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Monotoneity and homogeneity of Picard dimensions for signed radial densities

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Abstract. The Picard dimension $\dim P$ of a locally Hölder continuous function P on the punctured unit ball in the d-dimensional Euclidean space ($d \geq 2$) at the origin is the limit of the cardinal number of the set of extremal rays of the cone of nonnegative solutions of the stationary Schrödinger equation $(-\Delta + P(x))u(x) = 0$ on the punctured ball $0 < |x| < a$ with vanishing boundary values on the sphere $|x| = a$ as $a \downarrow 0$. In this paper the monotoneity of $\dim P$ in radial P in the sense that $\dim P \leq \dim Q$ for radial functions P and Q with $P \leq Q$ and the homogeneity of $\dim P$ for radial functions P in the sense that $\dim(cP) \geq \dim P$ $(0 < c \leq 1)$ or equivalently $\dim(cP) \leq \dim P$ $(c \geq 1)$ for radial P are established.

Key words: Picard dimension, Picard principle, Schrödinger equation.

1. Introduction

The purpose of this paper is to contribute to the study on structures of spaces of positive solutions of time independent Schrödinger equations around isolated singularities of their potentials. By translations we may restrict ourselves to the case where isolated singularities of potentials are situated over the origin 0 of the Euclidean space $\mathbf{R}^d$ of dimension $d \geq 2$. Here we denote by Ω_a the punctured ball $0 < |x| < a$ and Γ_a the sphere $|x| = a$ centered at the origin 0 of radius $a > 0$. A real valued locally Hölder continuous function $P(x) = P(x_1, \cdots, x_d)$ defined on $\Omega_a \cup \Gamma_a$ will be referred to as a *density* on $\Omega_a \cup \Gamma_a$, which is viewed as having an isolated singularity at the origin 0, either removable or essential. We consider a stationary Schrödinger equation whose potential is a density $P(x)$ on $\Omega_a \cup \Gamma_a$:

$$(-\Delta + P(x))u(x) = 0 \quad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right). \tag{1}$$

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By a solution u of (1) on Ω_a we mean a real valued C^2 function u satisfying the equation (1) on Ω_a. We denote by $P(\Omega_a)$ the space of all solutions of (1) on Ω_a, which forms a locally convex linear topological space equipped with the topology given by the uniform convergence on each compact subset of Ω_a. We denote by $PP(\Omega_a)$ the subclass of $P(\Omega_a)$ consisting of nonnegative members in $P(\Omega_a)$. The first P in $PP(\Omega_a)$ indicates the dependence of the class on the density P and the second P stands for the initial of the term positive (nonnegative) so that the class associated with another density Q is denoted by $QP(\Omega_a)$.

Since we are interested solely in the effect on the class $PP(\Omega_a)$ of the singular behavior of P at the origin 0, eliminating the influence on $PP(\Omega_a)$ of the boundary behavior of each solution in $PP(\Omega_a)$ on the relative boundary Γ_a of Ω_a, we consider the subclass

$$PP(\Omega_a;\Gamma_a)=\{u\in PP(\Omega_a)\cap C(\Omega_a\cup\Gamma_a):u\mid\Gamma_a=0\},$$

which forms a closed positive cone in $PP(\Omega_a)$ as a consequence of the Harnack inequality. We wish to study the cone $PP(\Omega_a;\Gamma_a)$ from the view point of its extremal rays. For the purpose it is convenient to consider the convex subset $PP_1(\Omega_a;\Gamma_a)$ which is the intersection of $PP(\Omega_a;\Gamma_a)$ with a closed hyperplane given by the equation $\ell(u)=1$ where ℓ is any strictly positive continuous linear functional on the closed linear span of $PP(\Omega_a;\Gamma_a)$:

$$PP_1(\Omega_a;\Gamma_a)=\{u\in PP(\Omega_a;\Gamma_a):\ell(u)=1\}.$$

We cannot exclude the trivial case $PP_1(\Omega_a;\Gamma_a)=\emptyset$ which occurs if and only if $PP(\Omega_a;\Gamma_a)=\{0\}$ which is seen to be equivalent to $PP(\Omega_a)=\{0\}$. Although $PP(\Omega_a;\Gamma_a)=\mathbf{R}^+PP_1(\Omega_a;\Gamma_a)$ for any ℓ unless $PP(\Omega_a;\Gamma_a)=\{0\}$, where $\mathbf{R}$ is the real number field and $\mathbf{R}^+=\{\xi\in\mathbf{R}:\xi\geq 0\}$, the convex structure of $PP_1(\Omega_a;\Gamma_a)$ does depend upon the choice of ℓ. However it is easy to see that the set theoretic structure of the set ex. $PP_1(\Omega_a;\Gamma_a)$ of extremal points in $PP_1(\Omega_a;\Gamma_a)$ is uniquely determined regardless how we choose ℓ and therefore we adopt the following special ℓ for a technical reason:

$$\ell(u)=-\frac{a}{s(\Gamma_a)}\int_{\Gamma_a}\frac{\partial u}{\partial n}ds$$

where ds is the area element on Γ_a, $s(\Gamma_a)$ the area of Γ_a and $\partial/\partial n$ the outer normal derivative on Γ_a considered in $\Omega_a\cup\Gamma_a$. Since each function in $PP(\Omega_a;\Gamma_a)$ or its closed linear span is of class C^2 on $\Omega_a \cup \Gamma_a$ as a consequence of the reflection principle, the above ℓ is certainly a well defined strictly positive continuous linear functional on the closed linear span of $PP(\Omega_a;\Gamma_a)$.

The Harnack principle yields that the convex set $PP_1(\Omega_a;\Gamma_a)$ is compact. As is well known compact convex sets are completely determined by sets of their all extremal points: the Krein-Milman theorem (cf. e.g. [7]) assures that

$$PP_1(\Omega_a;\Gamma_a) = \overline{\mathrm{co}}\,[\mathrm{ex.}\, PP_1(\Omega_a;\Gamma_a)] \tag{2}$$

where $\overline{\mathrm{co}}[X]$ is the closed convex hull of a subset X of $PP_1(\Omega_a;\Gamma_a)$; more precisely, the Choquet theorem (cf. e.g. [25]) implies the existence (cf. e.g. [21]) of a bijective correspondence $u \leftrightarrow \mu$ between $PP_1(\Omega_a;\Gamma_a)$ and the set of probability measures on ex. $PP_1(\Omega_a;\Gamma_a)$ such that

$$u = \int_{\mathrm{ex.}\, PP_1(\Omega_a;\Gamma_a)} v\, d\mu(v). \tag{3}$$

Thus the set ex. $PP_1(\Omega_a;\Gamma_a)$ is essential for the class $PP(\Omega_a;\Gamma_a)$. Following Bouligand the cardinal number $\#(\mathrm{ex.}\, PP_1(\Omega_a;\Gamma_a))$ of the set ex. $PP_1(\Omega_a;\Gamma_a)$ is referred to as the *Picard dimension* of the density P on Ω_a at the origin, $\dim(P,\Omega_a)$ in notation, i.e.

$$\dim(P,\Omega_a) = \#(\mathrm{ex.}\, PP_1(\Omega_a;\Gamma_a))$$

and we say that the *Picard principle* is valid for P on Ω_a at the origin 0 if $\dim(P,\Omega_a) = 1$ (cf. [5]). The reference to the name Picard comes from his classical result in 1923 that $\dim(P,\Omega_a) = 1$ for $P \equiv 0$, the classical harmonic case, formulated in our present setting; the result is actually found earlier in 1903 by Bôcher (cf. [8]). The cardinal number $\dim(P,\Omega_a)$ does not completely describe the dependence of the set theoretic behaviors of positive solutions of (1) at the origin 0 on the singular behavior of P at 0 since it also depends on the choice of $a > 0$. However it is seen that $\dim(P,\Omega_a)$ is a fixed cardinal number for all sufficiently small $a > 0$ (cf. [20], [17]; see Appendix at the end of this paper) and hence we can define the *Picard dimension* $\dim P$ of P at 0 by

$$\dim P = \lim_{a \downarrow 0} \dim(P,\Omega_a).$$

In the present paper we study positive solutions of (1) from the view point of the Picard dimension $\dim P$. It is known (cf. [22]) that the range of the mapping $P \mapsto \dim P$ covers the set of all finite cardinal numbers $0, 1, 2, \cdots$, the cardinal number $\aleph_0$ of the countably infinite set and the cardinal number $\aleph$ of continuum. We are mainly concerned with the following two problems in this paper.

1. Problem of Monotoneity: Does $P \leq Q$ imply $\dim P \leq \dim Q$?
2. Problem of Homogeneity: Does $\dim(cP) = \dim P$ hold for every real constant $c > 0$?

We say that a density $P(x)$ defined on an Ω_a is *radial* if $P(x)$ depends only on $|x|$. As a contribution of positive direction to the problem of monotoneity we have the following result ([19], [3]):

Theorem A *If P and Q are nonnegative radial densities with $P \leq Q$ on a punctured ball about* 0, *then the inequality* $\dim P \leq \dim Q$ *holds.*

The result is no longer true if P and Q are not supposed to be radial even if they are nonnegative. For example, there exists a nonnegative density Q such that $\dim Q = 1$ and $Q \geq P$ for any given nonnegative density P with $\dim P \geq 2$ (cf. [23], [24], [29], [2]). Concerning the problem of homogeneity we have the following positive result ([11], [3]):

Theorem B *If P is a nonnegative radial density on a punctured ball about* 0, *then* $\dim(cP) = \dim P$ *for every constant* $c > 0$.

Despite the case of the problem of monotoneity it seems quite difficult to prove or disprove the homogeneity of Picard dimensions for nonnegative but not necessarily radial densities, for which nothing has been published yet. The result which will be mentioned below somehow suggests the extreme diversity of the homogeneity question. The *main purpose* of this paper is to discuss what happens to Theorems A and B if we remove the assumption of nonnegativeness of radial densities. We say that a density P is *signed* if P is not necessarily of constant sign. As a lucky case we have a complete generalization of Theorem A as follows:

Theorem 1 *If P and Q are signed radial densities with $P \leq Q$ on a punctured ball about* 0, *then the inequality* $\dim P \leq \dim Q$ *holds.*

It is interesting to observe the following direct consequence of the above result for the reason that many densities appearing in the nuclear physics and engineering are negative and radial about their isolated singularities: If $P \leq 0$, then $\dim P$ is either 0 or 1. This follows from the inequality $0 \leq \dim P \leq \dim 0 = 1$. Theorem B, however, cannot be generalized to signed radial densities in its original formulation since we have a negative radial density P on Ω_1 such that $\dim P > \dim(cP)$ for every constant $c > 1$ ([9]; see §7 and §8 of the present paper). Hence the following weak form is the best possible generalization of Theorem B:

Theorem 2 *If P is a signed radial density on a punctured ball about* 0*, then* $\dim P \leq \dim(cP)$ *for every constant c with $0 < c \leq 1$, or equivalently,* $\dim P \geq \dim(cP)$ *for every constant c with $c > 1$.*

This result may certainly be viewed as a generalization of Theorem B. In fact, let P be any nonnegative radial density on a punctured ball about 0 and c any constant with $0 < c \leq 1$. Theorem 2 implies $\dim P \leq \dim(cP)$. On the other hand, since $P \geq cP$ by virtue of the fact $P \geq 0$, using Theorem A or 1 we have $\dim P \geq \dim(cP)$ and therefore we can conclude that $\dim P = \dim(cP)$. The case of $c \geq 1$ can be treated similarly.

The decisive factor which makes Theorems 1 and 2 valid lies in the fact that $\dim P$ takes only three values 0, 1 and $\aleph$ for every radial density P on a punctured ball about 0 (cf. [18], [16]). The proofs of this fact in the cited papers are both based heavily upon the concrete analysis of (3) using the Green's function of (1), i.e. the so-called Martin theory for (1) at 0. In our paper we present a much simpler proof based merely upon the relation (2), and actually a part of it, which is a subsidiary achievement of this paper.

The paper consists of 8 sections including this introduction and the proofs of the main results of this paper, Theorems 1 and 2 mentioned above, will be completed in the final §8 after a long series of auxiliary discussions in §§2–7. We also have an appendix at the end of this paper.

2. Reduction to the unit ball

We consider a punctured ball $\Omega_a = \{x \in \mathbf{R}^d : 0 < |x| < a\}$ and its boundary sphere $\Gamma_a = \{x \in \mathbf{R}^d : |x| = a\}$ centered at the origin 0 of the Euclidean space $\mathbf{R}^d$ of dimension $d \geq 2$ of radius $a > 0$. For simplicity we set $\Omega_1 = \Omega$, the punctured unit ball, and $\Gamma_1 = \Gamma$, the unit sphere. A real valued locally Hölder continuous function $P(x)$ on $\Omega_a \cup \Gamma_a$ will be referred to as a *density* on $\Omega_a \cup \Gamma_a$. With a density P on $\Omega_a \cup \Gamma_a$ we associate the linear space

$$P(\Omega_a) = \{u \in C^2(\Omega_a) : (-\Delta + P(x))u(x) = 0 \text{ on } \Omega_a\}$$

equipped with the topology of locally uniform convergence on Ω_a, i.e. uniform convergence on every compact subset of Ω_a, which forms a locally convex linear topological space. Each function u in $P(\Omega_a)$ is referred to as being *P-harmonic* on Ω_a. If a sequence $\{u_n\}$ in $P(\Omega_a)$ converges to a function u on Ω_a locally uniformly on Ω_a, then $u \in P(\Omega_a)$. We set

$$PP(\Omega_a) = \{u \in P(\Omega_a) : u \geq 0 \text{ on } \Omega_a\}$$

which is a closed subset of $P(\Omega_a)$. Here the first (second, resp.) P in $PP(\Omega_a)$ refers to the density P (the initial of 'positive' meaning nonnegative, resp.). Concerning the class $PP(\Omega_a)$ the Harnack inequality is important: For any compact subset K of Ω_a there exists a constant $c(K) \geq 1$ such that $c(K)^{-1}u(\xi) \leq u(\eta) \leq c(K)u(\xi)$ for any u in $PP(\Omega_a)$ and any pair (ξ, η) of points in K ([14]). The Harnack inequality is equivalent to the Harnack principle: If $\{u_n\}$ is a nondecreasing sequence in $PP(\Omega_a)$ convergent at a point in Ω_a, then $\{u_n\}$ converges to a u in $PP(\Omega_a)$ locally uniformly on Ω_a. We then consider

$$PP(\Omega_a; \Gamma_a) = \{u \in PP(\Omega_a) \cap C(\Omega_a \cup \Gamma_a) : u \mid \Gamma_a = 0\}.$$

If a sequence $\{u_n\}$ in $PP(\Omega_a; \Gamma_a)$ converges to a function u in $PP(\Omega_a)$ in the topology of $P(\Omega_a)$, then $u \in PP(\Omega_a; \Gamma_a)$ and $\{u_n\}$ converges to u locally uniformly on $\Omega_a \cup \Gamma_a$. In particular, $PP(\Omega_a; \Gamma_a)$ is a closed subset of $PP(\Omega_a)$ and hence of $P(\Omega_a)$. To see this we observe the existence of a $b \in (0, a)$ enough close to a such that there exists a P-harmonic function w on $\Omega_a \backslash (\Omega_b \cup \Gamma_b)$ with boundary values 0 on Γ_a and 1 on Γ_b. By the minimum principle, $u_n \leq \|u_n; L_\infty(\Gamma_b)\| w$ on $(\Omega_a \cup \Gamma_a) \backslash \Omega_b$ and a fortiori $u \leq \|u; L_\infty(\Gamma_b)\| w$ on $(\Omega_a \cup \Gamma_a) \backslash \Omega_b$ which shows that $u \in PP(\Omega_a; \Gamma_a)$. Again by the minimum principle, $|u_n - u| \leq \|u_n - u; L_\infty(\Gamma_b)\| w$ on $(\Omega_a \cup \Gamma_a) \backslash \Omega_b$, which proves the above assertion.

