkappa+1)} E_2^{-1/2} \frac{i\xi_2}{\|\xi'\|^2} \right\}^t\\
&\qquad (2.17)
\end{aligned}
$$

(see [13]).

We obtain the formulae for the next to the principal symbols of the operators $\gamma \operatorname{div} \Pi_\varkappa \operatorname{grad} \Lambda$, $\gamma_1 \Pi_\varkappa \operatorname{grad} \Lambda$, $\gamma_2 \Pi_\varkappa \operatorname{grad} \Lambda$ using calculus of ΨDOs and Lemmas 1 and 5.

It holds that

$$
\sigma_1(\gamma \operatorname{div} \Pi_\varkappa \operatorname{grad} \Lambda) = 0, \qquad (2.18)
$$

$$
\begin{aligned}
&\sigma_1(\gamma_1\Pi_\varkappa \operatorname{grad}\Lambda)\\
&=\frac{1}{4(\varkappa+1)}\sum_{l,m=1}^{2}\Big\{\mu_l\Big[k_m E_m^{-1}\frac{i\xi_m^2}{\|\xi'\|^5}\\
&\quad-E_m^{-2}\frac{\xi_m^2}{\|\xi'\|^4}\Big(\frac{\partial E_m}{\partial x_m}\xi_m^{-1}+\frac{1}{4}\frac{\partial E_m}{\partial x_l}\xi_l^{-1}\Big)\\
&\quad+\frac{1}{2}\sum_{p=1}^{2}E_m^{-1}E_p^{-1}\frac{\partial E_m}{\partial x_p}\Big(1+4E_m^{-1}\frac{\xi_m^2}{\|\xi'\|^2}\Big)\frac{\xi_p}{\|\xi'\|^4}\Big]\\
&\quad+\frac{3}{2}\frac{\partial\mu_l}{\partial x_m}E_m^{-1}\frac{\xi_m}{\|\xi'\|^4}\Big\}+\frac{1}{8(\varkappa+1)}N\sum_{l=1}^{2}\Big\{E_l^{-2}\Big(\frac{\partial E_l}{\partial x_l}\Big)\frac{i\xi_l}{\|\xi'\|^3}\\
&\quad-\frac{k_l}{\|\xi'\|^2}-\frac{1}{2}\sum_{m=1}^{2}E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\Big(1+3E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\Big)\frac{i\xi_m}{\|\xi'\|^3}\Big)\Big\}
\end{aligned}\tag{2.19}
$$

$$
\begin{aligned}
&\sigma_1(\gamma_2\Pi_\varkappa \operatorname{grad}\Lambda)\\
&=\frac{1}{4(\varkappa+1)}\sum_{l=1}^{2}\Big\{\mu_l\Big[-\frac{2ik_l}{\|\xi'\|^2}+\sum_{m=1}^{2}\Big(\frac{ik_m}{\|\xi'\|^2}\Big(3E_m^{-1}\frac{\xi_m^2}{\|\xi'\|^2}-1\Big)\\
&\quad-E_m^{-2}\frac{\xi_m^2}{\|\xi'\|^3}\Big(\frac{\partial E_m}{\partial x_m}\xi_m^{-1}-\frac{3}{4}\frac{\partial E_m}{\partial x_l}\xi_l^{-1}\Big)\Big)\\
&\quad+\frac{1}{2}\sum_{m,p=1}^{2}E_m^{-1}E_p^{-1}\frac{\partial E_m}{\partial x_p}\Big(1+3E_m^{-1}\frac{\xi_m^2}{\|\xi'\|^2}\Big)\frac{\xi_p}{\|\xi'\|^3}\Big]\\
&\quad+\frac{1}{2}\sum_{m=1}^{2}\frac{\partial\mu_l}{\partial x_m}E_m^{-1}\frac{\xi_m}{\|\xi'\|^3}+N\frac{k_l}{\|\xi'\|}\Big(\frac{3}{2}E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}-2\Big)\Big\}
\end{aligned}\tag{2.20}
$$

We obtain the formulae for the next to the principal symbols of the operators $\gamma_3\Pi_\varkappa\operatorname{grad}\Lambda$ and $\gamma_4\Pi_\varkappa\operatorname{grad}\Lambda$ using the boundary conditions (0.6), (0.7) and also the equalities (2.19) and (2.20).

