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Trace scaling automorphisms of certain stable AF algebras

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Abstract. Trace scaling automorphisms of a stable AF algebra with dimension group totally ordered are outer conjugate if the scaling factors are the same (not equal to one).

Key words: AF algebra, automorphism, Rohlin property, outer conjugacy.

1. Introduction

This is a continuation of [8], where we showed a UHF version of a well-known result of A. Connes [5] that trace scaling automorphisms of the AFD type II_{∞} factor with the same non-trivial scale are outer conjugate with each other. In this paper we show the same result for stable AF algebras with totally ordered dimension group.

The key idea remains the same as in [8] and hence as in [5]: Define and prove a Rohlin property for such automorphisms and analyse them using this property. We are now familiar with the unital case (see [3, 2, 16, 13, 14, 15]). What we did in [8] is to evade non-unital C^*-algebras and deal with *partial* unital endomorphisms of unital (UHF) algebras. What we do here is to define a suitable Rohlin property for automorphisms of non-unital C^*-algebras and prove it. We will define it by borrowing an idea due to Rørdam [18], where corner endomorphisms are treated, and prove it by using an argument in [13, 14], where automorphisms of unital AF algebras are treated. Our main contribution is to find a passage from the *non-unital* case to the *unital* case in proving the Rohlin property, which is done in Section 2.

In Section 3 we shall show that our definition of Rohlin property is the *right* one, i.e., this is at least strong enough to prove the stability or 1 cohomology property [5, 9, 10]. Note that it is this property that we actually need.

Another idea used in [5, 8], a technique envolving tensor products, is

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no longer applicable here. For example if A is an AF algebra with totally ordered dimension group and $A \otimes A$ has the same property, then A must be UHF. However by using an intertwining argument we shall show that any two automorphisms α and β with the Rohlin property of an AF algebra are outer conjugate if α_* and β_* are equal as automorphisms of the dimension group. More precisely, the conclusion is that for any $\epsilon > 0$ there is an automorphism σ of A and a unitary U in $A + \mathbb{C}1$ (or in A if A is unital) such that $\|U - 1\| < \epsilon$ and $\alpha = \operatorname{Ad} U \circ \sigma \circ \beta \circ \sigma^{-1}$. In general we cannot take 1 for U or cannot conclude conjugacy of α and β. This extends Theorem 2 of [14] and Theorem 7 of [8]. (Note that even in the UHF case the conclusion is stronger than the one in [8].)

Let $\lambda \neq 1$ be a positive number and let G_λ be the subgroup of $\mathbb{R}$ generated by λ^n, $n \in \mathbb{Z}$. If A is the stable AF algebra whose dimension group is G_λ and α is an automorphism of A such that α_* acts on the dimension group by multiplication by λ, then from the above result the crossed product $A \rtimes_\alpha \mathbb{Z}$ depends only on λ. But we have now a more general theorem in this direction: it follows from [18, 11, 17] that $A \rtimes_\alpha \mathbb{Z}$, being a purely infinite simple C^*-algebra, is isomorphic to a stable Cuntz algebra.

2. Rohlin property

Let A be a non-unital C^*-algebra and let α be an automorphism of A. We assume that A has an approximate unit consisting of projections. Based on [18] we define a Rohlin property for α as follows:

Definition 2.1 The automorphism α has the Rohlin property if for any $k \in \mathbb{N}$ there are positive integers $k_1, \ldots, k_m \geq k$ satisfying the following condition: For any projections E, e in A, any unitary U in $A + \mathbb{C}1$, any finite subset $\mathcal{F}$ of $A_E = EAE$, and $\epsilon > 0$ with

$$e \leq E,\ \operatorname{Ad} U \circ \alpha(e) \leq E,\ e \in \mathcal{F},\ \operatorname{Ad} U \circ \alpha(e) \in \mathcal{F},$$

there exists a family $\{e_{i,j};\ i = 1, \ldots, m,\ j = 0, \ldots, k_i - 1\}$ of projections in A such that

$$\sum_i \sum_j e_{i,j} = E,$$
$$\|\operatorname{Ad} U \circ \alpha(e_{i,j} e) - e_{i,j+1} \operatorname{Ad} U \circ \alpha(e)\| < \epsilon,$$
$$\|[x, e_{i,j}]\| < \epsilon,$$

for $i = 1, \ldots, m$, $j = 0, \ldots, k_i - 1$ and $x \in \mathcal{F}$ where $e_{i,k_i} = e_{i,0}$. The projections $\{e_{i,j}\}$ will be called a set of Rohlin towers.

If we apply the same definition to a unital C^*-algebra A, then the Rohlin property for the unital case [13, 14], where $E = 1 = e$ is preassumed, implies the present definition. (We just have to cut down by E a set of Rohlin towers obtained for $E = 1 = e$ which almost commutes with E, e and use functional calculus to get the desired set of Rohlin towers.) The following is an easy consequence whose proof we omit:

Proposition 2.2 *Suppose that there is an increasing sequence $\{P_n\}$ of projections in A such that $\|P_n x - x\| \to 0$ for any $x \in A$ and $\alpha(P_n) = P_n$. Then α has the Rohlin property if and only if the restriction of α to $A_{P_n} = P_n A P_n$ has the Rohlin property for any n.*

We are, however, interested in the situation where the above proposition does not apply.