We denote by j the natural symmetric selfmapping of the double $\widehat{\Omega}_a$ of Ω_a about Γ_a onto itself. Let $\widehat{P}$ be the natural extension of a density P on Ω_a to $\widehat{\Omega}_a$. Take any P-harmonic function u on $\Omega_a \backslash (\Omega_c \cup \Gamma_c)$ having boundary values 0 on Γ_a, where $0 < c < b < a$ and consider the $\widehat{P}$-harmonic function $\widehat{u}$ on a subregion $\{(\Omega_a \cup \Gamma_a)\backslash(\Omega_b \cup \Gamma_b)\} \cup j\{(\Omega_a \cup \Gamma_a)\backslash(\Omega_b \cup \Gamma_b)\}$ with boundary values $\widehat{u} = u$ on Γ_b and $\widehat{u} = -u \circ j$ on $j(\Gamma_b)$. Since $\widehat{u} + \widehat{u} \circ j \equiv 0$, we see that $\widehat{u} = 0$ on Γ_a. This shows that $\widehat{u} \equiv u$ on $(\Omega_a \cup \Gamma_a)\backslash\Omega_b$. The fact that $\widehat{u}$ is of class C^2 assures that u is of class C^2 on $(\Omega_a \cup \Gamma_a)\backslash(\Omega_c \cup \Gamma_c)$. We have seen, in particular, that $\mathrm{cls}[PP(\Omega_a;\Gamma_a)] \subset C^2(\Omega_a \cup \Gamma_a)$, where $\mathrm{cls}[X]$ is the closed linear span of $X \subset P(\Omega_a)$. Thus the following functional is well defined on $\mathrm{cls}[PP(\Omega_a;\Gamma_a)]$:

$$\ell_a(u) = -\frac{a}{s(\Gamma_a)} \int_{\Gamma_a} \frac{\partial u}{\partial n} ds \tag{4}$$

for $u \in \mathrm{cls}[PP(\Omega_a;\Gamma_a)]$, where ds is the area element on Γ_a, $s(\Gamma_a)$ the area of Γ_a and $\partial/\partial n$ the outer normal derivation on Γ_a considered for $\Omega_a \cup \Gamma_a$. Using the polar coordinate (r,ω) of $x \in \mathbf{R}^d\backslash\{0\}$ where $r = |x|$ and $\omega = |x|^{-1}x \in \Gamma = \Gamma_1$, (4) can be written as

$$\ell_a(u) = -\frac{a}{\omega_d} \int_{\Gamma} \left[\frac{\partial}{\partial r} u(r\omega)\right]_{r=a} d\omega$$

where $d\omega$ is the area element on Γ and $\omega_d = \omega(\Gamma)$ is the area of Γ. It is easily seen that ℓ_a is positive and linear on $\mathrm{cls}[PP(\Omega_a;\Gamma_a)]$. Take the function w introduced in the first paragraph. Take any u in $\mathrm{cls}[PP(\Omega_a;\Gamma_a)]$. By the minimum principle, we have $|u| \leq \|u; L_\infty(\Gamma_b)\| w$ on $(\Omega_a \cup \Gamma_a)\backslash\Omega_b$ and hence $|\partial u/\partial n| \leq -\|u; L_\infty(\Gamma_b)\|(\partial w/\partial n)$ on Γ_a, which implies $|\ell_a(u)| \leq \ell_a(w)\|u; L_\infty(\Gamma_b)\|$. This proves that ℓ_a is continuous on $\mathrm{cls}[PP(\Omega_a;\Gamma_a)]$. Suppose that $\ell_a(u) = 0$ for a nonnegative P-harmonic function u on $\Omega_a\backslash(\Omega_b \cup \Gamma_b)$ with boundary values 0 on Γ_a. Since $-\partial u/\partial n \geq 0$ on Γ_a, we see that $\partial u/\partial n = 0$ on Γ_a. By the uniqueness of solutions of the Cauchy problem we conclude that $u \equiv 0$ on $\Omega_a\backslash(\Omega_b \cup \Gamma_b)$. We have thus seen that ℓ_a is strictly positive. We set

$$PP_1(\Omega_a;\Gamma_a) = \{u \in PP(\Omega_a;\Gamma_a) : \ell_a(u) = 1\},$$

which can be empty. We now maintain that $PP_1(\Omega_a;\Gamma_a)$ forms a compact convex subset of $P(\Omega_a)$. For the proof, again take the function w introduced in the first paragraph. Let $\{u_n\}$ be an arbitrary sequence in $PP_1(\Omega_a;\Gamma_a)$. We only have to show that $\{u_n\}$ contains a convergent subsequence in $PP_1(\Omega_a;\Gamma_a)$. Fix a point $\zeta \in \Gamma_b$. We first assert that $\sup_{n\in\mathbf{N}} u_n(\zeta) < \infty$ where $\mathbf{N}$ is the set of positive integers. If this is not the case, then, by choosing a subsequence if necessary, we may assume that $u_n(\zeta) \to \infty$. The Harnack inequality yields that $\lambda_n = \inf_{\Gamma_b} u_n \to \infty$. By the minimum principle, we see that $u_n \geq \lambda_n w$ on $(\Omega_a \cup \Gamma_a) \backslash \Omega_b$ and $1 = \ell_a(u_n) \geq \lambda_n \ell_a(w) > 0$, a contradiction. Hence $\{u_n\}$ is locally bounded on Ω_a. By the Poisson type representation of each u_n on every small ball in Ω_a we see that $\{u_n\}$ is equicontinuous at each point of Ω_a. Thus $\{u_n\}$ forms a normal family and the assertion follows.

For each density P on $\Omega_a \cup \Gamma_a$ $(a > 0)$ we define the *Picard dimension* $\dim(P, \Omega_a)$ of P considered on Ω_a to be the cardinal number $\#(\text{ex.}\, PP_1(\Omega_a; \Gamma_a))$ of the set $\text{ex.}\, PP_1(\Omega_a; \Gamma_a)$ of extremal points of the compact convex set $PP_1(\Omega_a; \Gamma_a)$:

$$\dim(P, \Omega_a) = \#(\text{ex.}\, PP_1(\Omega_a; \Gamma_a)).$$

For a density P on a punctured ball centered at the origin 0 there exists a $c > 0$ such that $\Omega_a \cup \Gamma_a$ is contained in the punctured ball for $0 < a < c$. We will see later that there exists a $b \in (0, c)$ such that $\dim(P, \Omega_a)$ is a fixed constant cardinal number for every $a \in (0, b]$. A proof of this is appended at the end of this paper. But, since the fact will be used only for densities P which are radial, another proof of independent interest will be given for such densities in §8. Hence we can define the *Picard dimension* $\dim P$ of P at the origin 0 by

$$\dim P = \lim_{a \downarrow 0} \dim(P, \Omega_a),$$

which describes, in essence, how many positive solutions of the Schrödinger equation (1) at the origin exist.

It is convenient to reduce the study of densities on $\Omega_a \cup \Gamma_a$ to that on $\Omega \cup \Gamma = \Omega_1 \cup \Gamma_1$, the closed unit punctured ball. Take any density P on $\Omega_a \cup \Gamma_a$ and set

$${}^aP(x) = a^2 P(ax)$$

for every x in $\Omega \cup \Gamma$. It is clear that ${}^aP$ is a density on $\Omega \cup \Gamma$. We denote by u_a the a-dilation of u: if u is a function on Ω_a or on $\Omega_a \cup \Gamma_a$, then u_a is the function on Ω or on $\Omega \cup \Gamma$ given by

$$u_a(x) = u(ax)$$

for every x in Ω or in $\Omega \cup \Gamma$. The mapping $u \mapsto u_a$ gives a natural isomorphism of $P(\Omega_a)$ onto ${}^aP(\Omega)$ as linear topological spaces; it is a bicontinuous bijection preserving positiveness, addition and positive scalar multiplication of $PP(\Omega_a)$ onto ${}^aPP(\Omega)$, and similarly of $PP(\Omega_a; \Gamma_a)$ onto ${}^aPP(\Omega; \Gamma)$. By the choice of normalizing functional ℓ_a we see that $u \mapsto u_a$ is also a bijection preserving convex combinations of $PP_1(\Omega_a; \Gamma_a)$ onto ${}^aPP_1(\Omega; \Gamma)$, which maps ex. $PP_1(\Omega_a; \Gamma_a)$ onto ex. ${}^aPP_1(\Omega; \Gamma)$ bijectively. Hence

$$\dim(P, \Omega_a) = \dim({}^aP, \Omega).$$

For this reason we mainly study densities P on Ω so that the compact convex set $PP_1(\Omega; \Gamma) = \{u \in PP(\Omega; \Gamma) : \ell(u) = 1\}$ will be centrally considered where $\ell = \ell_1$ is given by

$$\ell(u) = -\frac{1}{\omega_d}\int_\Gamma \frac{\partial u}{\partial n} d\omega = -\frac{1}{\omega_d}\int_\Gamma \left[\frac{\partial}{\partial r} u(r\omega)\right]_{r=1} d\omega.$$

3. Singularity indices of limit form

A function f on Ω_a $(0 < a \le \infty)$ is said to be *radial* if $f(r\omega)$ $(r \in (0, a), \omega \in \Gamma)$ depends only on r. In this case we define a function $f(r)$ on $(0, a)$ by $f(r) = f(r\omega)$. Conversely, a function $f(r)$ on $(0, a)$ gives rise to a radial function $f(x)$ on Ω_a defined by $f(x) = f(|x|)$. Hereafter in this paper all the densities $P(x)$ on $\Omega \cup \Gamma$ in consideration will be supposed to be radial unless otherwise is explicitly stated. A radial density $P(x)$ on $\Omega \cup \Gamma$ determines and is determined by a locally Hölder continuous function $P(r)$ on $(0, 1]$ such that $P(x) = P(|x|)$. It is convenient to view $P(x)$ ($P(r)$, resp.) as being the restriction to $\Omega \cup \Gamma$ ($(0, 1]$, resp.) of a density $P(x)$ (a locally Hölder continuous function $P(r)$, resp.) on $\mathbf{R}^d \backslash \{0\}$ ($(0, \infty)$, resp.). For definiteness we set $P(x) = |x|^{-4} P(|x|^{-2}x)$ ($P(r) = r^{-4}P(r^{-1})$, resp.) for $|x| \ge 1$ ($r \ge 1$, resp.). With a radial density $P(x)$ on $\Omega \cup \Gamma$ (and hence on $\mathbf{R}^d \backslash \{0\}$) we associate an ordinary differential operator L_P given by

$$L_P w(r) = -\left(w''(r) + \frac{d-1}{r} w'(r)\right) + P(r)w(r)$$

where $w' = dw/dr$ and $w'' = d^2w/dr^2$. The unique existence of solutions of Cauchy problem for the linear differential equation $L_P w = 0$ is of fundamental importance in our study: For any $c \in (0, \infty)$ and any pair $(a, b) \in \mathbf{R} \times \mathbf{R}$ there exists a unique solution w of $L_P w = 0$ on $(0, \infty)$ such that $(w(c), w'(c)) = (a, b)$. In particular, we see that any solution w of $L_P w = 0$ on any interval $(\alpha, \beta) \subset (0, \infty)$ is uniquely extended to a solution of $L_P w = 0$ on $(0, \infty)$. A radial function $u(x) = u(|x|)$ belongs to $P(\Omega)$ if and only if $L_P u(r) = 0$ on $(0, 1)$.

The Laplacian $\Delta = \Delta_x = \Delta_{r\omega}$ is decomposed into the form $\Delta_x =$

$\Delta_r + r^{-2}\Delta_\omega$ according to the polar decomposition $x = r\omega$ where $\Delta_r = \partial^2/\partial r^2 + (d-1)r^{-1}\partial/\partial r$ and Δ_ω is the Laplace-Beltrami operator on Γ with respect to the natural Riemannian metric on Γ induced by the Euclidean metric on $\mathbf{R}^d$. For any $w \in C^2(0,\infty)$ we have $L_P w(r) = (-\Delta_r + P(r))w(r)$. A spherical harmonic S_n of degree $n = 0, 1, \cdots$ on Γ is a proper function of $-\Delta_\omega$ of the proper value $n(n+d-2)$: $-\Delta_\omega S_n = n(n+d-2)S_n$ on Γ. We define an orthonormal basis $\{S_{nj}(\omega) : j = 1, \cdots, N(n)\}$ in $L_2(\Gamma) = L_2(\Gamma, d\omega)$, where $d\omega$ is the area element in Γ, for spherical harmonics of degree $n = 0, 1, \cdots$. It is seen that $N(0) = 1$ and $S_{01}(\omega) = 1/\sqrt{\omega_d}$, where $\omega_d = \int_\Gamma d\omega$, the area of Γ. The addition theorem implies that

$$\sum_{j=1}^{N(n)} S_{nj}(\omega)^2 = \frac{N(n)}{\omega_d}. \tag{5}$$

Then $\{S_{nj}(\omega) : j = 1, \cdots, N(n); n = 0, 1, \cdots\}$ forms a complete orthonormal system for $L_2(\Gamma)$. For spherical harmonics, see e.g. [15] and [28].