Thus we get

$$
\begin{aligned}
&\sigma_1(\gamma_3\Pi_\varkappa \operatorname{grad}\Lambda)\\
&=\frac{1}{(\varkappa+1)}\Big\{E_1^{-1/2}\sum_{l=1}^{2}\Big[-\frac{1}{4}E_l^{-2}\frac{\xi_l}{\|\xi'\|^3}\Big(\frac{\partial E_l}{\partial x_l}\xi_1+\frac{\partial E_l}{\partial x_1}\xi_l\Big)\\
&\quad-\frac{1}{4}\frac{ik_l}{\|\xi'\|^2}\Big(\xi_l-3E_l^{-1}\frac{\xi_1\xi_l^2}{\|\xi'\|^2}\Big)-\frac{1}{2}\frac{ik_1\xi_1}{\|\xi'\|^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}\sum_{m=1}^{2}E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\left(1+3E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\frac{\xi_1\xi_m}{\|\xi'\|^3}\Bigg],\\
&E_2^{-1/2}\sum_{l=1}^{2}\Bigg[-\frac{1}{4}E_l^{-2}\frac{\xi_l}{\|\xi'\|^3}\left(\frac{\partial E_l}{\partial x_l}\xi_2+\frac{\partial E_l}{\partial x_2}\xi_l\right)\\
&-\frac{1}{4}\frac{ik_l}{\|\xi'\|^2}\left(\xi_l-3E_l^{-1}\frac{\xi_2\xi_l^2}{\|\xi'\|^2}\right)-\frac{1}{2}\frac{ik_2\xi_2}{\|\xi'\|^2}\\
&+\frac{1}{8}\sum_{m=1}^{2}E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\left(1+3E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\frac{\xi_2\xi_m}{\|\xi'\|^3}\Bigg],\\
&\sum_{l=1}^{2}\Bigg[k_l\left(-\frac{1}{8}+\frac{3}{8}E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\frac{1}{\|\xi'\|^2}+\frac{1}{8}E_l^{-2}\frac{\partial E_l}{\partial x_l}\frac{i\xi_l}{\|\xi'\|^3}\\
&-\frac{1}{16}\sum_{m=1}^{2}E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\left(1+3E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right)\frac{i\xi_m}{\|\xi'\|^3}\Bigg]\Bigg\}^t,
\end{aligned}\tag{2.21}
$$

$$
\begin{aligned}
&\sigma_1(\gamma_4\Pi_\varkappa \operatorname{grad}\Lambda)\\
&=\frac{1}{(\varkappa+1)}\Bigg\{\frac{1}{2}\sum_{l=1}^{2}\frac{k_l}{\|\xi'\|}\left(E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}-1\right),\\
&E_1^{-1/2}\sum_{l=1}^{2}\Bigg[-\frac{1}{4}E_l^{-2}\frac{\xi_l}{\|\xi'\|^4}\left(\frac{\partial E_l}{\partial x_l}\xi_1+\frac{\partial E_l}{\partial x_1}\xi_l\right)+\frac{1}{4}ik_lE_l^{-1}\frac{\xi_l^2\xi_1}{\|\xi'\|^5}\\
&+\frac{1}{8}\sum_{m=1}^{2}E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\left(1+4\frac{\xi_l^2}{\|\xi'\|^2}\right)\frac{\xi_1\xi_m}{\|\xi'\|^4}\Bigg],\\
&E_2^{-1/2}\sum_{l=1}^{2}\Bigg[-\frac{1}{4}E_l^{-2}\frac{\xi_l}{\|\xi'\|^4}\left(\frac{\partial E_l}{\partial x_l}\xi_2+\frac{\partial E_l}{\partial x_2}\xi_l\right)+\frac{1}{4}ik_lE_l^{-1}\frac{\xi_l^2\xi_2}{\|\xi'\|^5}\\
&+\frac{1}{8}\sum_{m=1}^{2}E_l^{-1}E_m^{-1}\frac{\partial E_l}{\partial x_m}\left(1+4\frac{\xi_l^2}{\|\xi'\|^2}\right)\frac{\xi_2\xi_m}{\|\xi'\|^4}\Bigg]\Bigg\}^t
\end{aligned}\tag{2.22}
$$