Let A be a simple stable AF algebra and let α be an automorphism of A. Let $\{A_n\}$ be an increasing sequence of finite-dimensional subalgebras of A such that the union $\cup_n A_n$ is dense in A.

Remark 2.3 In this situation for any $\epsilon > 0$ there is a subsequence $\{n_k\}$ of positive integers and a unitary $U \in A + 1$ such that $\|U - 1\| < \epsilon$ and $\mathrm{Ad}\, U \circ \alpha(A_k) \subset A_{k+1}$, $(\mathrm{Ad}\, U \circ \alpha)^{-1}(A_k) \subset A_{k+1}$ for any k. This can be proved by using the following fact inductively: If B is a finite-dimensional subalgebra of A and $\epsilon > 0$, there is an $n \in \mathbb{N}$ and a unitary $U \in A + 1$ such that $\|U - 1\| < \epsilon$ and $\mathrm{Ad}\, U(B) \subset A_n$.

Hence by slightly perturbing α and passing to a subsequence of $\{A_n\}$ we may assume that $\alpha^{-1}(A_n) \subset A_{n+1}$, $\alpha(A_n) \subset A_{n+1}$ for any n. We fix a nonzero projection $E \in A_1$.

Let e be a projection in $\cup_k A_k$ and U a unitary in $\cup_k A_k + \mathbb{C}1$ such that $e \le E$ and $\mathrm{Ad}\, U \circ \alpha(e) \le E$. Since $A_E = EAE$ is simple there is a $k \in \mathbb{N}$ such that e, $\mathrm{Ad}\, U \circ \alpha(e)$, $U \in A_k + \mathbb{C}1$ and the multiplication by e (resp. $f = \mathrm{Ad}\, U \circ \alpha(e)$) induces an isomorphism of $(A \cap A_k')_E$ onto $(A \cap A_k')_e$ (resp. $(A \cap A_k')_f$) or the central support of e (resp. f) in $(A_k)_E$ is E. We define a homomorphism $\phi(\alpha, e, U)$ of $(A \cap A_{k+1}')_E$ into $(A \cap A_k')_E$ by

$$\phi(\alpha, e, U)(x) f = \mathrm{Ad}\, U \circ \alpha(xe) = \alpha(x) f.$$

This is indeed well-defined: Since

$$[\operatorname{Ad} U \circ \alpha(xe), b] = \operatorname{Ad} U \circ \alpha([xe, (\operatorname{Ad} U \circ \alpha)^{-1}(b)]) = 0$$

for $b \in (A_k)_f = fA_kf$, we have that

$$\operatorname{Ad} U \circ \alpha(xe) \in (A \cap A_k')_f = A_f \cap ((A_k)_f)'.$$

We note that $\phi(\alpha, e, U)$ is essentially independent of e and U in the sense that if $\phi(\alpha, e_1, U_1)$ is another one then $\phi(\alpha, e, U) = \phi(\alpha, e_1, U_1)$ on the common domain D. Because if $e_1 \le e$ and $U_1 = U$, this follows since

$$(\phi(\alpha, e, U)(x) - \phi(\alpha, e_1, U_1)(x)) \operatorname{Ad} U \circ \alpha(e_1) = 0$$

for any $x \in D$; if $e = e_1$, this follows since

$$(\phi(\alpha, e, U)(x) - \operatorname{Ad}(UU_1^*) \circ \phi(\alpha, e, U_1)(x)) \operatorname{Ad} U \circ \alpha(e) = 0$$

for any $x \in D$ and UU_1^* commutes with $\phi(\alpha, e, U_1)(x)$; and if $e_1 = \operatorname{Ad} V(e)$ and $U_1 = U\alpha(V^*)$ with V a unitary in $\cup_n A_n + \mathbb{C}1$, this follows since $\operatorname{Ad} U \circ \alpha(e) = \operatorname{Ad} U_1 \circ \alpha(e_1)$.

Thus we denote by $\tilde{\alpha}$ the homomorphism induced by these $\phi(\alpha, e, U)$; this is a homomorphism of $(A \cap A_{k+1}')_E$ into $(A \cap A_k')_E$ for some k. The same computation shows that $\tilde{\alpha}$ maps $(A \cap A_{n+1}')_E$ into $(A \cap A_n')_E$ for $n \ge k$. Since $\phi(\alpha^{-1}, \operatorname{Ad} U \circ \alpha(e), \alpha^{-1}(U^*))$ is well-defined if so is $\phi(\alpha, e, U)$, $\widetilde{\alpha^{-1}}$ is defined at least on $(A \cap A_{k+2}')_E$, and satisfies that $\tilde{\alpha} \circ \widetilde{\alpha^{-1}} = \mathrm{id}$, $\widetilde{\alpha^{-1}} \circ \tilde{\alpha} = \mathrm{id}$ on $(A \cap A_{k+2}')_E$. A similar computation shows that $(\tilde{\alpha})^n = \widetilde{\alpha^n}$ on $(A \cap A_l')_E$ for a sufficiently large l (depending on n).