With a radial density $P(x) = P(|x|)$ on $\Omega \cup \Gamma$ we associate a sequence $\{P_n\}$ $(n = 0, 1, \cdots)$ of radial densities $P_n(x) = P_n(|x|)$ on $\Omega \cup \Gamma$ defined by

$$P_n(r) = P(r) + \frac{n(n+d-2)}{r^2} \quad (n = 0, 1, \cdots).$$

Consider the Fourier coefficients $c_{nj}(r)$ of a $u(r\omega)$ in $P(\Omega)$ with respect to the complete orthonormal system $\{S_{nj}(\omega) : j = 1, \cdots, N(n); n = 0, 1, \cdots\}$ in $L_2(\Gamma)$:

$$\begin{aligned} c_{nj}(r) &= (u, S_{nj})_{L_2(\Gamma)} \\ &= \int_\Gamma u(r\omega) S_{nj}(\omega) d\omega \quad (j = 1, \cdots, N(n); n = 0, 1, \cdots). \end{aligned}$$

We show that c_{nj} is a solution of $L_{P_n} w = 0$ on $(0,1)$ $(j = 1, \cdots, N(n); n = 0, 1, \cdots)$. In fact, we have

$$\begin{aligned} & L_{P_n} c_{nj}(r) \\ &= \int_\Gamma \left(-\Delta_r + P(r) + \frac{n(n+d-2)}{r^2}\right) u(r\omega) \cdot S_{nj}(\omega) d\omega \\ &= \int_\Gamma \left(-\Delta_{r\omega} + P(r) + \frac{1}{r^2}\Delta_\omega + \frac{n(n+d-2)}{r^2}\right) u(r\omega) \cdot S_{nj}(\omega) d\omega \\ &= \int_\Gamma \left(\frac{1}{r^2}\Delta_\omega u(r\omega) \cdot S_{nj}(\omega) + u(r\omega)\frac{n(n+d-2)}{r^2} S_{nj}(\omega)\right) d\omega \end{aligned}$$

$$= \frac{1}{r^2} \int_\Gamma (\Delta_\omega u(r\omega) \cdot S_{nj}(\omega) - u(r\omega)\Delta_\omega S_{nj}(\omega))\, d\omega = 0.$$

The last equality follows from the Green's formula applied to Γ whose boundary is empty. If, in addition, u has boundary values zero on Γ, then it is readily seen that $c_{nj}(1) = 0$. Hence we have seen the following

Proposition 3 *If u belongs to $PP(\Omega)$ ($PP(\Omega;\Gamma)$, resp.) for a radial density P on $\Omega \cup \Gamma$, then c_{nj} are radial and belong to $P_nP(\Omega)$ ($P_nP(\Omega;\Gamma)$, resp.) $(j = 1, \cdots, N(n); n = 0, 1, \cdots)$.*

A density P on $\Omega \cup \Gamma$ is said to be *elliptic* (*nonelliptic*, resp.) if $PP(\Omega) = \{0\}$ ($PP(\Omega) \neq \{0\}$, resp.). Although the notion is defined for general densities on $\Omega \cup \Gamma$, we are interested only in the case P is radial. We use the *P-subunit* f_P associated with a radial density P on $\Omega \cup \Gamma$ characterized as the unique solution of $L_P w = 0$ on $(0, 1]$ with $(f_P(1), f'_P(1)) = (0, -1)$ (cf. [4], [16], etc.). The P-subunit is used to judge whether P is elliptic or not:

Proposition 4 (Ellipticity and nonellipticity criterion). *The following four conditions are equivalent by pairs*: (a) *P is nonelliptic, i.e. $PP(\Omega) \neq \{0\}$*; (b) $PP(\Omega;\Gamma) \neq \{0\}$; (c) $f_P(r) > 0$ $(0 < r < 1)$; (d) *there exists a solution w of $L_P w = 0$ such that $w(r) > 0$ $(0 < r < 1)$.*

Proof. Two implications (b) from (c) and (a) from (b) are clear. We next show that (a) implies (d). Take a $u \in PP(\Omega)\backslash\{0\}$ and set

$$w(r) := \frac{1}{\sqrt{\omega_d}} \int_\Gamma u(r\omega) d\omega = (u, S_{01})_{L_2(\Gamma)} = c_{01}(r) > 0 \quad (0 < r < 1)$$

which is a solution of $L_P w = L_{P_0} w = 0$ on $(0, 1)$. Thus (d) is valid. Finally we maintain that (d) implies (c) which completes the proof. Suppose there exists a solution w of $L_P w = 0$ such that $w > 0$ on $(0, 1)$. Then either $w(1) = 0$ or $w(1) > 0$. In the former case we see that $w'(1) < 0$ and thus $f_P(r) = -w(r)/w'(1) > 0$ $(0 < r < 1)$. In the latter case we consider a so-called d'Alembert transform

$$w_0(r) = w(r) \int_r^1 \frac{dt}{t^{d-1} w(t)^2}$$

of $w(r)$, which is also a solution of $L_P w = 0$. Since $w_0(1) = 0$ and $w_0(r) > 0$ $(0 < r < 1)$, we again see that $w'_0(1) < 0$ and thus $f_P(r) = -w_0(r)/w'_0(1) > 0$ $(0 < r < 1)$. □

Corollary 5 *A radial density P on $\Omega \cup \Gamma$ is elliptic if and only if* $\dim(P, \Omega) = 0$.

Proof. If P is elliptic, then $PP(\Omega) = \{0\}$ which implies $PP(\Omega; \Gamma) = \{0\}$ and hence $PP_1(\Omega; \Gamma) = \emptyset$. A fortiori $\dim(P, \Omega) = 0$. Conversely, if $\dim(P, \Omega) = 0$, then $\mathrm{ex.}\, PP_1(\Omega; \Gamma) = \emptyset$. By the Krein-Milman theorem, $PP_1(\Omega; \Gamma) = \emptyset$ (cf. §2) and therefore $PP(\Omega; \Gamma) = \{0\}$. Proposition 4 assures that P is elliptic. $\square$

Next we study the dependence of the P-subunit f_P on the radial density P. In particular, we will see that $P \mapsto f_P$ is increasing. The following assertion including this will play one of key roles in our study of Picard dimensions.

Proposition 6 (Comparison principle). *If P and Q are radial densities on $\Omega \cup \Gamma$ such that $P \leq Q$ ($P < Q$, resp.) on $0 \leq \rho < r = |x| < 1$ and $f_P > 0$ on $(\rho, 1)$, then f_Q/f_P is decreasing (strictly decreasing, resp.) on $(\rho, 1)$ and $\lim_{r\uparrow 1} f_Q(r)/f_P(r) = 1$. In particular, $f_Q \geq f_P$ ($f_Q > f_P$, resp.) on $(\rho, 1)$.*

Proof. For simplicity we set $w(r) := f_Q(r)/f_P(r)$ $(0 \leq \rho < r < 1)$. By a simple computation we obtain

$$\frac{d}{dr}\left(r^{d-1} f_P(r)^2 \frac{d}{dr} w(r)\right) = r^{d-1}(Q(r) - P(r)) f_P(r) f_Q(r). \tag{6}$$

It is readily seen that

$$\lim_{r\uparrow 1} r^{d-1} f_P(r)^2 \frac{d}{dr} w(r) = 0. \tag{7}$$

Since $f_Q(1) = 0$ and $f_Q'(1) = -1$, there exists a $t \in (0,1)$ sufficiently close to 1 such that $f_Q > 0$ on $(t, 1)$. Therefore $\tau := \inf\{t \in (\rho, 1) : f_Q > 0$ on $(t, 1)\}$ belongs to $[\rho, 1)$. We now maintain that $\tau = \rho$. Contrariwise suppose $\rho < \tau < 1$. Then $f_Q(\tau) = 0$ and $f_Q > 0$ on $(\tau, 1)$. Hence (6) implies that $r^{d-1} f_P(r)^2 w'(r)$ is increasing (strictly increasing, resp.) on $(\tau, 1)$ and afortiori (7) assures that $w'(r) \leq 0$ ($w'(r) < 0$, resp.) on $(\tau, 1)$. By the l'Hospital rule, $\lim_{r\uparrow 1} w(r) = \lim_{r\uparrow 1} f_Q'(r)/f_P'(r) = (-1)/(-1) = 1$. Thus $w(\tau) \geq 1$. But $f_Q(\tau) = 0$ and $f_P(\tau) > 0$ implies that $w(\tau) = 0$, a contradiction. Therefore we must have $\tau = \rho$. The rest of the assertion is now clear. $\square$

Corollary 7 *If $f_{P_j} > 0$ on $(0,1)$ for some $j = 0, 1, \cdots$, then $f_{P_k}(r)/f_{P_j}(r) > f_{P_k}(s)/f_{P_j}(s)$ for any r and s with $0 < r < s < 1$ and any $k > j$.*

Corollary 8 *If $f_{P_j} > 0$ on $(0,1)$ for some $j = 0, 1, \cdots$, then $f_{P_j} < f_{P_k}$ $(j < k)$ on $(0,1)$.*

For proofs of the above two corollaries we only have to observe that $P_j < P_k$ $(j < k)$ on $(0,1)$ and then apply Proposition 6.

Suppose now that a radial density P is nonelliptic on $\Omega \cup \Gamma$ so that $f_P > 0$ on $(0,1)$. By Corollary 8 we have

$$0 < f_P = f_{P_0} < f_{P_1} < f_{P_2} < \cdots < f_{P_{j-1}} < f_{P_j} < \cdots \quad (j = 1, 2, \cdots)$$

and therefore Corollary 7 assures that f_{P_0}/f_{P_j} is increasing on $(0,1)$. Hence

$$\alpha_j(P) = \lim_{r \downarrow 0} \frac{f_P(r)}{f_{P_j}(r)} \in [0,1) \quad (j = 1, 2, \cdots)$$

exists. In particular $\alpha(P) := \alpha_1(P)$ is referred to as the *singularity index of limit form* of P. Clearly

$$\begin{aligned} 1 > \alpha(P) = \alpha_1(P) \geq \alpha_2(P) \geq \cdots \geq \alpha_j(P) \geq \alpha_{j+1}(P) \\ \geq \cdots \quad (j = 1, 2, \cdots) \end{aligned}$$

and hence we have obtained the following

Proposition 9 *If $\alpha(P) = 0$, then $\alpha_j(P) = 0$ $(j = 1, 2, \cdots)$.*

Once more we confirm that the singularity index $\alpha(P)$ can be defined only for nonelliptic radial densities P on $\Omega \cup \Gamma$. It will be seen that $\dim(P, \Omega)$ for a radial density P on $\Omega \cup \Gamma$ is determined by whether $\alpha(P) = 0$ or $\alpha(P) > 0$. In this sense it is important to be able to compute $\alpha(P)$ concretely. We exhibit the simplest case in the following

Example 10 The constant function 0 is a density on $\Omega \cup \Gamma$ which is referred to as the *harmonic density* since 0-harmonicity is nothing but the classical harmonicity. By solving $L_0 w = 0$ concretely we see that the 0-subunit f_0 is given as follows:

$$f_0(r) = \begin{cases} \log \dfrac{1}{r} & (d = 2) \\ \dfrac{1}{d-2}\left(\dfrac{1}{r^{d-2}} - 1\right) & (d \geq 3), \end{cases}$$

and the 0_1-subunit (i.e. $(d-1)/|x|^2$-subunit) f_{0_1} is given as follows:

$$f_{0_1}(r) = \frac{1}{d}\left(\frac{1}{r^{d-1}} - r\right).$$

Thus the harmonic density 0 is nonelliptic and $\alpha(0) = 0$. Hence by Propositions 4 and 6 nonnegative radial densities P on $\Omega \cup \Gamma$ are seen to be nonelliptic.

4. Fundamental theorem on radial densities

We know that the range of $\dim(P, \Omega)$ for general densities P on $\Omega \cup \Gamma$ covers the set $\{0\} \cup \mathbf{N} \cup \{\aleph_0, \aleph\}$ of cardinal numbers which implies the diversity of behavior of $\dim(P, \Omega)$. However the range of $\dim(P, \Omega)$ for radial densities P on $\Omega \cup \Gamma$ is very simple as in the following theorem. The theorem is not new but the proof given below is surprisingly simple and elementary compared with the known ones (cf. [18], [16]).

Theorem 11 (Fundamental theorem). *The range of* $\dim(P, \Omega)$ *for radial densities* P *on* $\Omega \cup \Gamma$ *consists of three cardinal numbers* 0, 1, *and* $\aleph$, *the cardinal number of continuum. More precisely,* $\dim(P, \Omega) = 0$ *if and only if* P *is elliptic on* Ω; *if* P *is nonelliptic, then* $\dim(P, \Omega) = 1$ *or* $\aleph$ *according as* $\alpha(P) = 0$ *or* $\alpha(P) > 0$.

Proof. By Corollary 5, $\dim(P, \Omega) = 0$ if and only if a radial density P on $\Omega \cup \Gamma$ is elliptic. Hence we assume that P is nonelliptic and show that $\dim(P, \Omega) = 1$ or $\aleph$ according as $\alpha(P) = 0$ or $\alpha(P) > 0$.

First we assume that $\alpha(P) = 0$ and show that $\dim(P, \Omega) = 1$. For the purpose we only have to show that $u = f_P = f_{P_0}$ for any $u \in PP_1(\Omega; \Gamma)$. For an arbitrary $r \in (0, 1]$ let the Fourier expansion of $u(r\omega)$ in ω be

$$u(r\omega) = c_{01}(r)S_{01}(\omega) + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{N(n)} c_{nk}(r)S_{nk}(\omega)\right)$$

in $L_2(\Gamma)$. Recall that $c_{nk}(r)$ is a solution of $L_{P_n}w = 0$ on $(0, 1)$ with $c_{nk}(1) = 0$. Hence $c_{nk}(r) = -c'_{nk}(1)f_{P_n}(r)$ on $(0, 1]$. In particular, $S_{01}(\omega) = \omega_d^{-1/2}$ implies

$$c_{01}(r) = \int_\Gamma u(r\omega)S_{01}(\omega)d\omega = \omega_d^{-1/2}\int_\Gamma u(r\omega)d\omega$$

and therefore we see that

$$c'_{01}(1) = \omega_d^{-1/2} \int_\Gamma \left[\frac{\partial}{\partial r} u(r\omega) \right]_{r=1} d\omega = -\omega_d^{1/2} \ell(u) = -\omega_d^{1/2}.$$

Thus $c_{01}(r) = -c'_{01}(1) f_P(r) = \omega_d^{1/2} f_P(r)$ and $c_{01}(r) S_{01}(\omega) = f_P(r)$. Therefore we obtain

$$u(r\omega) = f_P(r) - \sum_{n=1}^{\infty} \left(\sum_{k=1}^{N(n)} c'_{nk}(1) S_{nk}(\omega) \right) f_{P_n}(r)$$

in $L_2(\Gamma)$. Multiplying $(N(n)/\omega_d)^{1/2} \pm S_{nk}(\omega) \geq 0$ to the both sides of the above and then integrating over Γ with respect to $d\omega$ we see that

$$\begin{aligned} 0 &\leq \int_\Gamma u(r\omega) \left((N(n)/\omega_d)^{1/2} \pm S_{nk}(\omega) \right) d\omega \\ &= \left(u(r\omega), (N(n)/\omega_d)^{1/2} \pm S_{nk}(\omega) \right)_{\omega, L_2(\Gamma)} \\ &= (N(n)\omega_d)^{1/2} f_P(r) \mp c'_{nk}(1) f_{P_n}(r) \end{aligned}$$

and a fortiori we conclude that

$$|c'_{nk}(1)| \leq (N(n)\omega_d)^{1/2} \frac{f_P(r)}{f_{P_n}(r)}$$

for every $k = 1, \cdots, N(n)$ and $n = 1, 2, \cdots$. On letting $r \downarrow 0$ we deduce that

$$|c'_{nk}(1)| \leq (N(n)\omega_d)^{1/2} \alpha_n(P) \leq (N(n)\omega_d)^{1/2} \alpha(P) = 0$$

and $c'_{nk}(1) = 0$ $(k = 1, \cdots, N(n); n = 1, 2, \cdots)$. This proves that $u = f_P$.

Next we prove that $\alpha(P) > 0$ implies $\dim(P, \Omega) = \aleph$. We denote by $\mathcal{O}^d$ the group of all orthogonal transformations τ of $\mathbf{R}^d$. The transformation τ may be identified with the orthogonal matrix of type $d \times d$. Observe that a function w on Ω is radial if and only if $w \circ \tau = w$ on Ω for every $\tau \in \mathcal{O}^d$. We note that $\mathrm{ex}.\, PP_1(\Omega; \Gamma)$ is closed under $\mathcal{O}^d$, i.e. for any $w \in \mathrm{ex}.\, PP_1(\Omega; \Gamma)$ and every $\tau \in \mathcal{O}^d$ we have $w \circ \tau \in \mathrm{ex}.\, PP_1(\Omega; \Gamma)$. Since P is nonelliptic, $PP(\Omega; \Gamma) \neq \{0\}$ and hence $PP_1(\Omega; \Gamma) \neq \emptyset$. As we have seen in §2, $PP_1(\Omega; \Gamma)$ is a nonempty compact convex subset of the locally convex linear topological space $P(\Omega)$. The Krein-Milman theorem, or a part of it, assures that

$$\mathrm{ex}.\, PP_1(\Omega; \Gamma) \neq \emptyset.$$

This simple observation is a crucial part in our proof.