To complete the proof of Theorem 1 we consider the equality (2.2). It implies the following formula for the next to the principal symbol of the ΨDO L_j $(j=1,2,3,4)$

$$
\begin{aligned}
&\sigma_1(L_j)\\
&=\sigma_1(\gamma \operatorname{div}\Pi_\varkappa \operatorname{grad}\Lambda)\\
&\quad-\Big\{\sigma_1(\gamma \operatorname{div}\Lambda_j)\sigma_0(\gamma_j\Pi_\varkappa \operatorname{grad}\Lambda)+\sigma_0(\gamma \operatorname{div}\Lambda_j)\sigma_1(\gamma_j\Pi_\varkappa \operatorname{grad}\Lambda)
\end{aligned}
$$

$$+ \sum_{|\alpha|=1} (-i)\partial_\xi^\alpha \sigma_0(\gamma \mathrm{div}\,\Lambda_j)\partial_x^\alpha \sigma_0(\gamma_j \Pi_\varkappa \mathrm{grad}\,\Lambda)\Big\}$$

$$+ \delta_{2j}\Big\{\sigma_1(\gamma \mathrm{div}\,\Lambda_2)N + \sum_{|\alpha|=1} (-i)\partial_\xi^\alpha \sigma_0(\gamma \mathrm{div}\,\Lambda_2)\partial_x^\alpha N\Big\}$$

$$+ \delta_{4j}\sigma_1(\gamma \mathrm{div}\,\Lambda_4)\bar{i}.$$

Now using the expressions (2.4)–(2.7), (2.9)–(2.22) and the latter formula, we obtain

$$\sigma_1(L_1) = \frac{1}{(\varkappa+2)^2}\sum_{l=1}^{2}\frac{k_l}{\|\xi'\|}\left(E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2} - 1\right) \tag{2.23}$$

$$\sigma_1(L_2) = \frac{1}{\varkappa^2}\sum_{l=1}^{2}\frac{k_l}{\|\xi'\|}\left(1 - E_l^{-1}\frac{\xi_l^2}{\|\xi'\|^2}\right) \tag{2.24}$$

$$\sigma_1(L_3) = \frac{1}{(\varkappa+1)^2}\sum_{l=1}^{2}E_l^{-1}\frac{k_l\xi_l^2}{\|\xi'\|^3} \tag{2.25}$$

$$\sigma_1(L_4) = \frac{1}{(\varkappa+1)^2}\sum_{l=1}^{2}\frac{k_l}{\|\xi'\|} \tag{2.26}$$

The formulae for the principal symbols of the operators K_j follow from the equalities (2.3) and (2.23)–(2.26). This completes the proof of Theorem 1. □

Theorem 1 implies Corollary from Introduction.

Proof of Corollary from Introduction

We denote by $F^{-1}(\sigma_0(K_j))$ the inverse Fourier transformation of $\sigma_0(K_j)$:

$$F^{-1}(\sigma_0(K_j))(x',y') = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\langle y',\xi'\rangle}\sigma_0(K_j))(x',\xi')d\xi'$$

where $\langle y',\xi'\rangle = y_1\xi_1 + \cdots + y_{n-1}\xi_{n-1}$. Let B_j be a ΨDO induced by the symbol $\sigma_0(K_j)(x',\xi')$. It holds

$$B_j\varphi = \int_\Gamma F^{-1}(\sigma_0(K_j))(x',x'-y')\varphi(y')dy' \tag{2.27}$$

(see [9]), where φ is a function defined on Γ.