Let ω be a free ultrafilter on $\mathbb{N}$ and let A_E^ω be the quotient C^*-algebra of $l^\infty(\mathbb{N}, A_E)$ by the ideal $I_\omega = \{(x_n) \mid \lim_{n\to\omega} \|x_n\| = 0\}$. Embedding A_E into $l^\infty(\mathbb{N}, A_E)$ and so into A_E^ω as constant functions, let $A_{E\omega} = A_E^\omega \cap A_E'$.

Let $x = (x_n) \in A_{E\omega}$. Then we can find an increasing sequence $\{k_n\}$ in $\mathbb{N}$ and $x_n' \in (A \cap A_{k_n}')_E$ such that $k_n \to \infty$ and $\lim_{n\to\omega} \|x_n - x_n'\| = 0$. We define a homomorphism $\tilde{\alpha}_\omega$ of $A_{E\omega}$ into itself by

$$\tilde{\alpha}_\omega(x) = (\tilde{\alpha}(x_n')).$$

This is indeed easily checked to be well-defined. In the same way we can define $\tilde{\beta}_\omega$ for $\beta = \alpha^{-1}$ and show that $\tilde{\alpha}_\omega \circ \tilde{\beta}_\omega = \mathrm{id}$ and $\tilde{\beta}_\omega \circ \tilde{\alpha}_\omega = \mathrm{id}$. Thus $\tilde{\alpha}_\omega$ is an automorphism of $A_{E\omega}$.

Let τ be a densely-defined non-zero lower semi-continuous trace on A and assume that τ is unique up to a constant multiple. Since $\tau \circ \alpha$ is again such a trace, there is a $\lambda > 0$ such that $\tau \circ \alpha = \lambda\tau$. We normalize τ by $\tau(E) = 1$. We can define a state τ_ω on $A_{E\omega}$ by $\tau_\omega(x) = \lim \tau(x_n)$ for $x = (x_n) \in A_{E\omega}$. Note that τ_ω is tracial. We shall show that τ_ω is invariant under $\tilde{\alpha}_\omega$.

Let $x = (x_n) \in A_{E\omega}$, where $x_n \in (A \cap A_{k_n})'_E$ for some non-decreasing sequence $\{k_n\}$ with $k_n \to \infty$. If $\tilde{\alpha} = \phi(\alpha, e, U)$ on $(A \cap A'_{k+1})_E$, $\{P_i\,;\ i = 1, \dots, N\}$ is the set of minimal projections in the center of $EA_{k+1}E$, and $k_n > k + 1$, then

$$\begin{aligned}
\tau(\tilde{\alpha}(x_n)) &= \sum_{i=1}^{N} \tau(P_i \tilde{\alpha}(x_n)) \\
&= \sum \frac{\tau(P_i)}{\tau(P_i \operatorname{Ad} U \circ \alpha(e))} \tau(P_i \operatorname{Ad} U \circ \alpha(x_n e)) \\
&= \sum \frac{\tau(P_i)}{\tau(P_i \alpha(e))} \tau \circ \alpha(\alpha^{-1}(P_i) e x_n) \\
&= \sum \frac{\tau(P_i)}{\tau(\alpha^{-1}(P_i) e)} \tau(\alpha^{-1}(P_i) e x_n)
\end{aligned}$$

which, when n is large, is almost equal to

$$\sum \frac{\tau(P_i)}{\tau(\alpha^{-1}(P_i) e)} \tau(\alpha^{-1}(P_i) e) \tau(x_n) = \tau(x_n)$$

since τ is factorial. Thus we obtain that $\tau_\omega \circ \tilde{\alpha}_\omega = \tau_\omega$.

Without loss of generality we assume, from now on, that $\tilde{\alpha}$ is defined as $\phi(\alpha, e, 1)$ on $(A \cap A'_2)_E$, i.e., e, $\alpha(e) \in A_1$, $e \leq E$, $\alpha(e) \leq E$, and the central supports of e and $\alpha(e)$ in $(A_1)_E$ are E.

The above argument carries over to the weak closure $\mathcal{R}$ of $\pi_\tau(A_E)$. Note that $\mathcal{R}^\omega$ is defined as the quotient of $l^\infty(\mathbb{N}, \mathcal{R})$ by

$$I = \{(x_n) \mid \lim_{n \to \omega} \|x_n\|_\tau = 0\}$$

where $\|a\|_\tau = \tau(a^* a)^{1/2}$, and τ is regarded as the tracial state on $\mathcal{R}$ induced from τ on A_E. Since $\tau \circ \tilde{\alpha} \mid (A \cap A'_2)_E$ is equivalent to $\tau \mid (A \cap A'_2)_E$, $\tilde{\alpha}$ extends to a homomorphism of $\mathcal{R} \cap \pi_\tau(A_{2E})'$ into $\mathcal{R} \cap \pi_\tau(A_{1E})'$, and for $x \in l^\infty(\mathbb{N}, \mathcal{R} \cap \pi_\tau(A_{2E})')$, $\lim_{n \to \omega} \|x_n\|_\tau = 0$ if and only if $\lim \|\tilde{\alpha}(x_n)\|_\tau = 0$. In this way we have the automorphism $\tilde{\alpha}_\omega$ of $\mathcal{R}_\omega = \mathcal{R}^\omega \cap \mathcal{R}'$ induced by $\tilde{\alpha}$ which satisfies that $\tau_\omega \circ \tilde{\alpha}_\omega = \tau_\omega$.