We now assert that any u in ex. $PP_1(\Omega;\Gamma)$ is not radial. Contrariwise assume that u is radial so that $u = f_P$. Set

$$u^{\pm}(r\omega) = f_P(r) \pm \alpha(P) f_{P_1}(r)(\omega_d/N(1))^{1/2} S_{11}(\omega).$$

By a direct computation we see that $(-\Delta+P)u^{\pm} = 0$ on Ω. We also see that $u^{\pm} \in C(\Omega \cup \Gamma)$ and $u^{\pm} = 0$ on Γ. Moreover, since $\alpha(P)(f_{P_1}/f_P)$ is strictly decreasing and $0 \leq \alpha(P)(f_{P_1}(r)/f_P(r)) < \lim_{s\downarrow 0} \alpha(P)(f_{P_1}(s)/f_P(s)) = 1$ for $0 < r \leq 1$, by $|(\omega_d/N(1))^{1/2} S_{11}(\omega)| \leq 1$, we see that, for $0 < r < 1$,

$$\begin{aligned} u^{\pm}(r\omega) &= f_P(r)\left(1 \pm \alpha(P)\frac{f_{P_1}(r)}{f_P(r)}(\omega_d/N(1))^{1/2} S_{11}(\omega)\right) \\ &\geq f_P(r)\left(1 - \alpha(P)\frac{f_{P_1}(r)}{f_P(r)}\right) > 0. \end{aligned}$$

Therefore we see that $u^{\pm} \in PP(\Omega;\Gamma)$. In addition to this, by

$$\int_\Gamma S_{11}(\omega)d\omega = \omega_d^{1/2}\int_\Gamma S_{01}(\omega)S_{11}(\omega)d\omega = 0,$$

we see that

$$\begin{aligned} \ell(u^{\pm}) &= -\frac{1}{\omega_d}\int_\Gamma \left[\frac{\partial}{\partial r}u^{\pm}(r\omega)\right]_{r=1} d\omega \\ &= -f_P'(1) \mp \frac{1}{\omega_d}\alpha(P) f_{P_1}'(1)(\omega_d/N(1))^{1/2}\int_\Gamma S_{11}(\omega)d\omega = 1 \end{aligned}$$

and therefore $u^{\pm} \in PP_1(\Omega;\Gamma)$. Clearly $u^+ \neq u^-$ and $u = f_P = (u^+ + u^-)/2$, contradicting $u \in$ ex. $PP_1(\Omega;\Gamma)$. Thus any u in ex. $PP_1(\Omega;\Gamma)$ is not radial.

Let X be the 2-dimensional subspace of $\mathbf{R}^d$ spanned by two vectors $(\delta_{i1},\cdots,\delta_{id})$ $(i = 1, 2)$ and $Y = \Omega \cap X$, the punctured unit disc in X. We view the unit circle $T = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$ in the complex plane $\mathbf{C}$ as a multiplicative group. With each $\zeta \in T$ we associate a $\tau(\zeta)$ in $\mathcal{O}^d$ whose matrix representation is

$$\tau(\zeta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \cdots & 0 \\ \sin\theta & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & E & \\ 0 & 0 & & & \end{pmatrix} \qquad (\theta = \arg\zeta)$$

where E is the unit matrix of type $(d-2)\times(d-2)$. Then $\mathcal{O}_X^d = \{\tau(\zeta) \in \mathcal{O}^d : \zeta \in T\}$ is isomorphic to T as multiplicative groups.

We maintain that there exists a $u \in \mathrm{ex}.\,PP_1(\Omega;\Gamma)$ such that $u|Y$ is not radial on Y. For the purpose take an arbitrary $w \in \mathrm{ex}.\,PP_1(\Omega;\Gamma)$. Since w is not radial on Ω, there exist two points x and y in Ω such that $|x| = |y|$ and $w(x) \neq w(y)$. Choose a $\tau_0 \in \mathcal{O}^d$ with $\tau_0(x) \in X$ and $\tau_0(y) \in X$. Then set $u = w \circ \tau_0^{-1}$. Clearly $u \in \mathrm{ex}.\,PP_1(\Omega;\Gamma)$ and u is not radial on Y. In fact, if we take a $\tau_1 \in \mathcal{O}_X^d$ with $\tau_1(\tau_0(x)) = \tau_0(y)$, then $u(\tau_0(x)) = w(x) \neq w(y) = u(\tau_0(y)) = u \circ \tau_1(\tau_0(x))$ shows that $u \circ \tau_1 \neq u$ on Y.

Fixing the above $u \in \mathrm{ex}.\,PP_1(\Omega;\Gamma)$ we consider $\mathcal{O}_X^d(u) = \{\tau \in \mathcal{O}_X^d : u \circ \tau = u \text{ on } Y\}$. Then $\mathcal{O}_X^d(u)$ is a normal subgroup of an Abelian group $\mathcal{O}_X^d$. For each $\tau^* \in \mathcal{O}_X^d/\mathcal{O}_X^d(u)$ we define $u_{\tau^*} = u \circ \tau$ $(\tau \in \tau^*)$ which is also an element of $\mathrm{ex}.\,PP_1(\Omega;\Gamma)$. Then $\tau^* \mapsto u_{\tau^*}$ is an injection of $\mathcal{O}_X^d/\mathcal{O}_X^d(u)$ into $\mathrm{ex}.\,PP_1(\Omega;\Gamma)$ and thus we obtain the inequality

$$\#\left(\mathcal{O}_X^d/\mathcal{O}_X^d(u)\right) \leq \#(\mathrm{ex}.\,PP_1(\Omega;\Gamma)) = \dim(P,\Omega). \tag{8}$$

With the above u we associate a function $v = \sigma(u)$ of $z = x+iy \in \mathbf{C}$ by $v(z) = v(x+iy) = u(x,y,0,\cdots,0)$. For each $\zeta \in T$ we consider v_ζ given by $v_\zeta(z) = v(\zeta z)$ on $0 < |z| < 1$. Set $T(u) = \{\zeta \in T : v_\zeta = v \text{ on } 0 < |z| < 1\}$. Since

$$\begin{aligned} v_\zeta(z) &= v(\zeta z) = u\left(\tau(\zeta)\, {}^t(x,y,0,\cdots,0)\right) \\ &= (u \circ \tau(\zeta))(x,y,0,\cdots,0), \end{aligned}$$

where ${}^t(x,y,0,\cdots,0)$ is the transposed matrix of $(x,y,0,\cdots,0)$, we have seen that $v_\zeta = \sigma(u \circ \tau(\zeta))$. This shows that $\zeta \in T(u)$ is equivalent to $\tau(\zeta) \in \mathcal{O}_X^d(u)$. The isomorphism $\zeta \mapsto \tau(\zeta)$ of T onto $\mathcal{O}_X^d$ sends $T(u)$ onto $\mathcal{O}_X^d(u)$ and thus $T/T(u)$ is isomorphic to $\mathcal{O}_X^d/\mathcal{O}_X^d(u)$. This with (8) and the trivial fact $\dim(P,\Omega) \leq \aleph$ yield the inequality

$$\#(T/T(u)) \leq \dim(P,\Omega) \leq \aleph. \tag{9}$$

Consider the subgroup $T_0 = \{\zeta \in T : \zeta^n = 1 \text{ for some } n \in \mathbf{Z}\}$ of T where $\mathbf{Z}$ is the set of integers. Clearly $\#T_0 = \aleph_0$, the cardinal number of countably infinite sets. For each fixed $0 < r < 1$ let

$$v(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n(r)e^{in\theta}$$

be the complex Fourier expansion of $v(re^{i\theta})$ as the function of θ. Then, for each $\zeta \in T(u)$, we have

$$v_\zeta(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n(r)\zeta^n e^{in\theta}.$$

Since $v \equiv v_\zeta$, we must have

$$c_n(r) = c_n(r)\zeta^n \quad (0 < r < 1,\ n \in \mathbf{Z}).$$

Since $v(z)$ is not radial on $0 < |z| < 1$, there exist an $r \in (0,1)$ and an $n \in \mathbf{Z}$ such that $c_n(r) \neq 0$. Hence $\zeta^n = 1$ and $\zeta \in T_0$, i.e. $T(u) \subset T_0$. Therefore

$$\#(T(u)) \leq \#T_0 = \aleph_0$$

implies that

$$\#(T/T(u)) = (\#(T/T(u))) \cdot (\#T(u)) = \#T = \aleph$$

and thus (9) implies that $\dim(P, \Omega) = \aleph$. $\square$

We say that the *Picard principle* is valid for P on Ω if $\dim(P, \Omega) = 1$.

Corollary 12 *The Picard principle is valid for a radial density P on $\Omega \cup \Gamma$ if and only if P is nonelliptic and $\alpha(P) = 0$.*

We have seen in Example 10 that the harmonic density 0 is nonelliptic and $\alpha(0) = 0$. Hence the Picard principle is valid for 0, i.e. $\dim(0, \Omega) = 1$. This is the classical principle of positive singularities due to Bôcher and Picard.

5. Singularity indices of integrated form

Besides the singularity index $\alpha(P)$ of limit form of a nonelliptic radial density P on $\Omega \cup \Gamma$ we introduce another type of singularity index which is sometimes more manageable than $\alpha(P)$. We call the nonnegative number $\beta(P)$ the *singularity index of integrated form* of a nonelliptic radial density P on $\Omega \cup \Gamma$ given by

$$\begin{aligned} 1/\beta(P) &= \iint_{0 \leq s \leq t \leq 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_P(t)}{f_P(s)}\right)^2 ds dt \quad \text{(double integral form)} \\ &= \int_0^1 t^{d-3} f_P(t)^2 \left(\int_0^t \frac{ds}{s^{d-1} f_P(s)^2}\right) dt \\ &\qquad \text{(iterated integral form)} \end{aligned}$$

under the convention $1/0 = \infty$ (cf. [16]). In general $\alpha(P)$ and $\beta(P)$ do not coincide with each other. However, by Theorem 11, it is only important whether $\alpha(P) = 0$ or $\alpha(P) > 0$ and in this respect $\alpha(P)$ and $\beta(P)$ are essentially identical:

Proposition 13 *The singularity index $\alpha(P) = 0$ for a nonelliptic radial density P on $\Omega \cup \Gamma$ if and only if the singularity index $\beta(P) = 0$.*

Proof. We set $w(r) = f_{P_1}(r)/f_P(r)$ for $0 < r < 1$ and observe that

$$\frac{d}{dr}\left(r^{d-1} f_P(r)^2 \frac{d}{dr} w(r)\right) = (d-1) r^{d-3} f_P(r)^2 w(r).$$

Integrating both sides of the above over the interval $(r, 1)$ $(r > 0)$ and noting that $f_P(1) = 0$ we obtain

$$-r^{d-1} f_P(r)^2 \frac{d}{dr} w(r) = (d-1) \int_r^1 t^{d-3} f_P(t)^2 w(t) dt.$$

Dividing both sides by $-r^{d-1} f_P(r)^2$, changing variable r to s and then integrating over the interval $(r, 1)$ $(r > 0)$ we have

$$w(1) - w(r) = -(d-1) \int_r^1 \frac{1}{s^{d-1} f_P(s)^2} \left(\int_s^1 t^{d-3} f_P(t)^2 w(t) dt\right) ds.$$

By l'Hospital rule, $w(1) = \lim_{r \uparrow 1}(f'_{P_1}(r)/f'_P(r)) = (-1)/(-1) = 1$. Hence

$$w(r) = 1 + (d-1) \iint_{r \le s \le t \le 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_P(t)}{f_P(s)}\right)^2 w(t) ds dt \quad (r > 0). \tag{10}$$

In view of Corollary 8 we see that $w(t) \ge 1$ for $0 < t \le 1$. Replacing $w(t)$ by 1 in the double integral of (10) we obtain

$$w(r) \ge 1 + (d-1) \iint_{r \le s \le t \le 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_P(t)}{f_P(s)}\right)^2 ds dt.$$

Recall that $\lim_{r \downarrow 0} w(r) = 1/\alpha(P)$. On letting $r \downarrow 0$ in the above we have

$$1/\alpha(P) \ge 1 + (d-1)/\beta(P) \ge (d-1)/\beta(P)$$

or $\beta(P) \ge (d-1)\alpha(P)$. This proves that $\beta(P) = 0$ implies $\alpha(P) = 0$.

We next prove that $\beta(P) > 0$ implies $\alpha(P) > 0$. For the purpose we set

$$\gamma(r,\rho) = \iint_{r<s<\rho, s\le t\le 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_P(t)}{f_P(s)}\right)^2 dsdt$$

for $0 \le r < \rho \le 1$ so that $\gamma(0,1) = 1/\beta(P) < \infty$. Therefore $\gamma(0,\rho) \downarrow 0$ as $\rho \downarrow 0$. Then there exists a $\rho \in (0,1)$ such that $\gamma(0,\rho) < 1/2(d-1)$ so that $\gamma(r,\rho) < 1/2(d-1)$ for every r with $0 < r < \rho$. By (10) we see that

$$\begin{aligned} w(r) &= 1 + (d-1)\left(\iint_{\rho\le s\le t\le 1} \right. \\ &\qquad \left. + \iint_{r<s<\rho, s\le t\le 1}\right) \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_P(t)}{f_P(s)}\right)^2 w(t) dsdt \\ &= w(\rho) + (d-1)\iint_{r<s<\rho, s\le t\le 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_P(t)}{f_P(s)}\right)^2 w(t) dsdt. \end{aligned}$$

By Corollary 7 we see that $w(t)$ is strictly decreasing on $(0,1)$ so that $w(t) < w(r)$ for $r < s \le t \le 1$. Hence the double integral of the right most side of the above identities is dominated by $w(r)\gamma(r,\rho)$ and hence

$$\begin{aligned} w(r) &\le w(\rho) + (d-1)w(r)\gamma(r,\rho) \\ &\le w(\rho) + (d-1)w(r)(1/2(d-1)) = w(\rho) + w(r)/2 \end{aligned}$$

or $w(r) \le 2w(\rho)$. On letting $r \downarrow 0$ we deduce $1/\alpha(P) \le 2w(\rho)$ or $\alpha(P) \ge 1/2w(\rho) > 0$. □

Using the singularity index $\beta(P)$ just introduced we are now ready to prove the monotoneity of $\dim(P,\Omega)$ which is a preliminary version of the monotoneity of $\dim P$ to be shown in §8.

Theorem 14 (Monotoneity). *If P and Q are radial densities on $\Omega \cup \Gamma$ with $P \le Q$ on $\Omega \cup \Gamma$, then the inequality* $\dim(P,\Omega) \le \dim(Q,\Omega)$ *holds.*

Proof. If P is elliptic, then $\dim(P,\Omega) = 0$. Since $\dim(Q,\Omega) \ge 0$, we can conclude that $\dim(P,\Omega) \le \dim(Q,\Omega)$. Hence we only have to treat the case P is nonelliptic which is equivalent to that $f_P > 0$ on $(0,1)$. By Proposition 6, $f_P \le f_Q$ so that $f_Q > 0$ on $(0,1)$ which means that Q is nonelliptic. By Corollary 5, $\dim(P,\Omega) \ge 1$ and $\dim(Q,\Omega) \ge 1$. If $\dim(P,\Omega) = 1$, then $\dim(Q,\Omega) \ge 1$ assures that $\dim(P,\Omega) \le \dim(Q,\Omega)$. Therefore, by Theorem 11, we only have to show that $\dim(Q,\Omega) = \aleph$ if $\dim(P,\Omega) = \aleph$, or equivalently, $\alpha(P) > 0$ implies $\alpha(Q) > 0$. By Proposition 6, $f_Q(s)/f_P(s) \ge$

$f_Q(t)/f_P(t)$ for $s \le t$ and thus

$$\frac{f_P(t)}{f_P(s)} \ge \frac{f_Q(t)}{f_Q(s)} \quad (0 < s \le t < 1).$$

Therefore we deduce that

$$\begin{aligned} 1/\beta(Q) &= \iint_{0\le s\le t\le 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_Q(t)}{f_Q(s)}\right)^2 dsdt \\ &\le \iint_{0\le s\le t\le 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{f_P(t)}{f_P(s)}\right)^2 dsdt = 1/\beta(P) < \infty \end{aligned}$$

which implies that $\beta(Q) > 0$. □

Corollary 15 *The Picard dimensions* $\dim(P,\Omega) \le 1$ *for radial densities* $P \le 0$ *on* $\Omega \cup \Gamma$ *and* $\dim(P,\Omega) \ge 1$ *for radial densities* $P \ge 0$ *on* $\Omega \cup \Gamma$.