It is well known that

$$F^{-1}((\xi_1^2+\xi_2^2)^{-1/2}) = (2\pi)^{-1}(x_1^2+x_2^2)^{-1/2} \tag{2.28}$$

$$F^{-1}(\xi_1^2(\xi_1^2+\xi_2^2)^{-3/2}) = (2\pi)^{-1}x_2^2(x_1^2+x_2^2)^{-3/2} \tag{2.29}$$

The proof of Corollary follows directly from Theorem 1 and equalities (2.27)–(2.29). □

3. Proof of the Theorem 2 from Introduction

Since K_j, $j=1,2,3,4$ is a ΨDO of order -1 on Γ it follows that K_j is a compact operator in $L^2(\Gamma)$ and hence its eigenvalues $\lambda_k \to 0$ as $k \to \infty$. Moreover, according to [1], the following formula of asymptotic distribution of these eigenvalues takes place:

$$N(\lambda) := \sum_{|\lambda_k|>\lambda} 1 = \tilde{C}_j\lambda^{-2} + o(\lambda^{-2}), \qquad \text{as} \quad \lambda \to 0, \tag{3.1}$$

where $\tilde{C}_j = (4\pi)^{-2}\int_\Gamma \int_S (\sigma_0(K_j)(x',\xi'))^2 dS\, d\Gamma$, S is a boundary of the unit circle in the plane (ξ_1,ξ_2), i.e. $S = \{\ (\xi_1,\xi_2) \in \mathbb{R}^2 : \xi_1^2+\xi_2^2 = 1\ \}$.

First we calculate the coefficient $\tilde{C}_1$. By Theorem 1 from the Introduction we get

$$(\varkappa+2)^2\int_S (\sigma_0(K_1)(x',\xi'))^2 dS = \int_S \frac{(k_1E_2^{-1}\xi_2^2+k_2E_1^{-1}\xi_1^2)^2}{(E_1^{-1}\xi_1^2+E_2^{-1}\xi_2^2)^3}dS. \tag{3.2}$$

Using the polar coordinates we can rewrite (3.2) in the form:

$$\int_0^{2\pi}\frac{(k_1E_2^{-1}\cos^2\varphi+k_2E_1^{-1}\sin^2\varphi)^2}{(E_1^{-1}\sin^2\varphi+E_2^{-1}\cos^2\varphi)^3}d\varphi. \tag{3.3}$$

We use the following formulae [4, sect. 3.642]:

$$I_1 := \int_0^{\pi/2}\frac{\sin^{n-1}x\cos^{n-1}x\,dx}{(a^2\cos^2x+b^2\sin^2x)^n} = \frac{B\left(\frac{n}{2},\frac{n}{2}\right)}{2(ab)^n}, \tag{3.4}$$

$$I_2 := \int_0^{\pi/2}\frac{\sin^{2n}x\,dx}{(a^2\cos^2x+b^2\sin^2x)^{n+1}} = \frac{(2n-1)!!\pi}{2^{n+1}n!ab^{2n+1}}, \tag{3.5}$$

$$I_3 := \int_0^{\pi/2}\frac{\cos^{2n}x\,dx}{(a^2\cos^2x+b^2\sin^2x)^{n+1}} = \frac{(2n-1)!!\pi}{2^{n+1}n!a^{2n+1}b}, \tag{3.6}$$

where $ab > 0$, $n = 1, 2, \cdots$ and

$$B(x,x) = \frac{1}{2^{2x-2}} \int_0^1 \left(1-t^2\right)^{x-1} dt. \tag{3.7}$$

In particular, for $n = 3$ we have from (3.7) and (3.4)

$$B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\pi}{8}, \qquad I_1 = \frac{\pi}{16\,(ab)^3}. \tag{3.8}$$

