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Some solvability criteria for finite groups

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Abstract. We prove a variant of Kegel-Wielandt's theorem. We use this to give some solvability criteria for factorizable finite groups.

Key words: permutable subgroups, 2-nilpotent subgroups, simple groups.

Let G be a finite group and $G=HK$, where H and K are subgroups of G. There are a number of results which deduce the solvability of G from suitable conditions on H and K. In particular, two of these results are:

Huppert - Itô [4], *If H is supersolvable and K is cyclic of odd order, then $G=HK$ is solvable.*

Kegel - Wielandt ([5], p. 674, Satz 4.3), *If H and K are nilpotent, then $G=HK$ is solvable.*

In the first part of this communication, using classification theorems of simple groups we prove the following:

Theorem A *Let G be a finite group, $H\leq G$ such that $|G:H|=p^{a}$ with p an odd prime number. If H is 2-nilpotent, then G is solvable.*

In the second part, we consider the following definition: A subgroup H of a group G is said to be *semi-normal* in G if there exists a subgroup K of G such that $G=HK$ and H permutes with every subgroup of K.

The solvability of the normal closure of a solvable semi-normal subgroup cannot be concluded in general: In the alternating group $G=\mathbf{A}_{5}$ the subgroups H of index 5 are solvable and semi-normal in G, but the normal closure $H^{G}=G$ is not solvable.

This makes the following theorem interesting:

Theorem B *Let G be a finite group and $H\leq G$ a semi-normal subgroup.*

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(a) *If H is 2-nilpotent, then the normal closure H^G of H in G is solvable.*
(b) *If the order $|H|$ of H is odd, then $|H^G|$ is odd.*

1. Preliminary Results

We prepare the proof of the theorems.

Lemma 1.1 (Fisman [3]) *Let $G = HK$ be a group with H and K proper solvable subgroups such that $(|H|, |K|) = 1$. Then the composition factors of G belong to one of the following types:*
(a) *Cyclic of prime order*
(b) $\mathbf{PSL}(2, 2^n)$, $n \geq 2$,
(c) $\mathbf{PSL}(2, q)$ *with* $q \equiv -1 \bmod (4)$,
(d) $\mathbf{PSL}(3, 3)$,
(e) $\mathbf{M}_{11}$.

Lemma 1.2 (Finkel and Lundgren [2]) *Let $G = HK$ be a group with H and K subgroups such that $(|H|, |K|) = 1$. Let $R \trianglelefteq H$ with $|H : R| = 2^i$ and $R = R_2 \times R_{2'}$ where R_2 is the Sylow-2-subgroup and $R_{2'}$ is the Hall-$2'$-subgroup of R. Assume further that, H/R is abelian or dihedral. If K is nilpotent of odd order, then G is solvable.*

2. Semi-normal subgroups

A subgroup H of a group G is said to be semi-normal in G if there exists a subgroup K of G such that $G = HK$ and H permutes with every subgroup of K. In this case, we say that K is an s-supplemnt of H in G. Clearly, every normal subgroup of G is semi-normal. Also every subgroup of prime index is semi-normal (see [1], [7], [8]).

Lemma 2.1 *Let H be a semi-normal subgroup of a group G.*
(a) *If $H \leq L \leq G$, then H is semi-normal in L.*
(b) *If $N \trianglelefteq G$, then HN/N is semi-normal in G/N.*
(c) *Let K be an s-supplement of H in G and $L \leq K$. Then H permutes with every conjugate of L in G.*

Proof. For (a) and (b) see (SU, [7]).
(c) Let $g \in G$. Put $g = kh$ with $k \in K$ and $h \in H$. Since L^k is a subgroup

of K we have that

$$L^gH = L^{kh}H = (L^kH)^h = (HL^k)^h = HL^{kh} = HL^g.$$

□

Lemma 2.2 *Let G be a finite group, H and K subgroups of G. If H permutes with every conjugate of K in G, then $H^K \cap K^H$ is subnormal in G.*

For a proof see ([6], p.221, Th. 7.2.5).

3. Proof of the theorems

Proof of Theorem A Suppose the theorem is false and let G be a counterexample of smallest order. We assume that H is of minimal order. Since H is solvable, we may assume $(|H|, p) = 1$. Hence $G = HK$, $K \in Syl_p(G)$ and $H \cap K = 1$.