Proof. We have seen in Example 10 that $\alpha(0) = 0$ or $\dim(0,\Omega) = 1$. Hence, by Theorem 14, $P \le 0$ implies that $\dim(P,\Omega) \le \dim(0,\Omega) = 1$ and similarly $P \ge 0$ implies that $\dim(P,\Omega) \ge \dim(0,\Omega) = 1$. □

6. Hyperbolicity and Parabolicity

There may or may not exist a function e_P for a general density P on $\Omega\cup\Gamma$ satisfying the following three conditions: (a) $e_P \in PP(\Omega) \cap C(\Omega\cup\Gamma)$; (b) $e_P|\Gamma = 1$; (c) $h \ge e_P$ on Γ_a for any $h \in PP(\Omega_a) \cap C(\Omega_a \cup \Gamma_a)$ $(0 < a \le 1)$ implies $h \ge e_P$ on $\Omega_a \cup \Gamma_a$. Such a function e_P, if exists, is unique and referred to as the *P-unit* on $\Omega \cup \Gamma$ (cf. [18]). A density P is said to be *hyperbolic* if the P-unit e_P exists on $\Omega\cup\Gamma$. Clearly hyperbolic densities are nonelliptic. A nonelliptic density P on $\Omega \cup \Gamma$ is said to be *parabolic* if it is not hyperbolic on $\Omega \cup \Gamma$. We do not use at all in this paper the known fact (cf. e.g. [27], [12]) that P is hyperbolic if and only if there exists the P-Green's function $G_P(x,y)$ on Ω which is the minimal positive solution of the equation $(-\Delta + P(x))u(x) = \delta_y$ (the Dirac delta). In this fashion the hyperbolicity and the parabolicity can be defined for general densities P on $\Omega\cup\Gamma$ but we are interested mainly in the case of radial densities P on $\Omega\cup\Gamma$. Hence we choose the following more concrete approach.

Suppose P is a nonelliptic radial density on $\Omega\cup\Gamma$ so that the P-subunit $f_P > 0$ on $(0,1)$. For each $R \in (0,1)$ we form a d'Alembert transform $e_{P,R}$

of f_P:

$$e_{P,R}(r) = f_P(r)\int_R^r \frac{dt}{t^{d-1}f_P(t)^2} \quad (R \le r < 1)$$

which is a solution of $L_P w = 0$ on $(R, 1)$ with the boundary data $e_{P,R}(R) = 0$ and $e_{P,R}(1) = 1$. The latter is deduced by using l'Hospital rule as follows:

$$\begin{aligned} e_{P,R}(1) &= \lim_{r\uparrow 1}\left(\int_R^r \frac{dt}{t^{d-1}f_P(t)^2}\right)/f_P(r)^{-1} \\ &= \lim_{r\uparrow 1}\left(\int_R^r \frac{dt}{t^{d-1}f_P(t)^2}\right)'/\left(f_P(r)^{-1}\right)' \\ &= \lim_{r\uparrow 1}\left(r^{1-d}f_P(r)^{-2}\right)/\left(-f_P(r)^{-2}f_P'(r)\right) \\ &= 1/\left(-f_P'(1)\right) = 1. \end{aligned}$$

The net $\{e_{P,R}\}_{R\downarrow 0}$ is strictly increasing and hence, by the Harnack principle, either $\{e_{P,R}\}_{R\downarrow 0}$ converges to a solution of $L_P w = 0$ locally uniformly on Ω or $\{e_{P,R}\}_{R\downarrow 0}$ diverges to ∞ locally uniformly on Ω. The former (the latter, resp.) occurs if

$$\int_0^r \frac{dt}{t^{d-1}f_P(t)^2} < \infty \tag{11}$$

is valid (invalid, resp.) for one and hence for every $r \in (0, 1)$. If (11) is valid, then we can define a function

$$w_P(r) = f_P(r)\int_0^r \frac{dt}{t^{d-1}f_P(t)^2} = \lim_{R\downarrow 0} e_{P,R}(r).$$

By a direct computation we see that w_P is a solution of $L_P w = 0$ on $(0, 1)$ and by the same way as above, we see that $w_P(1) = 1$. Clearly $w_P(x) = w_P(|x|)$ satisfies conditions (a) and (b) of the P-unit. Take any $h \in PP(\Omega_a) \cap C(\Omega_a \cup \Gamma_a)$ $(0 < a < 1)$ such that $h \ge w_P$ on Γ_a. Since $e_{P,R} \le w_P$ for any $R \in (0, a)$, we see that $e_{P,R} \le h$ on the boundary of the ring $R < |x| < a$ and therefore, by the minimum principle, $e_{P,R} \le h$ on $R < |x| < a$. On letting $R \downarrow 0$, we conclude that $w_P \le h$ on $\Omega_a \cup \Gamma_a$. This proves that w_P also satisfies the condition (c) of the P-unit. Hence w_P is the P-unit on $\Omega \cup \Gamma$, i.e. $w_P = e_P$. Conversely suppose the P-unit e_P exists on $\Omega \cup \Gamma$. Then, by the minimum principle, $e_{P,R} \leq e_P$ on $(\Omega \cup \Gamma)\backslash\Omega_R$ or

$$\int_R^r \frac{dt}{t^{d-1} f_P(t)^2} \leq \frac{e_P(r)}{f_P(r)}$$

for any fixed $r \in (0,1)$ and for every $R \in (0,r)$. On letting $R \downarrow 0$ we conclude that (11) is valid. We have thus established the following

Proposition 16 *A radial density P on $\Omega \cup \Gamma$ is hyperbolic if and only if* (11) *is valid for one and hence for any $0 < r < 1$ and in this case the P-unit e_P is radial and given by a d'Alembert transform of f_P :*

$$e_P(r) = f_P(r) \int_0^r \frac{dt}{t^{d-1} f_P(t)^2}.$$

We have thus obtained the classification of densities as follows: all the (radial) densities on $\Omega \cup \Gamma$ are classified into two categories: elliptic (radial) densities and nonelliptic (radial) densities; all the nonelliptic (radial) densities are then classified into two categories: parabolic (radial) densities and hyperbolic (radial) densities. The Picard dimension $\dim(P, \Omega) = 0$ for elliptic (radial) densities P and $\dim(P, \Omega) \geq 1$ for nonelliptic (radial) densities. In this respect the following fact is worth observing:

Corollary 17 *If a radial density P on $\Omega \cup \Gamma$ is parabolic, then its Picard dimension* $\dim(P, \Omega) = 1$.

Proof. By Proposition 16 we see that (11) is invalid so that $\int_0^t s^{1-d} f_P(s)^{-2} ds = \infty$ for any $0 < t < 1$. Then

$$\begin{aligned} 1/\beta(P) &= \int_0^1 t^{d-3} f_P(t)^2 \left(\int_0^t \frac{ds}{s^{d-1} f_P(s)^2} \right) dt \\ &\geq \int_0^1 t^{d-3} f_P(t)^2 \infty dt = \infty \end{aligned}$$

and, by Theorem 11 and Proposition 13, we see that $\dim(P, \Omega) = 1$. □

Corollary 18 *If P and Q are radial densities with $P \leq Q$ on $\Omega \cup \Gamma$, then the hyperbolicity of P implies that of Q.*

Proof. The hyperbolicity of P implies its nonellipticity and thus $f_P > 0$

on $(0,1)$. By Proposition 6, $f_P \leq f_Q$. Hence

$$\int_0^r \frac{dt}{t^{d-1} f_Q(t)^2} \leq \int_0^r \frac{dt}{t^{d-1} f_P(t)^2} < \infty$$

shows that Q satisfies (11) so that Q is hyperbolic. $\square$

Corollary 19 *A nonnegative radial density P on $\Omega \cup \Gamma$ is hyperbolic.*

Proof. Since $P \geq 0$, in view of Corollary 18, we only have to show that the harmonic density 0 is hyperbolic. By Example 10, we see that

$$\int_0^r \frac{dt}{t^{2-1} f_0(t)^2} = \int_0^r \frac{dt}{t(\log t)^2} < \infty$$

for the case $d = 2$ and

$$\int_0^r \frac{dt}{t^{d-1} f_0(t)^2} = \int_0^r \frac{(d-2)^2}{t^{d-1}(t^{2-d}-1)^2} dt < \infty$$

for the case $d \geq 3$. In any case the harmonic density 0 satisfies the condition (11) and then it is hyperbolic. $\square$

In passing we remark the following. If a radial density P on $\Omega \cup \Gamma$ is hyperbolic so that there exists the P-unit e_P on $\Omega \cup \Gamma$, then we can recover the P-subunit f_P by forming a d'Alembert transform of e_P:

$$f_P(r) = e_P(r) \int_r^1 \frac{dt}{t^{d-1} e_P(t)^2}.$$

To establish the above identity we denote by $w(r)$ the right hand side of the above on $(0,1)$. By a direct computation we see that $L_P w = 0$ on $(0,1)$. It is clear that $w(1) = 0$ and, since

$$w'(r) = e'_P(r) \int_r^1 \frac{dt}{t^{d-1} e_P(t)^2} - \frac{1}{r^{d-1} e_P(r)},$$

we see that $w'(1) = -1$. Thus we must conclude that $w(r) = f_P(r)$.

The following technical fact will play an important role in the proof of the homogeneity of Picard dimensions for radial densities on $\Omega \cup \Gamma$ in §7.

Lemma 20 *If P is a nonelliptic radial density on $\Omega \cup \Gamma$, then the density cP is hyperbolic on $\Omega \cup \Gamma$ for every $0 < c < 1$.*

Before giving a proof we need to recall some fundamentals related to the operator

$$L_P w(r) = -\frac{1}{r^{d-1}}\left(r^{d-1}w'(r)\right)' + P(r)w(r)$$

(cf. e.g. [6], [13]). A function $u \in C^2(a,b)$ is said to be a *supersolution* of $L_P w = 0$ on an interval $(a,b) \subset (0,1)$ if $L_P u \geq 0$ on (a,b). Suppose there exists a supersolution S of $L_P w = 0$ on $(0,1)$ with $S > 0$ on $(0,1)$. The following *minimum principle* is valid. Namely, if $s \in C^2[a,b]$ is a supersolution of $L_P w = 0$ on (a,b) for an interval $[a,b] \subset (0,1)$ with $s(a)$ and $s(b) \geq 0$, then $s \geq 0$ on (a,b). If moreover, either $s(a) \geq 0$, $s(b) \geq 0$ and $s(a) + s(b) > 0$ or $s(a) \geq 0$, $s(b) \geq 0$ and s is not a solution of $L_P w = 0$ on (a,b), then $s > 0$ on (a,b). Still assuming the existence of a supersolution S of $L_P w = 0$ with $S > 0$ on $(0,1)$, we have the *solvability of Dirichlet problem* (cf. e.g. Chapters 2 and 3 in [6]): for any $\varphi \in C[a,b]$ there exists a $u \in C^2[a,b]$ uniquely determined by $(\varphi(a), \varphi(b))$ which is a solution of $L_P w = 0$ on (a,b) such that $(u(a), u(b)) = (\varphi(a), \varphi(b))$ where $[a,b]$ is an arbitrary interval in $(0,1)$. Such a u will be denoted by $P_\varphi^{(a,b)}$. We also state the *Harnack principle* in the following form. Let (a,b) be an arbitrary interval in $(0,1)$ and $c \in (a,b)$. If $\{u_n\}$ is a sequence of positive solution u_n of $L_P w = 0$ on (a,b) such that $u_n(c) = k$, a constant, $(n = 1, 2, \cdots)$, then there exists a subsequence of $\{u_n\}$ that converges to a solution of $L_P w = 0$ locally uniformly on (a,b). We are ready to begin

Proof of Lemma 20. Since P is nonelliptic, the P-subunit $f_P > 0$ on $(0,1)$. By a simple computation we obtain the following equality:

$$L_{cP} f_P^c = c(1-c)\left(f_P'\right)^2 f_P^{c-2}. \tag{12}$$

On setting $S = f_P^c$ we see that S is a supersolution of $L_{cP} w = 0$ on $(0,1)$ such that $S > 0$ on $(0,1)$. Choose sequences $\{a_n\}$ and $\{b_n\}$ such that $0 < a_n < b_n < 1$, $a_n \downarrow 0$ and $b_n \uparrow 1$ and consider $g_n = (cP)_S^{(a_n,b_n)}$. By the minimum principle, $g_n \leq S$ and $g_{n+1} \leq g_n$. Hence $g = \lim_{n\to\infty} g_n$ is a nonnegative solution of $L_{cP} w = 0$ on $(0,1)$. Then let $s = S - g$ on $(0,1)$ which is a nonnegative supersolution of $L_{cP} w = 0$ on $(0,1)$. By $f_P'(1) = -1$, (12) shows that $L_{cP} f_P^c \not\equiv 0$ on $(0,1)$ and therefore s is not a solution of $L_{cP} w = 0$ on $(0,1)$. Thus $s > 0$ on $(0,1)$. We now show that s satisfies the following minimal property: If h is a solution of $L_{cP} w = 0$ on $(0,1)$ with

$0 \leq h \leq s$ on $(0,1)$, then $h \equiv 0$ on $(0,1)$. In fact, observe that

$$(cP)_s^{(a_n,b_n)} = (cP)_S^{(a_n,b_n)} - (cP)_g^{(a_n,b_n)} = g_n - g \downarrow 0 \quad (n \uparrow \infty).$$

By the minimum principle, $0 \leq h \leq s$ implies that

$$0 \leq h = (cP)_h^{(a_n,b_n)} \leq (cP)_s^{(a_n,b_n)} \downarrow 0 \quad (n \uparrow \infty),$$

which yields $h \equiv 0$ on $(0,1)$.