For $n = 2$ from (3.5) and (3.6) it follows that

$$I_2 = \frac{3\pi}{2^4 ab^5}, \qquad I_3 = \frac{3\pi}{2^4 a^5 b}. \tag{3.9}$$

Using (3.2) and (3.3), we get

$$(\varkappa+2)^2 \int_S (\sigma_0(K_1)(x',\xi'))^2 dS$$
$$= (\pi/4)\sqrt{E_1E_2}(3k_1^2 + 3k_2^2 + 2k_1k_2).$$

Thus

$$\tilde{C}_1 = (32\pi)^{-1}(\varkappa+2)^{-2} \int_\Gamma \sqrt{E_1E_2}(3k_1^2 + 3k_2^2 + 2k_1k_2)\, d\Gamma.$$

The formulae for the coefficients $\tilde{C}_2, \tilde{C}_3, \tilde{C}_4$ are obtained in the same way.

In view of the equivalence of the j-th BPET to the corresponding boundary integral equation with operator K_j, [13, Theorem 3 of Introduction] the eigenvalues λ_k of the operator K_j and the eigenvalues ω_k of the j-th BPET are related by the equality $\lambda_k = (\alpha_j + \omega_k)/(\omega_k - \varkappa)$. Solving the inequality

$$|\lambda_k| = \left| \frac{\alpha_j + \omega_k}{\alpha_j + \omega_k - (\alpha_j + \varkappa)} \right| > \lambda$$

we get since $\omega_k < \varkappa$ that

$$\alpha_j + \omega_k < -\frac{(\alpha_j+\varkappa)\,\lambda}{1-\lambda} \quad \text{or} \quad \alpha_j + \omega_k > \frac{(\alpha_j+\varkappa)\,\lambda}{1+\lambda}.$$

Define the number $\sharp\,\{\omega_k : \omega_k \in A\} := \sum_{\omega_k \in A} 1$. Then

$$\sharp\{\omega_k : \omega_k \notin\,]\alpha_j - (\alpha_j+\varkappa)\lambda(1-\lambda)^{-1},\ \alpha_j + (\alpha_j+\varkappa)\lambda(1+\lambda)^{-1}\,[\,\}$$
$$= \tilde{C}_j\lambda^{-2} + o(\lambda^{-2}). \tag{3.10}$$

Let $\varepsilon = (\alpha_j + \varkappa)\,\lambda\,(1-\lambda)^{-1}$ then $\lambda^{-1} = (\alpha_j + \varkappa)\,\varepsilon^{-1} + 1$ and the equality (3.10) can be rewritten in the form

$$\sharp\,\{\omega_k : \omega_k \notin \,]\alpha_j - \varepsilon\,,\ \alpha_j + \varepsilon - \varepsilon_1\,[\,\} = C_j\varepsilon^{-2} + o(\varepsilon^{-2}), \qquad (3.11)$$

where $\varepsilon_1 = 2\varepsilon^2(\alpha_j + \varkappa + 2\varepsilon)^{-1}$ and $C_j = \tilde{C}_j(\alpha_j + \varkappa)^2$, $j = 1,2,3,4$.

Let now $\varepsilon = (\alpha_j + \varkappa)\,\lambda\,(1+\lambda)^{-1}$ then $\lambda^{-1} = (\alpha_j + \varkappa)\,\varepsilon^{-1} - 1$ and the equality (3.10) can be rewritten in the form

$$\sharp\,\{\omega_k : \omega_k \notin \,]\alpha_j - \varepsilon - \varepsilon_2\,,\ \alpha_j + \varepsilon\,[\,\} = C_j\varepsilon^{-2} + o(\varepsilon^{-2}). \qquad (3.12)$$

where $\varepsilon_2 = 2\varepsilon^2(\alpha_j + \varkappa - 2\varepsilon)^{-1}$.