Lemma 2.4 *Suppose that $\tau \circ \alpha = \lambda\tau$ with $\lambda \neq 1$. If $\mathcal{R} = \pi_\tau(A_E)^{-w}$, $\mathcal{R}_\omega$,*

$\tilde{\alpha}_\omega$, τ_ω *etc. are as above, then any non-zero power of* $\tilde{\alpha}_\omega$ *is properly outer.*

Proof. The proof is similar to the proofs of Lemmas 1 and 2 of [8]. Denote by $\overline{\alpha}$ the automorphism of $\pi_\tau(A)''$ induced by α. Fix $n \geq 2$ and let $B = \pi_\tau(A_{nE}) \subset \mathcal{R}$. For a non-zero projection $f \in \mathcal{R} \cap B'$ we assert that

$$\inf\{\|p\tilde{\alpha}(p)\|;\ 0 \neq p = p^* = p^2 \in \mathcal{R} \cap B',\ p \leq f\} = 0.$$

Suppose that the above infimum is positive, say $\delta > 0$. We may suppose that f is in one factor direct summand of $\mathcal{R} \cap B'$. Let f_1 be a minimal projection in B such that $f_1 f \neq 0$. Then any projection $\tilde{p} \in \mathcal{R}$ with $\tilde{p} \leq f f_1$ is of the form $\tilde{p} = f_1 p$ with p a projection in $f(\mathcal{R} \cap B')f$. Hence for any $z \in \mathcal{R}$ we have

$$\inf\{\|pf_1 z\overline{\alpha}(f_1)\overline{\alpha}(p)\|;\ 0 \neq p = p^* = p^2 \in \mathcal{R} \cap B',\ p \leq f\} = 0,$$

since $\overline{\alpha}$ is an outer automorphism of $\pi_\tau(A)''$. There is a finite set $\{V_1, \ldots, V_k\}$ of unitaries in B such that $\sum_i V_i f_1 V_i^*$ is the central support of f_1 in B. Since

$$\sum_{i,j} V_i p f_1 V_i^* \overline{\alpha}(V_j)\overline{\alpha}(f_1)\overline{\alpha}(p)\overline{\alpha}(V_j^*) = p\overline{\alpha}(p)$$

and

$$\|p\tilde{\alpha}(p)\| = \|p\tilde{\alpha}(p)\overline{\alpha}(e)\| = \|p\overline{\alpha}(p)\overline{\alpha}(e)\| = \|p\overline{\alpha}(p)\|,$$

there exist i, j such that

$$\|pf_1 V_i^* \overline{\alpha}(V_j)\overline{\alpha}(f_1)\overline{\alpha}(p)\| \geq \delta/k^2,$$

which is a contradiction. By using this we can show that α_ω is properly outer (cf. Lemma 2 of [8] and [5]).

By applying the same argument to $\tilde{\alpha}^n$, we obtain that $(\tilde{\alpha}_\omega)^n$ is properly outer for any $n \neq 0$. □

Let $D = A_E$ and suppose that D has a unique tracial state. Let B be a finite-dimensional subalgebra of D. We say $x \in D$ is *independent* of B if $x \in B'$ and

$$\tau(xy) = \tau(x)\tau(y), \quad y \in B.$$

Lemma 2.5 *Let* $\{P_1, \ldots, P_N\}$ *be the set of minimal central projections of*

B. Then $x \in D \cap B'$ is independent of B if and only if

$$\tau(xP_i) = \tau(x)\tau(P_i), \quad i = 1, \ldots, N.$$

Proof. It suffices to show the *if* part. Since $P_i D P_i \cong P_i B P_i \otimes (D \cap B')P_i$, it follows that for $y \in P_i B P_i$

$$\frac{\tau(xy)}{\tau(P_i)} = \frac{\tau(xP_i)}{\tau(P_i)} \frac{\tau(y)}{\tau(P_i)}.$$

Hence $\tau(xy) = \tau(x)\tau(y)$. This completes the proof. □

Lemma 2.6 *Suppose that the dimension group $K_0(A)$ is totally ordered and identified with a subgroup of $\mathbb{R}$ and that $K_0(A) = \lambda K_0(A)$ for some $\lambda \neq 1$. For any central sequence $\{f_k\}$ of projections in A_E there is a central sequence $\{f'_k\}$ of projections in A_E such that $f'_k \leq f_k$, $\tau(f_k - f'_k) \to 0$, and for any n there is a k_n satisfying that f'_k is independent of A_{nE} for any $k \geq k_n$.*