Since there exists a strictly positive supersolution s of $L_{cP}w = 0$ on $(0,1)$, we can find a unique solution u_n of $L_{cP}w = 0$ on (a_n, b_n) and a unique solution v_n of $L_{cP}w = 0$ on (a, b_n) such that $(u_n(a_n), u_n(b_n)) = (0, s(b_n))$ and $(v_n(a), v_n(b_n)) = (s(a), 0)$ for every $n = 1, 2, \cdots$, where $a \in (a_1, b_1)$. Clearly $0 < u_n < s$ on (a_n, b_n) and $0 < v_n < s$ on (a, b_n). Then set

$$w_n = \frac{s(a)}{u_n(a)} u_n \quad (n = 1, 2, \cdots).$$

Since $w_n(a) = s(a)$ $(n = 1, 2, \cdots)$, by the Harnack principle, we can assume that $\{w_n\}$ converges to a strictly positive solution w of $L_{cP}w = 0$ locally uniformly on $(0,1)$, by choosing a subsequence of $\{w_n\}$ if necessary. Hence $w \in (cP)P(\Omega) \backslash \{0\}$ and thus we see that (cP)-subunit $f_{cP} > 0$ on $(0,1)$. We now show that $w(1) > 0$. Otherwise we must have $w(1) = 0$. By the minimum principle, $v_n \leq v_{n+1} \leq s$ and w and $v = \lim_{n\to\infty} v_n$ is a solution of $L_{cP}w = 0$ on $(a,1)$ such that $v \leq s$ and w and $v(a) = w(a) = s(a)$ and $v(1) = w(1) = 0$. If $v \equiv w$ on $(a,1)$, then $w = v \leq s$ on $(a,1)$. Since $w_n \leq s$ on (a_n, a), we have $w \leq s$ on $(0,a)$. Thus $0 \leq w \leq s$ on $(0,1)$. The minimal property of s must imply $w \equiv 0$, a contradiction. Hence $v < w$ on $(a,1)$ or $w - v > 0$ on $(a,1)$ and $w - v = 0$ at 1. Thus

$$w - v = -(w'(1) - v'(1)) f_{cP}$$

which is again a contradiction since $w(a) - v(a) = 0$ and thus $f_{cP}(a) = 0$.

Finally we set $E = (1/w(1))w$ so that E is a strictly positive solution of $L_{cP}w = 0$ on $(0, 1+\varepsilon)$ with a suitable $\varepsilon > 0$ and $E(1) = 1$. By the minimum principle, $e_{cP,R} \leq E$ on $[R,1]$ for any $R \in (0,1)$ and the (cP)-unit

$$e_{cP} = \lim_{R \downarrow 0} e_{cP,R} \leq E$$

can be defined and therefore cP is hyperbolic on Ω. Although not needed, we can show that we actually have $e_{cP} = E$. □

7. Singularity indices for hyperbolic densities

In view of Corollary 18 we see that the hyperbolicity of a radial density $P = P_0$ on $\Omega \cup \Gamma$ implies that of the radial density P_j on $\Omega \cup \Gamma$ given for $0 < r < 1$ by

$$P_j(r) = P(r) + \frac{j(j+d-2)}{r^2} \quad (j = 0, 1, \cdots)$$

(cf. §3). Hence we can consider the P_j-unit e_{p_j} on $\Omega \cup \Gamma$ $(j = 1, 2, \cdots)$. We have introduced the numbers $\alpha_j(P) = \lim_{r \downarrow 0} f_P(r)/f_{P_j}(r)$ $(j = 1, 2, \cdots)$ and in particular the singularity index of limit form $\alpha(P) = \alpha_1(P) = \lim_{r \downarrow 0} f_P(r)/f_{P_1}(r)$ of a nonelliptic radial density P on $\Omega \cup \Gamma$ in §3. Here we show that f_{P_j} and f_P in the definition of $\alpha_j(P)$ can be replaced by $1/e_{P_j}$ and $1/e_P$ if P is moreover hyperbolic.

Proposition 21 ([18]). *The numbers $\alpha_j(P)$ associated with a hyperbolic radial density P on $\Omega \cup \Gamma$ can also be given by*

$$\alpha_j(P) = \lim_{r \downarrow 0} \frac{e_{P_j}(r)}{e_P(r)} \quad (j = 1, 2, \cdots)$$

and in particular the singularity index of limit form $\alpha(P) = \alpha_1(P)$ of P is given by

$$\alpha(P) = \lim_{r \downarrow 0} \frac{e_{P_1}(r)}{e_P(r)}.$$

Proof. For a $j = 1, 2, \cdots$, the P_j-unit e_{P_j} is expressed as

$$\begin{aligned} e_{P_j}(r) &= f_{P_j}(r) \int_0^r \frac{dt}{t^{d-1} f_{P_j}(t)^2} \\ &= f_{P_j}(r) \int_0^r \frac{1}{t^{d-1} f_P(t)^2} \left(\frac{f_P(t)}{f_{P_j}(t)} \right)^2 dt. \end{aligned}$$

By Corollary 7, $f_P(t)/f_{P_j}(t) \leq f_P(r)/f_{P_j}(r)$ $(0 < t \leq r)$ and thus

$$\begin{aligned} e_{P_j}(r) &\leq f_{P_j}(r) \int_0^r \frac{1}{t^{d-1} f_P(t)^2} \left(\frac{f_P(r)}{f_{P_j}(r)} \right)^2 dt \\ &= \frac{f_P(r)}{f_{P_j}(r)} f_P(r) \int_0^r \frac{dt}{t^{d-1} f_P(t)^2} = \frac{f_P(r)}{f_{P_j}(r)} e_P(r). \end{aligned}$$

Hence we have the following inequality:

$$\frac{e_{P_j}(r)}{e_P(r)} \leq \frac{f_P(r)}{f_{P_j}(r)} \quad (0 < r < 1,\ j = 1, 2, \cdots).$$

If $\alpha_j(P) = \lim_{r\downarrow 0} f_P(r)/f_{P_j}(r) = 0$, then the above inequality implies that $\lim_{r\downarrow 0} e_{P_j}(r)/e_P(r) = 0$ and a fortiori we conclude that $\alpha_j(P) = \lim_{r\downarrow 0} e_{P_j}(r)/e_P(r)$. If $\alpha_j(P) > 0$, then by l'Hospital rule, we deduce

$$\begin{aligned}
&\lim_{r\downarrow 0} \frac{e_{P_j}(r)}{e_P(r)} \\
&= \lim_{r\downarrow 0} \left(f_{P_j}(r) \int_0^r \frac{dt}{t^{d-1} f_{P_j}(t)^2} \right) \Big/ \left(f_P(r) \int_0^r \frac{dt}{t^{d-1} f_P(t)^2} \right) \\
&= \left(\lim_{r\downarrow 0} \frac{f_{P_j}(r)}{f_P(r)} \right) \cdot \left(\lim_{r\downarrow 0} \left(\int_0^r \frac{dt}{t^{d-1} f_{P_j}(t)^2} \right) \Big/ \left(\int_0^r \frac{dt}{t^{d-1} f_P(t)^2} \right) \right) \\
&= \alpha_j(P)^{-1} \cdot \lim_{r\downarrow 0} \left(\frac{1}{r^{d-1} f_{P_j}(r)^2} \right) \Big/ \left(\frac{1}{r^{d-1} f_P(r)^2} \right) \\
&= \alpha_j(P)^{-1} \cdot \alpha_j(P)^2 = \alpha_j(P).
\end{aligned}$$

□

We also show that, in the definition of the singularity index of integrated form $\beta(P)$ of a nonelliptic radial density P on $\Omega \cup \Gamma$, the P-subunit f_P can be replaced by the P-unit e_P if P is moreover hyperbolic. One must note that the integrating regions are different in these two expressions.

Proposition 22 *The singularity index of integrated form $\beta(P)$ of a hyperbolic radial density P on $\Omega \cup \Gamma$ takes the following form:*

$$\begin{aligned}
1/\beta(P) &= \iint_{0\leq t\leq s\leq 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{e_P(t)}{e_P(s)} \right)^2 ds dt \quad (\textit{double integral form}) \\
&= \int_0^1 t^{d-3} e_P(t)^2 \left(\int_t^1 \frac{ds}{s^{d-1} e_P(s)^2} \right) dt \\
&\qquad\qquad (\textit{iterated integral form}).
\end{aligned}$$

Proof. Rewrite $1/\beta(P)$ by using $e_P(t) = f_P(t) \int_0^t s^{1-d} f_P(s)^{-2} ds$ and

$f_P(t) = e_P(t) \int_t^1 s^{1-d} e_P(s)^{-2} ds$ as follows:

$$\begin{aligned}
1/\beta(P) &= \int_0^1 t^{d-3} f_P(t) \left(f_P(t) \int_0^t \frac{ds}{s^{d-1} f_P(s)^2} \right) dt \\
&= \int_0^1 t^{d-3} f_P(t) e_P(t) dt = \int_0^1 t^{d-3} e_P(t) f_P(t) dt \\
&= \int_0^1 t^{d-3} e_P(t) \left(e_P(t) \int_t^1 \frac{ds}{s^{d-1} e_P(s)^2} \right) dt \\
&= \int_0^1 t^{d-3} e_P(t)^2 \left(\int_t^1 \frac{ds}{s^{d-1} e_P(s)^2} \right) dt \\
&= \iint_{0 \le t \le s \le 1} \frac{t^{d-3}}{s^{d-1}} \left(\frac{e_P(t)}{e_P(s)} \right)^2 dsdt.
\end{aligned}$$

□

In the course of the above proof we have seen the following mixed integral expression of $\beta(P)$ by using both of e_P and f_P:

Corollary 23 *If P is a hyperbolic radial density on $\Omega \cup \Gamma$, then*

$$1/\beta(P) = \int_0^1 t^{d-3} e_P(t) f_P(t) dt.$$

Another form of mixed expression is derived from the above mixed expression. The expression will play an essential role in the proof of the homogeneity of Picard dimension given below.

Corollary 24 *If P is a hyperbolic radial density on $\Omega \cup \Gamma$, then*

$$1/\beta(P) = \int_0^1 \frac{dr}{r^2 \left(\dfrac{e'_P(r)}{e_P(r)} - \dfrac{f'_P(r)}{f_P(r)} \right)}.$$

Proof. Differentiate the both sides of $e_P(r) = f_P(r) \int_0^r t^{1-d} f_P(t)^{-2} dt$ with respect to r. Then we obtain

$$e'_P(r) = \frac{f'_P(r)}{f_P(r)} e_P(r) + \frac{1}{r^{d-1} f_P(r)}$$

or

$$r^{d-1} \left(\frac{e'_P(r)}{e_P(r)} - \frac{f'_P(r)}{f_P(r)} \right) = \frac{1}{e_P(r) f_P(r)}.$$

Substituting $e_P(r)f_P(r)$ in Corollary 23 by the above expression of $e_P(r)f_P(r)$ we deduce

$$\begin{aligned} 1/\beta(P) &= \int_0^1 r^{d-3} \frac{dr}{r^{d-1}\left(\frac{e_P'(r)}{e_P(r)} - \frac{f_P'(r)}{f_P(r)}\right)} \\ &= \int_0^1 \frac{dr}{r^2\left(\frac{e_P'(r)}{e_P(r)} - \frac{f_P'(r)}{f_P(r)}\right)}. \end{aligned}$$

□

Having established Lemma 20 and Corollary 24 we are ready to prove a form of the homogeneity of the Picard dimension $\dim(P, \Omega)$ in radial densities P which is a preliminary version of the homogeneity of the Picard dimension $\dim P$ at the origin established in §8.

Theorem 25 (Homogeneity). *If P is a radial density on $\Omega \cup \Gamma$, then* $\dim(cP, \Omega) \geq \dim(P, \Omega)$ *for any* $0 < c \leq 1$, *or equivalently,* $\dim(cP, \Omega) \leq \dim(P, \Omega)$ *for any* $c > 1$.

Proof. Since the two assertions in the above statement are clearly equivalent, we only have to prove that $\dim(cP, \Omega) \geq \dim(P, \Omega)$ for $0 < c \leq 1$. If P is elliptic, then the conclusion is trivial. Hence we may assume that P is nonelliptic. Then cP is hyperbolic by Lemma 20 and in particular nonelliptic and thus $\dim(cP, \Omega) \geq 1$. If P is parabolic on $\Omega \cup \Gamma$, then, by Corollary 17, $\dim(P, \Omega) = 1$ and a fortiori $\dim(P, \Omega) \leq \dim(cP, \Omega)$.

Therefore we only have to treat the case P is hyperbolic on $\Omega \cup \Gamma$. Again by Lemma 20, cP is also hyperbolic on $\Omega \cup \Gamma$. Observe that

$$L_{cP} f_P(r)^c = c(1-c) f_P(r)^{c-2} \left(f_P'(r)\right)^2 \geq 0$$

on $(0,1)$ so that $f_P(r)^c$ is a supersolution of $L_{cP}w = 0$ on $(0,1)$. Fix an arbitrary number $0 < s < 1$. Applying the minimum principle to $f_P(r)^c/f_P(s)^c - f_{cP}(r)/f_{cP}(s)$ on $(s,1)$ as a supersolution of $L_{cP}w = 0$ of the variable r we see that

$$\frac{f_P(r)^c}{f_P(s)^c} \geq \frac{f_{cP}(r)}{f_{cP}(s)} \quad (0 < s < r < 1)$$

or equivalently

$$\frac{f_P(s)^c}{f_{cP}(s)} \leq \frac{f_P(r)^c}{f_{cP}(r)} \quad (0 < s < r < 1).$$

This shows that $f_P(r)^c/f_{cP}(r)$ or $\log(f_P(r)^c/f_{cP}(r))$ is an increasing function of r on $(0,1)$ and thus

$$\frac{d}{dr} \log \frac{f_P(r)^c}{f_{cP}(r)} \geq 0 \quad (0 < r < 1)$$

and a fortiori we can conclude that

$$c\frac{f_P'(r)}{f_P(r)} \geq \frac{f_{cP}'(r)}{f_{cP}(r)} \quad (0 < r < 1). \tag{13}$$

Fix an arbitrary $R \in (0,1)$ and consider

$$e_{P,R}(r) = f_P(r) \int_R^r \frac{dt}{t^{d-1} f_P(t)^2}$$

which converges to the P-unit $e_P(r)$ on $\Omega \cup \Gamma$. Similarly

$$e_{cP,R}(r) = f_{cP}(r) \int_R^r \frac{dt}{t^{d-1} f_{cP}(t)^2}$$

converges to the (cP)-unit e_{cP} on $\Omega \cup \Gamma$. Observe that

$$L_{cP} e_{P,R}(r)^c = c(1-c) e_{P,R}(r)^{c-2} \left(e_{P,R}'(r)\right)^2 \geq 0$$

on $(0,1)$ so that $e_{P,R}(r)^c$ is a supersolution of $L_{cP}w = 0$ on $(R,1)$. Fix an arbitrary number $s \in (0,1)$ and then choose an arbitrary $R \in (0,s)$. The minimum principle applied to the supersolution $e_{P,R}(r)^c/e_{P,R}(s)^c - e_{cP,R}(r)/e_{cP,R}(s)$ of $L_{cP}w = 0$ on (R,s) implies that

$$\frac{e_{P,R}(r)^c}{e_{P,R}(s)^c} \geq \frac{e_{cP,R}(r)}{e_{cP,R}(s)} \quad (0 < R \leq r < s \leq 1).$$

On letting $R \downarrow 0$ we deduce

$$\frac{e_P(s)^c}{e_{cP}(s)} \leq \frac{e_P(r)^c}{e_{cP}(r)} \quad (0 < r \leq s \leq 1)$$

which shows that $e_P(r)^c/e_{cP}(r)$ or $\log(e_P(r)^c/e_{cP}(r))$ is decreasing on $(0,1)$

so that

$$\frac{d}{dr}\log\frac{e_P(r)^c}{e_{cP}(r)} \le 0 \quad (0 < r < 1)$$

and hence we can conclude that

$$c\frac{e'_P(r)}{e_P(r)} \le \frac{e'_{cP}(r)}{e_{cP}(r)} \quad (0 < r < 1). \tag{14}$$

From (13) and (14) it follows that

$$c\left(\frac{e'_P(r)}{e_P(r)} - \frac{f'_P(r)}{f_P(r)}\right) \le \frac{e'_{cP}(r)}{e_{cP}(r)} - \frac{f'_{cP}(r)}{f_{cP}(r)} \quad (0 < r < 1).$$