Subtracting (3.12) from (3.11) we get

$$\begin{aligned}\sharp\{\omega_k : \omega_k \,\in\, &[\alpha_j - \varepsilon - \varepsilon_2\,,\ \alpha_j - \varepsilon\,] \cup [\alpha_j + \varepsilon - \varepsilon_1\,,\ \alpha_j + \varepsilon]\,\} \\ &= o(\varepsilon^{-2}).\end{aligned}$$

It follows from this equality and from (3.11) that

$$\sharp\,\{\omega_k : \omega_k \notin \,]\alpha_j - \varepsilon\,,\ \alpha_j + \varepsilon\,[\,\} = C_j\varepsilon^{-2} + o(\varepsilon^{-2}).$$

This completes the proof of Theorem 2. $\square$

4. On spectral properties of operator $\mathrm{div}\Delta_0^{-1}\mathrm{grad}$

The spectrum $\mathrm{sp}(T)$ of the operator $T = \mathrm{div}\Delta_0^{-1}$ grad in $L^2\,(\Omega)\,/\mathbb{R}$ has been considered in [2]. It has been proved that $\mathrm{sp}(T) \subseteq]0,1]$. It has been proved also that the point 1 is an eigenvalue of T.

It is more naturally from the point of view of application (see [12], [13]) to consider the restriction of T to the space $\{u \in L^2\,(\Omega)\,/\mathbb{R} :\ \Delta u = 0 \text{ in } \Omega\}$. By [13, Theorem 1, p.270] the restriction equals $\frac{1}{2}\,(I - K)$, where K is a compact operator. Hence the spectrum of the restriction is discrete and $1/2$ is the unique limit point of its eigenvalues. It follows immediately from [13, Theorem 1, p.270] that the spectrum of the restriction $\subset]0,1[$.

Remark. (Correction to an earlier paper). There is an error in the proof of the following assertion [13, p.272]: the point $\lambda = -1$ is not an eigenvalue of the operator K and consequently the equation $\mathrm{div}\Delta_0^{-1}\mathrm{grad}p = p$ has a nontrivial solution $p \neq 0$, which is a harmonic function. (In [13, p.272] it was erroneously written $p{+}\mathrm{div}\Delta_0^{-1}\mathrm{grad}p = 0$, i.e. $\mathrm{div}\Delta_0^{-1}\mathrm{grad}p = -p$).

To prove the assertion suppose now that it is false. Then the point 1 is an eigenvalue of the restriction of T and a nontrivial harmonic function

p is a corresponding eigenfunction, i.e. $\mathrm{div}\Delta_0^{-1}\mathrm{grad}p = p$, $p \neq 0$. Denote by u the function $\Delta_0^{-1}\mathrm{grad}p$. Then $u \in [C^\infty(\overline{\Omega})]^3$ and satisfies the equations $\Delta u - \mathrm{graddiv}u = 0$ in Ω and $u = 0$ on Γ. Since $\Delta u - \mathrm{graddiv}u = -\mathrm{rot}^2 u$, it follows that $\mathrm{rot}^2 u = 0$. Since $0 = (\mathrm{rot}^2 u, u) = \|\mathrm{rot}u\|^2_{[L^2(\Omega)]^3}$, $\mathrm{rot}u = 0$, i.e. $u \in \ker(\mathrm{rot})$. By [17, Appendix I, Proposition 1.1] $u = \mathrm{grad}\Phi$ and $\Phi \in C^\infty(\overline{\Omega})$. Since $\mathrm{div}\Delta_0^{-1}\mathrm{grad}p = \mathrm{div}u = p$ and $\Delta p = 0$, $\Delta^2\Phi = 0$ in Ω. Since $u = \mathrm{grad}\Phi$ and $u = 0$ on Γ, we obtain $\partial\Phi/\partial N = 0$ and $\Phi =$const on Γ. It follows that $\Phi =$const is a unique solution to the boundary value problem $\Delta^2\Phi = 0$ in Ω, $\partial\Phi/\partial N = 0$, $\Phi =$const on Γ. Hence $u = \mathrm{grad}\Phi = 0$ in Ω and $p = \mathrm{div}u = 0$ in Ω, a contradiction.