Proof. We may suppose that there is a k_n such that $f_k \in (A \cap A'_n)_E$ for $k \geq k_n$. Let $\{P_i^{(n)}\}$ be the set of minimal central projections in A_{nE}. We can find an $l_n \geq k_n$ such that

$$\left| \frac{\tau(P_i^{(n)} f_k)}{\tau(P_i^{(n)})} - \tau(f_k) \right| < \frac{1}{n}$$

for $k \geq l_n$ and all i. We will then define f'_k, $k \in \{l_n, \ldots, l_{n+1} - 1\}$ as follows: Let $g_k \in \cup_{n \in \mathbb{Z}} \lambda^n \mathbb{Z}$ be such that $\max\{0, \gamma - 1/n\} < g_k \leq \gamma$ where

$$\gamma = \min_i \frac{\tau(P_i^{(n)} f_k)}{\tau(P_i^{(n)})}.$$

Choose a subprojection $q_{k,i}$ of $P_i^{(n)} f_k$ in $(A \cap A'_n)_{P_i^{(n)} f_k}$ such that $\tau(q_{k,i})/\tau(P_i^{(n)}) = g_k$, and let $f'_k = \sum_i q_{k,i}$. This is possible because, when $(A_n)_{P_i^{(n)}}$ is isomorphic to the $m_i \times m_i$ matrix algebra,

$$K_0((A \cap A'_n)_{P_i^{(n)}}) = m_i K_0(A_{P_i^{(n)}})$$

and when $[P_i^{(n)}] = 1$, $K_0((A \cap A'_n)_{P_i^{(n)}})$ contains $\cup_{n \in \mathbb{Z}} \lambda^n \mathbb{Z}$ for any i. □

Lemma 2.7 *Suppose that $K_0(A)$ is totally ordered. Let $p \in (A \cap A'_n)_E$ be* *a projection independent of* $(A_n)_E$. *Then if* $n - m \geq 1$, $p, \tilde{\alpha}(p), \ldots, \tilde{\alpha}^m(p)$ *are all equivalent in* $(A \cap A'_{n-m})_E$.

Proof. Let $\{P_i\}$ be the set of minimal central projections in $EA_{n-1}E$. Then

$$\begin{aligned}\tau(\tilde{\alpha}(p)P_i) &= \frac{\tau(P_i)}{\tau(P_i\alpha(e))}\tau(\tilde{\alpha}(p)P_i\alpha(e)) \\ &= \frac{\tau(P_i)}{\tau(\alpha^{-1}(P_i)e)}\tau(\alpha^{-1}(P_i)ep) \\ &= \tau(P_i)\tau(p) = \tau(pP_i),\end{aligned}$$

since $\alpha^{-1}(P_i)e \in A_n$. Hence p is equivalent to $\tilde{\alpha}(p)$ in $(A \cap A'_{n-1})_E$. We just repeat this procedure. □

Theorem 2.8 *Let A be a stable AF algebrasuch that $K_0(A)$ is totally ordered and let α be an automorphism of A such that $\tau \circ \alpha = \lambda\tau$ where τ is a trace on A (unique up to constant multiple) and $\lambda \neq 1$. Then α has the Rohlin property.*

Proof. Let E, e be projections in A and U a unitary in $A + \mathbb{C}1$ such that $e \leq E$ and $\mathrm{Ad}\,U \circ \alpha(e) \leq E$. Let $\{A_n\}$ be an increasing sequence of finite-dimensional subalgebras of A such that the union $\cup_n A_n$ is dense in A and E, e, $\mathrm{Ad}\,U \circ \alpha(e) \in A_1$. By taking $\mathrm{Ad}\,U \circ \alpha$ instead of α we now assume that $U = 1$. For any $\delta > 0$ we find a unitary $V \in A + 1$ such that $\|V - 1\| < \delta$ and, by passing to a subsequence of $\{A_n\}$, $\mathrm{Ad}\,V \circ \alpha(A_n) \subset A_{n+1}$, $(\mathrm{Ad}\,V \circ \alpha)^{-1}(A_n) \subset A_{n+1}$ for any n (Remark 2.3). By perturbing V if necessary we may further assume that $\mathrm{Ad}\,V \circ \alpha(e) = \alpha(e)$. By taking a sufficiently small $\delta > 0$ we may take $\mathrm{Ad}\,V \circ \alpha$ for α. Thus we have the following situation: There exists an increasing sequence $\{A_n\}$ of finite-dimensional subalgebras of A such that the union $\cup_n A_n$ is dense in A, $\alpha(A_n) \subset A_{n+1}$, $\alpha^{-1}(A_n) \subset A_{n+1}$, $E, e, \alpha(e) \in A_1$, and $e, \alpha(e) \leq E$. The problem is to find a set of Rohlin towers as specified in Definition 2.1. But this follows from Lemmas 2.4, 2.6, and 2.7 based on the arguments given in [13, 14] because we just have to prove the (ordinary) Rohlin property for $\tilde{\alpha}$. Here is an outline. Since $\mathcal{R}_\omega$ is a finite von Neumann algebra [5, 2.2.1], we know by Lemma 2.3 and [5, 1.2.5] that $\tilde{\alpha}_\omega$ on $\mathcal{R}_\omega$ satisfies a Rohlin property. Then by approximating a Rohlin tower in $\mathcal{R}_\omega$ by projections in A_E we show that the *partial* endomorphism $\tilde{\alpha}$ of A_E satisfies an approximate Rohlin property (Lemmas 2.5 and 2.6 and [14]). But this suffices to conclude the Rohlin property by [13, Proof of 2.1]. □

Remark 2.9 In the situation of the previous theorem let α be an automorphism of A such that $\alpha_* = \mathrm{id}$ and any non-zero power is not weakly inner in the tracial representation. Then α has the Rohlin property. (See Proposition 2.2 or Theorem 4.1 of [14].)