Hence we deduce that

$$c\int_0^1 \frac{dr}{r^2\left(\frac{e'_{cP}(r)}{e_{cP}(r)} - \frac{f'_{cP}(r)}{f_{cP}(r)}\right)} \le \int_0^1 \frac{dr}{r^2\left(\frac{e'_P(r)}{e_P(r)} - \frac{f'_P(r)}{f_P(r)}\right)}$$

or, by Corollary 24, $c/\beta(cP) \le 1/\beta(P)$ and a fortiori we obtain that

$$c\beta(P) \le \beta(cP) \quad (0 < c \le 1).$$

Since P and cP are hyperbolic, $\dim(P,\Omega) \ge 1$ and $\dim(cP,\Omega) \ge 1$. If $\dim(P,\Omega) = 1$, then $\dim(cP,\Omega) \ge \dim(P,\Omega)$. If $\dim(P,\Omega) = \aleph$, then, by Theorem 11 and Proposition 13, $\beta(P) > 0$. The above inequality implies that $\beta(cP) > 0$ and again by Theorem 11 and Proposition 13 we conclude that $\dim(cP,\Omega) = \aleph$ and the inequality $\dim(cP,\Omega) \ge \dim(P,\Omega)$ is trivially valid. In view of Theorem 11, the proof is herewith complete. □

Corollary 26 ([11]). *If P is a nonnegative radial density on $\Omega \cup \Gamma$, then $\dim(cP,\Omega) = \dim(P,\Omega)$ for every $c > 0$.*

Proof. We only have to treat the case $0 < c \le 1$. By Theorem 25, we have $\dim(cP,\Omega) \ge \dim(P,\Omega)$. On the other hand, since $P \ge 0$, $cP \le P$ is valid on $\Omega \cup \Gamma$ and hence Theorem 14 assures that $\dim(cP,\Omega) \le \dim(P,\Omega)$. We thus obtain $\dim(cP,\Omega) = \dim(P,\Omega)$. □

From the view point of Corollary 26 it is a natural question to ask whether the inequality in Theorem 25 can be replaced by the equality. It is not too difficult to give a counter example P on $\Omega \cup \Gamma$ to show that in fact the inequality cannot be replaced by the equality. The importance of the following example lies in the fact that $\dim(cP,\Omega_a) < \dim(P,\Omega_a)$ $(c > 1)$

holds not only for $a = 1$ but also for every $0 < a \leq 1$. (Compare this with Lemma 20; there $0 < c < 1$.) This will play an important role in Assertions 36 and 37 in §8.

Example 27 ([9]). The *Imai density* $I(x)$ is a nonpositive radial density on $\Omega \cup \Gamma$ given by

$$I(x) = -\frac{1}{4|x|^2}\left((d-2)^2 + \frac{1}{\left(\log \frac{2}{|x|}\right)^2}\right),$$

for which $\dim(I, \Omega_a) = 1$ for every $0 < a \leq 1$ and $\dim(cI, \Omega_a) = 0$ for every $c > 1$ and every $0 < a \leq 1$.

Proof. A rather tedious calculation is needed but somehow it is not difficult to check that the pair $(p(r), p(r)q(r))$ determined by

$$p(r) = r^{-\frac{d-2}{2}}\left(\log \frac{2}{r}\right)^{1/2} \quad \text{and} \quad q(r) = \log \frac{2}{r}$$

is a system of fundamental solutions of $L_I w = 0$ on $(0, 1)$. For each $0 < a < 1$, set

$$^aI(r) = a^2 I(ar) = -\frac{1}{4r^2}\left((d-2)^2 + \frac{1}{\left(\log \frac{2}{ar}\right)^2}\right).$$

Then $(p(ar), p(ar)q(ar))$ is a system of fundamental solutions of $L_{^aI} w = 0$ on $(0, 1)$. Since $p(ar) > 0$ on $(0, 1)$, $^aIP(\Omega)\backslash\{0\} \neq \emptyset$ and hence ^aI is nonelliptic on $\Omega \cup \Gamma$. Hence $\dim(I, \Omega_a) = \dim(^aI, \Omega) \geq 1$ (cf. §2). On the other hand, by Corollary 15, $^aI \leq 0$ implies $\dim(I, \Omega_a) = \dim(^aI, \Omega) \leq 1$. Hence we have established that $\dim(I, \Omega_a) = 1$ for every $0 < a \leq 1$.

We need to introduce the following auxiliary density I_ε on $\Omega \cup \Gamma$:

$$I_\varepsilon(r) = -\frac{1}{4r^2}\left((d-2)^2 + \frac{1+\varepsilon^2}{\left(\log \frac{2}{r}\right)^2}\right) \qquad (\varepsilon > 0).$$

Then we see that

$$Ap(r)\sin\left(\frac{\varepsilon}{2}q(r)+B\right) \quad (A, B \in \mathbf{R})$$

is a general solution of $L_{I_\varepsilon}w = 0$ on $(0,1)$. For each $0 < a \leq 1$, set

$$^{a}I_\varepsilon(r) = a^2 I_\varepsilon(ar) = -\frac{1}{4r^2}\left((d-2)^2 + \frac{1+\varepsilon^2}{\left(\log\dfrac{2}{ar}\right)^2}\right).$$

Then the function

$$Ap(ar)\sin\left(\frac{\varepsilon}{2}q(ar)+B\right) \quad (A, B \in \mathbf{R})$$

is a general solution of $L_{^{a}I_\varepsilon}w = 0$ on $(0,1)$, which always takes both strictly positive and negative values in $(0,1)$ unless it is identically zero. Hence $^{a}I_\varepsilon P(\Omega) = \{0\}$ and $^{a}I_\varepsilon$ is elliptic on $\Omega \cup \Gamma$.

Take an arbitrary $c > 1$. We can find an $\varepsilon > 0$ such that $cI \leq I_\varepsilon$ on $(0,1)$. Actually we only have to choose ε in $(0, \sqrt{c-1})$. Then $^{a}(cI) \leq {}^{a}I_\varepsilon$. By Proposition 6 we see that $^{a}(cI)$ is elliptic along with $^{a}I_\varepsilon$. Therefore we now conclude that $\dim(cI, \Omega_a) = \dim(^{a}(cI), \Omega) = 0$ for every $c > 1$ and every $0 < a \leq 1$. □

8. Picard dimensions at the origin

In this section we prove the existence of a $b \in (0,1]$ for a given radial density P on $\Omega \cup \Gamma$ such that $\dim(P, \Omega_a) = \dim(P, \Omega_b)$ for every $a \in (0,b]$ so that we can define $\dim P = \lim_{a \downarrow 0} \dim(P, \Omega_a)$, the Picard dimension of P at the origin, so to speak. Then we will complete the proofs of the main theorems 1 and 2 of this paper mentioned in the introduction. We recall here the notation $^{a}P(x) = a^2 P(ax)$ and $u_a(x) = u(ax)$ (cf. §2). We start with the following simple fact:

Lemma 28 *Suppose P is a radial density on $\Omega \cup \Gamma$. If ^{b}P is nonelliptic for a $b \in (0,1]$, then ^{a}P is also nonelliptic for $a \in (0,b]$.*

Proof. Since ^{b}P is nonelliptic, there exists a $u \in {}^{b}PP(\Omega)\backslash\{0\}$ so that $u_{b^{-1}} \in PP(\Omega_b)\backslash\{0\} \subset PP(\Omega_a)\backslash\{0\}$ for any $a \in (0,b]$. Hence $u_{b^{-1}a} = (u_{b^{-1}})_a \in {}^{a}PP(\Omega)\backslash\{0\}$, which shows that ^{a}P is nonelliptic. □

The following claim is less trivial. (Compare this with Lemma 20.)

Lemma 29 *Suppose P is a radial density on $\Omega\cup\Gamma$. If ${}^bP$ is nonelliptic for a $b\in(0,1]$, then ${}^aP$ is hyperbolic for every $a\in(0,b)$.*

Proof. Since ${}^bP$ is nonelliptic, the ${}^bP$-subunit $f_{{}^bP}(r)>0$ on $(0,1)$ so that $s(r)=f_{{}^bP}(b^{-1}r)$ belongs to $PP(\Omega_b;\Gamma_b)\backslash\{0\}$ and $s>0$ on $\Omega_a\cup\Gamma_a$ for every fixed $a\in(0,b)$. Thus there exists a unique radial $e_{P,R}(\cdot\,;a)\in PP(\Omega_a\backslash(\Omega_R\cup\Gamma_R))$ $(0<R<a)$ with boundary values $e_{P,R}(a;a)=1$ and $e_{P,R}(R;a)=0$. By the minimum principle, $e_{P,R}(\cdot\,;a)\le s(a)^{-1}s$ and $\{e_{P,R}(\cdot\,;a)\}_{R\downarrow 0}$ is increasing. We can thus define

$$e_P(r;a)=\lim_{R\downarrow 0}e_{P,R}(r;a).$$

It is easily seen that $e_{{}^aP}(r)=e_P(ar;a)$ is the ${}^aP$-unit and ${}^aP$ is hyperbolic. □

The function $e_P(r;a)$ in the above proof will be referred to as the *P-unit* on Ω_a. We have also seen in the proof that $e_P(ar;a)$ is the ${}^aP$-unit $e_{{}^aP}(r)=e_{{}^aP}(r;1)$ on Ω.

Lemma 30 *Suppose P is a radial density on $\Omega\cup\Gamma$. If ${}^cP$ is nonelliptic for a $c\in(0,1]$, then ${}^aP$ and ${}^bP$ are hyperbolic and*

$$\beta({}^aP)\ge\beta({}^bP) \tag{15}$$

for every $0<a<b<c$.

Proof. By Lemma 29, ${}^aP$ and ${}^bP$ are hyperbolic and $e_{{}^aP}(r)=e_P(ar;a)$. Hence, by Proposition 22, we have

$$\begin{aligned}1/\beta({}^aP) &= \iint_{0\le t\le s\le 1}\frac{t^{d-3}}{s^{d-1}}\left(\frac{e_{{}^aP}(t)}{e_{{}^aP}(s)}\right)^2 dsdt\\ &= \iint_{0\le t\le s\le 1}\frac{t^{d-3}}{s^{d-1}}\left(\frac{e_P(at;a)}{e_P(as;a)}\right)^2 dsdt.\end{aligned}$$

On replacing variables at and as by t and s, respectively, we see that

$$1/\beta({}^aP)=\iint_{0\le t\le s\le a}\frac{t^{d-3}}{s^{d-1}}\left(\frac{e_P(t;a)}{e_P(s;a)}\right)^2 dsdt. \tag{16}$$

It is easy to see that $e_P(r;a) = e_P(r;b)/e_P(a;b)$ for $0 < a < b < c$. Hence

$$
\begin{aligned}
1/\beta({}^aP) &= \iint_{0\le t\le s\le a} \frac{t^{d-3}}{s^{d-1}} \left(\frac{e_P(t;b)}{e_P(s;b)}\right)^2 dsdt \\
&\le \iint_{0\le t\le s\le b} \frac{t^{d-3}}{s^{d-1}} \left(\frac{e_P(t;b)}{e_P(s;b)}\right)^2 dsdt.
\end{aligned}
$$

The last term is $1/\beta({}^bP)$ in view of (16), and we have seen that $1/\beta({}^aP) \le 1/\beta({}^bP)$ or $\beta({}^aP) \ge \beta({}^bP)$. $\square$

We are ready to show that $\dim(P, \Omega_a)$ is monotone in a:

Proposition 31 *Suppose P is a radial density on $\Omega \cup \Gamma$. For any pair (a, b) of real numbers with $0 < a \le b \le 1$,*

$$\dim(P, \Omega_a) \ge \dim(P, \Omega_b). \tag{17}$$

Proof. In view of $\dim(P, \Omega_a) = \dim({}^aP, \Omega)$ and $\dim(P, \Omega_b) = \dim({}^bP, \Omega)$, the inequality (17) is equivalent to the following inequality

$$\dim({}^aP, \Omega) \ge \dim({}^bP, \Omega) \quad (0 < a \le b \le 1). \tag{18}$$

This is trivially true if $\dim({}^bP, \Omega) = 0$ and thus we can assume $\dim({}^bP, \Omega) > 0$. If $\dim({}^bP, \Omega) = 1$, then ${}^bP$ is nonelliptic and hence, by Lemma 28, ${}^aP$ is nonelliptic. Thus $\dim({}^aP, \Omega) \ge 1$ and (18) is valid. By Theorem 11, the only possibility left is the case $\dim({}^bP, \Omega) = \aleph$. Once more by Theorem 11 and by Proposition 13, we must have $\beta({}^bP) > 0$. This with (15) implies $\beta({}^aP) > 0$. Thus, by the same reason as above, we obtain $\dim({}^aP, \Omega) = \aleph$. A fortiori (18) is also true. $\square$

We now show that $\dim(P, \Omega_a)$ is constant for sufficiently small $a > 0$:

Theorem 32 *Suppose P is a radial density on $\Omega \cup \Gamma$. There exists a $b \in (0, 1]$ such that $\dim(P, \Omega_a) = \dim(P, \Omega_b)$ for every $0 < a \le b$.*

Proof. In view of the fundamental theorem 11, $\gamma = \sup\{\dim(P, \Omega_a) = \dim({}^aP, \Omega) : 0 < a \le 1\}$ is 0, 1 or $\aleph$. If $\gamma = 0$, then (17) assures that we only have to set $b = 1$. If $\gamma = 1$, then there exists a $b \in (0, 1]$ with $\dim(P, \Omega_b) = 1$ so that again (17) assures that $\dim(P, \Omega_a) = \dim(P, \Omega_b)$ for every $a \in (0, b]$. Finally let $\gamma = \aleph$. There exists a $b \in (0, 1]$ such that $\dim(P, \Omega_b) = \aleph$. Hence, by (17), $\dim(P, \Omega_a) = \dim(P, \Omega_b) = \aleph$ for every $a \in (0, b]$. $\square$

As mentioned in the introduction we define the *Picard dimension* $\dim P$ *at the origin* 0 of a general density P on $\Omega \cup \Gamma$ by

$$\dim P = \lim_{a \downarrow 0} \dim(P, \Omega_a),$$

which is in fact a common fixed cardinal number $\dim(P, \Omega_a)$ for every small $a > 0$. A proof for this assertion will be given in Appendix at the end of this paper. However, as far as radial densities P concern, this fact is just established in the above Theorem 32.

Proof of Theorem 1 (Monotoneity). Suppose that radial densities P and Q on $\Omega \cup \Gamma$ satisfies $P \leq Q$ on Ω_b for some $0 < b < 1$. Then ${}^aP \leq {}^aQ$ on $\Omega \cup \Gamma$ for every $0 < a \leq b$. By Theorem 14, we have $\dim({}^aP, \Omega) \leq \dim({}^aQ, \Omega)$ or $\dim(P, \Omega_a) \leq \dim(Q, \Omega_a)$ for every $0 < a \leq b$. Therefore

$$\dim P = \lim_{a \downarrow 0} \dim(P, \Omega_a) \leq \lim_{a \downarrow 0} \dim(Q, \Omega_a) = \dim Q.$$

□

Example 33 The Picard dimension at the origin $\dim 0$ of the harmonic density 0 on $\Omega \cup \Gamma$ is one.

Proof. By Example 10, $\dim(0, \Omega) = 1$. Since ${}^a0 = 0$ for every $0 < a < 1$, $\dim(0, \Omega_a) = \dim({}^a0, \Omega) = \dim(0, \Omega) = 1$ and $\dim 0 = \lim_{a \downarrow 0} \dim(0, \Omega_a) = 1$. □

Proposition 34 *Suppose P is a radial density on $\Omega \cup \Gamma$. If P is nonnegative (nonpositive, resp.) in a punctured ball about the origin, then* $\dim P \geq 1$ ($\dim P \leq 1$, *resp.*).