First we prove that G is simple. Suppose $N \lhd G$. We have that $HN/N \cong H/H \cap N$ is 2-nilpotent and $|G/N : HN/N| = |G : HN|$ divides p^a.

If $N \neq 1$, then by the minimality of $|G|$, we have that G/N is solvable. Since $|N : H \cap N|$ divides p^a, the minimality of $|G|$ tells that N is solvable. So G is solvable, a contradiction. Hence G is simple. □

By lemma 1.1, G is isomorphic to one of the groups;

$$\mathbf{PSL}(2, 2^n)\ n \geq 2, \quad \mathbf{PSL}(2, q) \text{ with } q \equiv -1 \bmod (4),$$
$$\mathbf{PSL}(3, 3) \text{ or } \mathbf{M}_{11}.$$

Since $\mathbf{M}_{11}$ has not a 2-nilpotent subgroup of prime power index, we have that G is not isomorphic to $\mathbf{M}_{11}$.

If $G \cong \mathbf{PSL}(3, 3)$, then by ([4], p.189, Satz 7.4 Fall 1), we have that $|H| = 2^4 3^3$ and $K \in Syl_{13}(G)$. But, in this case H is not 2-nilpotent, because otherwise $H = \mathbf{N}_G(P)$, with $P \in Syl_3(G)$, a contradiction.

Since $\mathbf{PSL}(2, 2^n)$ has abelian Sylow-2-subgroups, lemma 1.2 tells that G is not isomorphic to $\mathbf{PSL}(2, 2^n)$.

Since $\mathbf{PSL}(2, q)$ with $q \equiv -1 \bmod (4)$ has dihedral Sylow-2-subgroups, lemma 1.2 tells that G is not isomorphic to $\mathbf{PSL}(2, q)$.

Proof of Theorem B Suppose the theorem is false and let G be a counterexample of smallest order.

Since H is semi-normal in G, there exists a subgroup K of G such that $G = HK$ and H permutes with every subgroup of K. We assume that K is chosen of minimal order. □

The following items are valid in G :

(i) *G has a unique minimal normal subgroup N and N is not solvable (in* (b), *N has even order).*

Suppose N_1 and N_2 are minimal normal subgroups of G, with $N_1 \neq N_2$. By Lemma 2.1 (b), the quotient HN_i/N_i is semi-normal in G/N_i ($i = 1, 2$). Since HN_i/N_i is 2-nilpotent (in (b), HN_i/N_i has odd order), by the minimality of $|G|$, we have that $H^G N_i/N_i$ is solvable (in (b), $H^G N_i/N_i$ has odd order). Hence $H^G \cong H^G/(N_1 \cap N_2 \cap H^G)$ is solvable (in (b), H^G has odd order), a contradiction.

So G has a unique minimal normal subgroup N. Since $N \leq H^G$ and H^G/N is solvable (in (b), H^G/N has odd order), we have that N is not solvable (in (b), N has even order).

(ii) *H centralizes every proper subgroup of K.*

Let L be a proper subgroup of K. By Lemma 2.1 (a), H is semi-normal in $HL = LH$. Since K is of minimal order, we have that HL is a proper subgroup of G and $H^{HL} = H^L$ is solvable (in (b), H^L has odd order), since the theorem holds in HL.

By Lemma 2.1 (c), H permutes with every conjugate of L in G. By Lemma 2.2, the intersection $H^L \cap L^H$ is subnormal in G. Since H^L is solvable (in (b), H^L has odd order), by (i) we have $H^L \cap L^H = 1$, since every subnormal solvable (in (b), subnormal odd order) subgroup is contained in a solvable (in (b), odd order) normal subgroup.Therefore, the commutator group $[H, L] \leq [H^L, L^H] \leq H^L \cap L^H = 1$.

(iii) *K is a cyclic p-group, for some prime number p and the maximal subgroup K_1 of K is normal in G.*

If K has two different maximal subgroups, then by (ii), H centralizes K and H is normal in G, a contradiction.

So K is a cyclic p-group, for some prime number p and $K_1 \trianglelefteq G$.

(iv) *The contradiction*

Item (i) shows that $|K| = p = |G : H|$. By Theorem A, we have that G is solvable. In the hypothesis (b), by (i) we have that $p = 2$ so $H \trianglelefteq G$.

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