Remark 2.10 In the above theorem we can make the Rohlin property more specific: In Definition 2.1 we may take $\{k, k+1\}$ for $\{k_1, \ldots, k_m\}$. This follows because of Lemma 2.7 (see [14]).

3. Stability

Theorem 3.1 *Let A be a (non-unital) AF algebra and let α be an automorphism of A with the Rohlin property. Let $\epsilon > 0$ and let B_1 be a finite-dimensional subalgebra of A. Then there is a finite-dimensional subalgebras B_2 of A such that for any unitary $U \in A \cap B_2' + 1$ there is a unitary $V \in A \cap B_1' + 1$ with $\|U - V\alpha(V^*)\| < \epsilon$.*

The following argument works if A is unital or non-unital; if A is unital, we should regard 1 as the unit of A, otherwise 1 as the unit adjoined to A.

Suppose that there is an increasing sequence $\{A_n\}$ of finite-dimensional subalgebras of A such that $\cup_n A_n$ is dense in A and $\alpha^{-1}(A_n) \subset A_{n+1}$, $\alpha(A_n) \subset A_{n+1}$ for any n. By Remark 2.3 the above theorem is an easy consequence of:

Lemma 3.2 *Under the above assumption let U be a unitary in $A \cap A_n' + 1$. Then for each $k \in \mathbb{N}$ there is a unitary V in $A \cap A_{n-2k}' + 1$ such that $\|U - V\alpha(V^*)\| < 4/k$, where $A_m = \{0\}$ for a non-positive m.*

Proof. Let U be a unitary in $A \cap A_n' + 1$. We may further suppose that there is an $m > n$ such that $U \in A_m + 1$. Let F be the identity of A_m and let

$$
\begin{aligned}
E &= \sup\{\alpha^m(F);\ -1 \le m \le 2k+1\}, \\
e &= \sup\{\alpha^m(F);\ -1 \le m \le 2k\},
\end{aligned}
$$

which are projections in A_{m+2k+1} with that $e, \alpha(e) \le E$. For $U_j = U\alpha(U)\cdots\alpha^{j-1}(U)$ with $j \ge 0$, we have that if $0 \le j \le k$,

$$U_j(1 - \alpha(e)) = 1 - \alpha(e).$$

For E, e and $\{k, k+1\}$ we find a set of Rohlin towers $\{e_{1,0}, \dots, e_{1,k-1}, e_{2,0}, \dots, e_{2,k}\}$ in $(A \cap A'_{m+2k+1})_E$. Let $W_t^{(1)}$, $W_t^{(2)}$ be paths of unitaries in $A_{m+k} \cap A'_{n-k} + 1$ such that

$$\begin{aligned} &W_0^{(i)} = 1, \\ &W_1^{(1)} = U_k,\ W_1^{(2)} = U_{k+1}, \\ &\|W_s^{(i)} - W_t^{(i)}\| \le \pi|s-t|, \quad s, t \in [0,1], \\ &W_t^{(i)}(1-e_0) = 1-e_0, \quad t \in [0,1], \end{aligned}$$

where $e_0 = \sup\{\alpha^m(F);\ 0 \le m \le k\}$. Set

$$V = \sum_{j=0}^{k-1} U_j \alpha^j(W^{(1)}_{1-j/(k-1)})e_{1,j} + \sum_{j=0}^{k} U_j \alpha^j(W^{(2)}_{1-j/k})e_{2,j} + 1 - E,$$

which is a unitary in $A \cap A'_{n-2k} + 1$. Since $\alpha^j(W_t^{(i)})(1-\alpha(e)) = 1-\alpha(e)$ for $0 \le j \le k$, we have

$$V(1-\alpha(e)) = 1-\alpha(e),\ \alpha(V)(1-\alpha(e)) = 1-\alpha(e).$$

Hence it follows that

$$\begin{aligned} &V\alpha(V^*)\alpha(e) \\ &\approx \sum_{j=0}^{k-1} U_{j+1}\alpha^{j+1}(W^{(1)}_{1-(j+1)/(k-1)})\alpha^{j+1}(W^{(1)}_{1-j/(k-1)})^*\alpha(U_j^*)e_{1,j+1}\alpha(e) \\ &\quad + \sum_{j=0}^{k} U_{j+1}\alpha^{j+1}(W^{(2)}_{1-(j+1)/k})\alpha^{j+1}(W^{(2)}_{1-j/k})^*\alpha(U_j^*)e_{2,j+1}\alpha(e), \end{aligned}$$

where the $k-1$'th summand in the first summation should be understood as

$$U_0 W_1^{(1)}\alpha^k(W_0^{(1)})^*\alpha(U_{k-1}^*)e_{1,0}\alpha(e) = Ue_{1,0}\alpha(e)$$

and the k'th term in the second as

$$U_0 W_1^{(2)}\alpha^{k+1}(W_0^{(2)})^*\alpha(U_k^*)e_{2,0}\alpha(e) = Ue_{2,0}\alpha(e).$$