Proof. Suppose $P \geq 0$ ($P \leq 0$, resp.) in Ω_b for some $b \in (0, 1]$. By Theorem 1 and Example 33, we see that $\dim P \geq \dim 0 = 1$ ($\dim P \leq \dim 0 = 1$, resp.). □

Proof of Theorem 2 (Homogeneity). It is clear that two assertions in Theorem 2 are equivalent. Hence we only have to prove that $\dim(cP) \geq \dim P$ holds for any radial density P on $\Omega \cup \Gamma$ and for any $0 < c \leq 1$. By Theorem 25, we have $\dim(c\,{}^aP, \Omega) \geq \dim({}^aP, \Omega)$ for every $0 < a \leq 1$ because ${}^aP$ is also a radial density on $\Omega \cup \Gamma$. In view of ${}^a(cP) = c\,{}^aP$, we see that $\dim(cP, \Omega_a) = \dim({}^a(cP), \Omega) = \dim(c\,{}^aP, \Omega)$ and trivially $\dim(P, \Omega_a) = \dim({}^aP, \Omega)$. Thus we see that $\dim(cP,\Omega_a) \geq \dim(P,\Omega_a)$ for every $0 < a \leq 1$. Therefore

$$\dim(cP) = \lim_{a\downarrow 0} \dim(cP,\Omega_a) \geq \lim_{a\downarrow 0} \dim(P,\Omega_a) = \dim P.$$

□

Proposition 35 ([11]). *Suppose P is a radial density on $\Omega\cup\Gamma$ such that $P \geq 0$ in a punctured ball about the origin. Then* $\dim(cP) = \dim P$ *for every $c > 0$.*

Proof. We only have to consider the case $0 < c \leq 1$. Theorem 2 assures that $\dim(cP) \geq \dim P$. On the other hand, since $cP \leq P$ in a punctured ball centered at the origin where $P \geq 0$, Theorem 1 implies $\dim(cP) \leq \dim P$. Thus we can deduce the identity $\dim(cP) = \dim P$. □

From the view point of the above Proposition 35 one might suspect that the inequality signs in $\dim(cP) \geq \dim P$ $(0 < c \leq 1)$ and $\dim(cP) \leq \dim P$ $(c > 1)$ of Theorem 2 are in reality able to be replaced by the equality signs. That this is not the case and therefore that Theorem 2 is the best possible generalization of proposition 35 to signed radial densities are seen by the following two assertions 36 and 37.

Assertion 36 *There exists a radial density P on $\Omega\cup\Gamma$ such that* $\dim P >$ $\dim(cP)$ *for every $c > 1$.*

Proof. Take the Imai density I in Example 27. Since $\dim(I,\Omega_a) = 1$ and $\dim(cI,\Omega_a) = 0$ for every $c > 1$ and every $0 < a \leq 1$, we have $\dim I = \lim_{a\downarrow 0} \dim(I,\Omega_a) = 1$ and $\dim(cI) = \lim_{a\downarrow 0} \dim(cI,\Omega_a) = 0$. Thus I qualifies to be a P in the above assertion. □

Assertion 37 *There exists a radial density P on $\Omega \cup \Gamma$ for any given $0 < c < 1$ such that* $\dim P < \dim(cP)$.

Proof. Take an arbitrary density Q satisfying Assertion 36. For an arbitrarily chosen $c \in (0,1)$ we consider $P = c^{-1}Q$. Since $c^{-1} > 1$, we have $\dim(c^{-1}Q) < \dim Q$ or, by $Q = cP$, $\dim P < \dim(cP)$. □

The above Assertion 37 is not as strong as Assertion 36 but actually it is recently shown by Imai [10] that the complete analogue to Assertion 36 is valid: There exists a radial density P on $\Omega\cup\Gamma$ such that $\dim P < \dim(cP)$ for every $0 < c < 1$. In the Assertions 36 and 37 the relevant densities involve those with zero Picard dimension at the origin and hence we are naturally led to ask the following problem whose partial answer corresponding to Assertion 37 is recently given in [30].

Problem 38 Does there exist a radial density P such that $\dim(cP) > \dim P \geq 1$ (for every $0 < c < 1$) ($1 \leq \dim(cP) < \dim P$ (for every $c > 1$), resp.) ?

Appendix

Let M be a connected, countable, orientable and noncompact Riemannian manifold of class C^∞ of dimension $d \geq 2$ and Δ the Laplace-Beltrami operator on M. A density P on M is a locally Hölder continuous function on M. We consider the time independent Schrödinger equation

$$(-\Delta + P(x))u(x) = 0 \tag{19}$$

whose potential is a density P on M. As in the text we denote by $P(G)$ the space of C^2-solutions of (19) on an open subset G of M. We also consider the space $PP(G) = \{u \in P(G) : u \geq 0 \text{ on } G\}$ as in the text. An *end* N of M is an open subset of M such that $M \backslash N$ is the closure of a nonempty relatively compact subregion of M and the relative boundary ∂N of N consists of a finite number of mutually disjoint smooth closed hypersurfaces. Again as in the text we are interested in the space

$$PP(N; \partial N) = \{u \in PP(N) \cap C(\overline{N}) : u|\partial N = 0\}.$$

For two ends N_1 and N_2 we say that $PP(N_1; \partial N_1)$ is *isomorphic* to $PP(N_2; \partial N_2)$ if there exists a positively homogeneous additive bijection τ of $PP(N_1; \partial N_1)$ onto $PP(N_2; \partial N_2)$. In this case we have

$$\#(\text{ex.}\, PP(N_1; \partial N_1)) = \#(\text{ex.}\, PP(N_2; \partial N_2))$$

where $\text{ex.}\, PP(N_i; \partial N_i)$ is the set of extremal rays of the positive cone $PP(N_i; \partial N_i)$ and $\#$ indicates the cardinal number. We prove

Theorem 40 *There exists an end N_0 of M such that $PP(N_1; \partial N_1)$ is isomorphic to $PP(N_2; \partial N_2)$ for all ends N_1 and N_2 of M contained in N_0 with their closures.*

Proof. If $PP(N; \partial N) = \{0\}$ for every end N of M, then clearly any choice of an end N_0 of M will do. If there is an end N_0 of M with $PP(N_0; \partial N_0) \neq \{0\}$, then we can show that the end N_0 is a required one: $PP(N_1; \partial N_1)$ is isomorphic to $PP(N_2;\partial N_2)$ for all ends N_1 and N_2 of M with $\overline{N}_1 \cup \overline{N}_2 \subset N_0$. Clearly we only have to prove that $PP(N_1;\partial N_1)$ is isomorphic to $PP(N_2;\partial N_2)$ when $\overline{N}_2 \subset N_1 \subset \overline{N}_1 \subset N_0$. We fix an exhaustion $\{M_n\}$ $(n=1,2,\cdots)$ of M with $M\backslash N_2 \subset M_1$, i.e. $\{M_n\}$ is a sequence of relatively compact subregions M_n of M such that $\overline{M}_n \subset M_{n+1}$ and ∂M_n consists of a finite number of mutually disjoint smooth closed hypersurfaces $(n = 1,2,\cdots)$ and $\cup_{n\geq 1}M_n = M$. We consider the third end $N_3 = M\backslash\overline{M}_1$ which satisfies $\overline{N}_3 \subset N_2$. We fix an $s \in PP(N_0;\partial N_0)\backslash\{0\}$ such that

$$s(x) \geq 1 \quad (x \in \overline{N}_1\backslash N_3).$$

Since s is strictly positive on $\overline{N}_1$, the minimum principle for solutions for (19) on any relatively compact open subset of N_1 is valid and the Dirichlet problem for the equation (19) for any relatively compact smooth open subset of N_1 is solvable (cf. e.g. Chapters 2 and 3 in [6]).

For each $\varphi \in C(\partial N_2)$ let $D_n\varphi \in P(M_n \cap N_2) \cap C(\overline{M}_n \cap \overline{N}_2)$ such that $D_n\varphi = \varphi$ on ∂N_2 and 0 on ∂M_n $(n = 1,2,\cdots)$. By the minimum principle, $D_n\varphi \leq D_{n+1}\varphi$ on $\overline{M}_n \cap \overline{N}_2$ and $D_n\varphi \leq (\sup_{\partial N_2}\varphi)s$ $(n = 1,2,\cdots)$ if $\varphi \geq 0$ on ∂N_2. Thus

$$D\varphi = \lim_{n\to\infty} D_n\varphi$$

exists on $\overline{N}_2$ and belongs to $PP(N_2) \cap C(\overline{N}_2)$. For general $\varphi \in C(\partial N_2)$ we can define

$$D\varphi = D\varphi^+ - D\varphi^-$$

where $\varphi^+ = \max(\varphi,0)$ and $\varphi^- = -\min(\varphi,0)$ on ∂N_2. Clearly $D: C(\partial N_2) \to P(N_2) \cap C(\overline{N}_2)$ is a linear operator and order-preserving: $\varphi_1 \leq \varphi_2$ implies $D\varphi_1 \leq D\varphi_2$.

Using the operator D we define an operator τ by

$$\tau u = u - Du.$$

If $u \in PP(N_1;\partial N_1)$, then $u - D_n u \geq 0$ on $\partial(M_n \cap N_2)$ and by the minimum principle the same inequality holds on $M_n \cap N_2$. On letting $n \uparrow \infty$ we see that $u - Du \geq 0$ on N_2 and belongs to $PP(N_2;\partial N_2)$. Hence $\tau : PP(N_1;\partial N_1) \to PP(N_2;\partial N_2)$ is positively homogeneous and additive operator. We will see that τ is injective and surjective, i.e. τ is bijective.

To see that τ is injective, let $\tau u = \tau v$ on N_2 for some u and v in $PP(N_1;\partial N_1)$. We need to show that $u = v$ or $w = u - v = 0$ on N_1.

For the purpose we only have to show that $w = 0$ on ∂N_2. Then $w = 0$ on $\partial(N_1 \backslash \overline{N}_2)$ and the minimum principle assures that $w = 0$ on $N_1 \backslash \overline{N}_2$. By the unicity principle we can conclude that $w = 0$ on N_1 as required. Contrary to the assertion we assume that $w \neq 0$ on ∂N_2. Considering $-w$ instead of w if necessary, we can assume that $\sup_{\partial N_2} w > 0$. Clearly

$$c = \inf\{\lambda \in \mathbf{R} : \lambda s \geq w \text{ on } \partial N_2\} > 0$$

and there exists an $x_0 \in \partial N_2$ such that $cs(x_0) = w(x_0)$. Since $cs - w \geq 0$ on $\partial(N_1 \backslash \overline{N}_2)$, we have $cs - w \geq 0$ on $N_1 \backslash \overline{N}_2$. Clearly, since $w = D_n w$ on ∂N_2, $cs - D_n w \geq 0$ on $\partial(N_2 \cap M_n)$ and hence $cs - D_n w \geq 0$ on $N_2 \cap M_n$. By letting $n \uparrow \infty$ we have $cs - Dw \geq 0$ on N_2, and since $w = Dw$ on N_2, $cs - w \geq 0$ on N_2. Hence $cs - w \geq 0$ on N_1 and $cs - w = cs > 0$ on ∂N_1. Thus $cs - w > 0$ on N_1 but $cs(x_0) - w(x_0) = 0$, a contradiction.

Finally we show that τ is surjective. For the purpose we need to find $u \in PP(N_1; \partial N_1)$ for an arbitrarily given $v \in PP(N_2; \partial N_2)$ such that $\tau u = v$, i.e. we need to solve the equation

$$u - Du = v \tag{20}$$

on N_2 with unknown $u \in PP(N_1; \partial N_1)$ for a given $v \in PP(N_2; \partial N_2)$. To solve (20) we consider an operator $K : C(\partial N_3) \to P(N_1 \backslash \overline{N}_3) \cap C(\overline{N}_1 \backslash N_3)$ given as follows. For any $\varphi \in C(\partial N_3)$ we let $K\varphi \in P(N_1 \backslash \overline{N}_3) \cap C(\overline{N}_1 \backslash N_3)$ such that $K\varphi = \varphi$ on ∂N_3 and $K\varphi = 0$ on ∂N_1. Then K is linear and order-preserving.

As the last one we define a linear operator T of $C(\partial N_3)$ into itself given by

$$T\varphi = (D((K\varphi)|\partial N_2))|\partial N_3$$

or more roughly $T\varphi = DK\varphi$ for all $\varphi \in C(\partial N_3)$ (cf. e.g. [26]). By the order-preservingness of D and K, we see that T is also order-preserving. We wish to solve

$$\varphi - T\varphi = v \tag{21}$$

on ∂N_3 with unknown $\varphi \in C(\partial N_3)$ for the given $v|\partial N_3 \in C(\partial N_3)$. If (21) is solved by a $\varphi \in C(\partial N_3)$ with $\varphi \geq 0$ on ∂N_3, then we define

$$u = \begin{cases} K\varphi & \text{on} \quad \overline{N}_1 \backslash N_3, \\ DK\varphi + v & \text{on} \quad \overline{N}_2. \end{cases}$$

Observe that $K\varphi$ and $DK\varphi + v$ belong to $P(N_2\backslash\overline{N}_3) \cap C(\overline{N}_2\backslash N_3)$ and

$$K\varphi - (DK\varphi + v) = \varphi - (T\varphi + v) = (\varphi - T\varphi) - v = 0$$

on ∂N_3 by (21) and

$$K\varphi - (DK\varphi + v) = K\varphi - (K\varphi + 0) = 0$$

on ∂N_2. By the minimum principle, $K\varphi = DK\varphi + v$ on $\overline{N}_2\backslash N_3$ and thus the definition of u above is well-defined on $\overline{N}_1$ and $u \in P(N_1) \cap C(\overline{N}_1)$ and $u \geq 0$ on N_1 and

$$u = DK\varphi + v = Du + v$$

on N_2 so that (20) is satisfied by this $u \in PP(N_1; \partial N_1)$.

Thus we only have to solve the abstract integral equation (21) by $\varphi \in C(\partial N_3)$ with $\varphi \geq 0$ on ∂N_3. Considering v as in $C(\partial N_3)$ with $v \geq 0$, we see that

$$0 \leq v \leq \|v\| s$$

on ∂N_3 where $\|v\| = \sup_{\partial N_3} |v|$, the norm on $C(\partial N_3)$. Applying the order-preserving operator T^n $(n = 1, 2, \cdots)$ we have

$$0 \leq T^n v \leq \|v\| T^n s$$

on ∂N_3. Since $s > 0$ on N_0 and $Ks = 0$ on ∂N_1 and $s = Ks$ on ∂N_3, we see that $Ks < s$ on ∂N_2 and therefore $DKs < Ds \leq s$ on ∂N_3, i.e. $Ts < s$ on ∂N_3. Then

$$q = \sup_{x \in \partial N_3} \frac{Ts(x)}{s(x)} \in (0, 1).$$

Hence $Ts \leq qs$ on ∂N_3. Inductively we see that $T^n s \leq q^n s$. Therefore

$$\|T^n v\| \leq q^n \|s\| \, \|v\|.$$

Thus the series

$$\varphi = \sum_{n=0}^{\infty} T^n v$$

has $\|s\| \, \|v\| \sum_{n=0}^{\infty} q^n$ as its majorant series and a fortiori $\varphi \in C(\partial N_3)$ and satisfies (21). Clearly $\varphi \geq 0$ along with $v \geq 0$ on ∂N_3. $\square$

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