Acknowledgment The authors thank Professor Y. Giga and the anonymous referee for useful suggestions. The first author was supported by the Rashi Foundation.

References

[1] Birman M.Sh. and Solomyak M.Z., *Asymptotics of the spectrum of pseudo-differential operators with anisotropic-homogeneous symbols.* Vestnik Leningrad Univ. Math. **10** (1982), 237–247.

[2] Gaultier M. and Lazaun M., *Quelques propriétés du spectre de l'opérateur* $-\mathrm{div}\,(-\Delta)^{-1}\,\mathrm{grad}p$ *dans* $L^2(\Omega)/\mathbb{R}$. C. R. Acad. Sci. Paris, **315**, Série I, (1992), 799–802.

[3] Giga Y., *Analyticity of the semigroup generated by the Stokes operator in* L_r *spaces.* Mathematische Zeitschrift **178** (1981), 297–329.

[4] Gradshtein I.S. and Ryzhik I.M., *Tables of integrals, sums, series and products.* Academic Press, New York, 1980.

[5] Grubb G., *Singular Green operators and their spectral asymptotics.* Duke Math. J. **51**(3) (1984), 477–528.

[6] Grubb G. and Geymonat G., *The essential spectrum of elliptic systems of mixed order.* Math. Ann. **227** (1977), 247–276.

[7] Grubb G. and Solonnikov V., *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods.* Mathematica Scandinavica, **69** (1991), 217–290.

[8] Gurtin M.E., *The linear theory of elasticity.* In: Flugge S. (ed.) Handbuch der Physik, Vol. YIa/2 pp.1–295 (ed. Truesdell, C.), Springer Berlin Heidelberg New York, 1972.

[9] Hörmander L., *Pseudo-differential operators.* Comm. Pure Appl. Math., **18**(3) (1965), 501–517.

[10] Hörmander L., *Pseudo-differential operators and non-elliptic boundary-value problems.* Ann. of Math. **83** (1966), 129–209.

[11] Hsiao G.C. and Wendland W.L., *On a boundary integral method for some exterior*

problems in elasticity. Proc. Tbilisi Univ. **257** (1985), 32–60.

[12] Kozhevnikov A.N., *On the operator of the linearized steady-state Navier-Stokes problem.* Math. USSR Sb., 53(1) (1986), 1–16.

[13] Kozhevnikov A.N., *The basic boundary value problems of the static elasticity theory and their Cosserat spectrum.* Mathematische Zeitschrift, **213** (1993), 241–274.

[14] Kupradze V.D., Gegelia T.G., Burchuladze T.V. and Basheleishvili H.O., *Three-dimensional problems of elasticity and thermoelasticity.* North-Holland Publishing Co., Amsterdam New York, 1979.

[15] Mikhlin S.G., *The spectrum of a family of operators in the theory of elasticity.* Russian Math. Surveys, **28** (1973), 45–88.

[16] Mikhlin S.G., *The problem of the minimum of a quadratic functional.* Holden-Day, San Francisko, Calif., 1965.

[17] Temam R., *Navier-Stokes equations. Theory and numerical analysis.* North-Holland, Amsterdam New York Oxford (1979).

Alexander Kozhevnikov
Department of Mathematics & Computer Sciences
University of Haifa
Haifa 31905, Israel
E-mail: kogevn@mathcs2.haifa.ac.il

Tatiana Skubachevskaya
Department of Applied Mathematics
Faculty of Applied Mathematics
& Informational Technologies
Moscow State Aviation Technological University
Petrovka 27, Moscow 103767, Russia