Hence we have that

$$\|V\alpha(V^*) - U\| \le \pi/k + \epsilon(2k+1),$$

where $\epsilon > 0$ is a small number depending on the Rohlin towers we used. This completes the proof. $\square$

4. Outer conjugacy

Theorem 4.1 *Let A be an AF algebra and let α and β be automorphisms of A with the Rohlin property. If $\alpha_* = \beta_*$ on $K_0(A)$, then for any $\epsilon > 0$ there is an automorphism σ of A and a unitary U in $A + 1$ such that $\alpha = \operatorname{Ad} U \circ \sigma \circ \beta \circ \sigma^{-1}$, $\|U - 1\| < \epsilon$, and $\sigma_* = \mathrm{id}$.*

Proof. Note that A can be either unital or non-unital.

Let $\epsilon > 0$ and let $\{x_n\}$ be a dense sequence in the unit ball of A. We shall construct inductively sequences $\{A_n\}$, $\{B_n\}$ of finite-dimensional subalgebras of A, sequences $\{u_n\}$, $\{v_n\}$ of unitaries in $A + 1$ such that

1. $A_n \ni_{1/n} x_1, \ldots, x_n$, $A_n \supset A_{n-1}$,
2. $B_n \supset A_n$, $B_n \ni v_n$,
3. $\beta_{2n}|A_{2n+1} = \operatorname{Ad} u_{2n+1} \circ \alpha_{2n-1}|A_{2n+1}$,
4. $\alpha_{2n-1}|A_{2n} = \operatorname{Ad} u_{2n} \circ \beta_{2n-2}|A_{2n}$,
5. $\|u_{2n+1} - v_{2n+1}\alpha_{2n-1}(v_{2n+1}^*)\| < 2^{-2n-1}\epsilon$, $v_{2n+1} \in B'_{2n-1}$,
6. $\|u_{2n} - v_{2n}\beta_{2n-2}(v_{2n}^*)\| < 2^{-2n}\epsilon$, $v_{2n} \in B'_{2n-2}$,
7. for any unitary $U \in A \cap \alpha_{2n-1}(A_{2n})' + 1$ there exists a unitary $V \in A \cap B'_{2n-1} + 1$ such that $\|U - V\alpha_{2n-1}(V^*)\| < 2^{-2n-1}\epsilon$,
8. for any unitary $U \in A \cap \beta_{2n}(A_{2n+1})' + 1$ there exists a unitary $V \in A \cap B'_{2n} + 1$ such that $\|U - V\beta_{2n}(V^*)\| < 2^{-2n-2}\epsilon$,

where $A_n \ni_\delta x$ means that there is a $y \in A_n$ with $\|x - y\| < \delta$ and

$$A_0 = \{0\},$$
$$\alpha_{2n+1} = \operatorname{Ad} u_{2n+1} \circ \alpha_{2n-1}, \quad \alpha_{-1} = \alpha,$$
$$\beta_{2n} = \operatorname{Ad} u_{2n} \circ \beta_{2n-2}, \quad \beta_0 = \beta.$$

We first construct A_1 according to (1). Having constructed

$$A_1, \ldots, A_{2n+1}, \ B_1, \ldots, B_{2n}, \ u_1, \ldots, u_{2n}, \ v_1, \ldots, v_{2n},$$

we proceed as follows: We choose u_{2n+1} according to (3). Since $\alpha_{2n-1}|A_{2n} = \beta_{2n}|A_{2n}$ from (4) and the definition of β_{2n} above, it follows that $u_{2n+1} \in \alpha_{2n-1}(A_{2n})'$ and so by (7) that there is a unitary $v_{2n+1} \in A \cap B'_{2n-1} + 1$

satisfying (5). We may assume that there is a B_{2n+1} satisfying (2) (for $2n+1$ in place of n). Having defined B_{2n+1} we define A_{2n+2} satisfying (1) and (7) (by using Theorem 3.1) and choose u_{2n+2} according to (4). Since $u_{2n+2} \in \beta_{2n}(A_{2n+1})'$, we define v_{2n+2} satisfying (6) by using (8) and assume that there is a B_{2n+2} satisfying (2). We define A_{2n+3} satisfying (1) and (8). This completes the induction.

We note that the union $\cup_n A_n$ is dense in A and define automorphisms σ_n of A by

$$\begin{aligned}\sigma_{2n} &= \mathrm{Ad}(v_{2n}v_{2n-2}\cdots v_2),\\ \sigma_{2n+1} &= \mathrm{Ad}(v_{2n+1}v_{2n-1}\cdots v_1),\end{aligned}$$

and define

$$\begin{aligned}\tilde{\sigma}_0 &= \lim_n \sigma_{2n},\\ \tilde{\sigma}_1 &= \lim_n \sigma_{2n+1}.\end{aligned}$$

Since $v_{n-2}, v_{n-4}, \ldots \in B_{n-2}$, $v_n \in B'_{n-2}$, and $\cup_n B_n$ is dense, they are well-defined. We let

$$w_{2n+1} = u_{2n+1}\alpha_{2n-1}(v_{2n+1})v^*_{2n+1}, \quad w_{2n} = u_{2n}\beta_{2n-2}(v_{2n})v^*_{2n}$$

and define unitaries $w'_n \in A+1$ by

$$\begin{aligned}w'_{2n} &= w_{2n}\mathrm{Ad}\, v_{2n}(w_{2n-2})\mathrm{Ad}(v_{2n}v_{2n-2})(w_{2n-4})\cdots \mathrm{Ad}(v_{2n}\cdots v_4)(w_2),\\ w'_{2n+1} &= w_{2n+1}\mathrm{Ad}\, v_{2n+1}(w_{2n-1})\cdots \mathrm{Ad}(v_{2n+1}\cdots v_3)(w_1).\end{aligned}$$

Since $\|w_n - 1\| < 2^{-n}\epsilon$, both $\{w'_{2n}\}$ and $\{w'_{2n+1}\}$ converge, say to $\tilde{w}_0$ and $\tilde{w}_1$ respectively. Then $\tilde{w}_i$'s are unitaries in $A+1$ such that $\|\tilde{w}_i - 1\| < \epsilon$.

Since $\alpha_{2n-1}|A_{2n} = \beta_{2n}|A_{2n}$, we have that

$$\mathrm{Ad}\, w'_{2n-1} \circ \sigma_{2n-1} \circ \alpha \circ \sigma^{-1}_{2n-1}|A_{2n} = \mathrm{Ad}\, w'_{2n} \circ \sigma_{2n} \circ \beta \circ \sigma^{-1}_{2n}|A_{2n},$$

which implies that

$$\mathrm{Ad}\, \tilde{w}_1 \circ \tilde{\sigma}_1 \circ \alpha \circ \tilde{\sigma}_1^{-1} = \mathrm{Ad}\, \tilde{w}_0 \circ \tilde{\sigma}_0 \circ \beta \circ \tilde{\sigma}_0^{-1}.$$

This completes the proof. □

Corollary 4.2 *Let A be a stable AF algebrasuch that $K_0(A)$ is totally ordered and let α, β be automorphisms of A such that $\tau \circ \alpha = \lambda\tau$ and $\tau \circ \beta = \lambda\tau$ where τ is a trace on A and $\lambda \neq 1$. Then for any $\epsilon > 0$*

there are an automorphism σ of A and a unitary U in $A+1$ such that $\|U-1\|<\epsilon$, $\sigma_*=\mathrm{id}$, and $\alpha=\mathrm{Ad}\,U\circ\sigma\circ\beta\circ\sigma^{-1}$.

Proof. This follows from Theorems 2.8 and 4.1. □

Remark 4.3 In the above corollary the exact conjugacy $\alpha=\sigma\circ\beta\circ\sigma^{-1}$ for some automorphism σ of A cannot be expected in general. For example if A is a prime AF algebrasuch that $A\cong A\otimes\mathcal{K}$, where $\mathcal{K}$ is the compact operators on $l^2(\mathbb{Z})$, and α is an automorphism of A such that α^n is properly outer for any $n\neq 0$, let α_1 be the automorphism of A defined as $\alpha\otimes\gamma$ through $A\otimes\mathcal{K}\cong A$ where $\gamma=\mathrm{Ad}\,U$ and U is the shift unitary on $l^2(\mathbb{Z})$. Then α_1 satisfies that for any $x,y\in A$,

$$\|\alpha_1^n(x)y\|\to 0$$

as $n\to\infty$. This property is preserved by conjugacy but not by outer conjugacy. (By [12] there exists a faithful α_1-covariant irreducible representation of A; by using Kadison's transitivity theorem in this irreducible representation it follows that for any $\epsilon>0$ there are an $x\in A$ and a unitary $U\in A+1$ such that $0\le x\le 1$, $\|x\|=1$, $\|U-1\|<\epsilon$, and $\|(\mathrm{Ad}\,U\circ\alpha_1)^n(x)x\|=1$.)

Remark 4.4 Let $\lambda\neq 1$ be a positive number and let

$$G_\lambda=\cup_{n\in\mathbb{Z}}\mathbb{Z}\lambda^n.$$

If $\{1,\lambda,\lambda^2,\ldots,\}$ are linearly independent over $\mathbb{Q}$ then the quotient $G_\lambda/(1-\lambda)G_\lambda$ is isomorphic to $\mathbb{Z}$ and otherwise if $\{f\in\mathbb{Z}[t]\mid f(\lambda)=0\}=p(t)\mathbb{Z}[t]$ with some $p(t)\in\mathbb{Z}[t]$, then $G_\lambda/(1-\lambda)G_\lambda\cong\mathbb{Z}/p(1)\mathbb{Z}$. If A is the stable AF algebra with dimension group G_λ and α is an automorphism of A with $\alpha_*=\lambda$, then the crossed product $A\rtimes_\alpha\mathbb{Z}$ has $G_\lambda/(1-\lambda)G_\lambda$ as K_0 and $\{0\}$ as K_1 by the Pimsner-Voiculescu exact sequence [1]. Hence $A\rtimes_\alpha\mathbb{Z}$ is isomorphic to a stable Cuntz algebra $\mathcal{O}_n\otimes\mathcal{K}$ where n is either finite or infinite [18, 11, 17, 6, 7].

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