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# Geometric characterization of Monge-Ampère equations

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**Abstract.** It is well known that a Monge-Ampère equation can be expressed in terms of exterior differential system—Monge-Ampère system, which is the ideal generated algebraically by a contact form and a 2-form and its exterior derivatives on a 5-dimensional contact manifold, and the system is independent of the choice of coordinate system. On the other hand, a single second order partial differential equation of one unknown function with two independent variables corresponds to the differential system on a hypersurface of Lagrange-Grassmann bundle over a 5-dimensional contact manifold obtained by restricting its canonical system to the hypersurface. We observe relations between Monge characteristic systems of Monge-Ampère equation and those of Monge-Ampère system and particularly analyze structure equations of those systems. This observation leads to the result—to characterize Monge-Ampère equation by the property that the certain differential system defined from the Monge characteristic system drops down to the contact manifold.

*Key words*: differential system, exterior differential system, partial differential equation, Monge-Ampere equation, Goursat equation, Monge characteristic system.

## 1. Introduction

An *exterior differential system* on a manifold $\Sigma$, or called *EDS* for short, consists of a differential ideal $\mathcal{I}$ on $\Sigma$, that is, an algebraic ideal of the differential algebra of differential forms on $\Sigma$ closed under exterior differentiation. Let $\mathcal{I} = \{\psi^1, \dots, \psi^n\}_{\mathrm{diff}}$ denote an EDS algebraically generated by differential forms $\psi^1, \dots, \psi^n$ and its derivatives $d\psi^1, \dots, d\psi^n$. For a point $p \in \Sigma$, $V$ is an *integral element* of an EDS $\mathcal{I}$ on $\Sigma$ if $V$ is a subspace of $T_p\Sigma$ such that $\psi|_V = 0$ for all $\psi \in \mathcal{I}$. An *integral manifold* of an EDS $\mathcal{I}$ on $\Sigma$ is an immersed submanifolder $\iota : M \hookrightarrow \Sigma$ such that $\iota^*\psi = 0$ for all $\psi \in \mathcal{I}$.

For a (classical) Monge-Ampère equation in coordinates description

$$Az_{xx} + 2Bz_{xy} + Cz_{yy} + D + E\big(z_{xx}z_{yy} - z_{xy}^2\big) = 0, \tag{1.1}$$

where each capital letter indicates a function of variables $x, y, z, z_x, z_y$, we consider the following EDS

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$$\mathcal{I} = \{\theta, \Psi\}_{\text{diff}},$$

where $\theta = dz - pdx - qdy$ and

$$\begin{aligned}\Psi = {} & A\,dp \wedge dy + B(-dp \wedge dx + dq \wedge dy) \\ & - C\,dq \wedge dx + D\,dx \wedge dy + E\,dp \wedge dq.\end{aligned} \tag{1.2}$$

Then a 2-dimensional integral manifold of $\mathcal{I}$ on which $dx \wedge dy$ is nonvanishing is locally the graph of a solution of the Monge-Ampère equation (1.1).

Let $J$ be a 5-dimensional contact manifold with contact form $\theta$ and $\Psi$ a 2-form on $J$ and suppose $\Psi \not\equiv 0 \pmod{\theta, d\theta}$. Then the EDS

$$\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$$

is called *a Monge-Ampère system* on $J$. By Darboux's Theorem, there exists a coordinate system $(x, y, z, p, q)$ of $J$ such that $\theta = dz - pdx - qdy$ and (1.2) holds (see [IL03]). Locally, a 2-dimensional integral manifold of a Monge-Ampère system on which $dx \wedge dy$ is nonvanishing is the graph of a solution of a Monge-Ampère equation (1.1).

We study regular single second order partial differential equations

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0 \tag{1.3}$$

of one unknown function with two independent variables. We regard them geometrically as differential systems $(R, D)$ where $R$ is a hypersurface in the Lagrange-Grassmann bundle $L(J)$ over a contact manifold $J$ of dimension 5 and $D$ is the restriction of the canonical system $E$ on $L(J)$ to $R$.

In this article, we give a geometric characterization of the class of Monge-Ampère equations in terms of Monge characteristic systems, in both hyperbolic and parabolic cases.

Here, a *hyperbolic* (resp. *parabolic*) PDE is an equation (1.3) which satisfies $AC - \frac{1}{4}B^2 < 0$ (resp. $AC - \frac{1}{4}B^2 = 0$) at any point, where $A = \frac{\partial F}{\partial z_{xx}}$, $B = \frac{\partial F}{\partial z_{xy}}$ and $C = \frac{\partial F}{\partial z_{yy}}$.

From the structure equations in each case we define the Monge characteristic systems $\mathcal{M}_i$ for PDE $(R, D)$, and the Monge characteristic systems $\mathcal{H}_i$ for Monge-Ampère system $\mathcal{I}$ in Section 2. We denote by $\mathrm{Ch}(\partial D)$ the Cauchy characteristic system of the first derived system $\partial D$ of $D$.

We observe relations between Monge characteristic systems $\mathcal{M}_i$ and $\mathcal{H}_i$ and particularly analyze structure equations of those systems: If $(R, D)$ is hyperbolic PDE, the first and second derived systems $\partial\mathcal{M}_i$, $\partial^2\mathcal{M}_i$ of $D$ are regular and of rank 4 and 5, and $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ coincides with pullbacks of $\mathcal{H}_i$ (Theorem 3.3 and Corollary 3.4). We observe also in parabolic case (Theorem 3.11).

This observation leads to main results in this paper.

One of main results is stated as follows: Suppose a regular second order partial differential equation $(R, D)$ of one unknown function with two independent variables is hyperbolic and suppose one of the differential systems on $R$, $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ or $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ drops down to $J$. Then $(R, D)$ coincides with the Monge-Ampère equation associated with a Monge-Ampère system $\mathcal{I}$ on $J$. Moreover $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ and $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ are pullbacks of the Monge characteristic systems $\mathcal{H}_i$ of $\mathcal{I}$.

We give a characterization of Monge-Ampère equations also in parabolic case (Theorem 4.6).

There are some earlier researches for Monge-Ampère equation. For example, V. V. Lychagin ([Lyc79]) discussed non-linear second-order differential operator and a generalization of Monge-Ampère equations by using non-linear second-order differential operators.

Especially, R. B. Gardner and N. Kamran ([GK93]) investigated invariants which characterized Monge-Ampère equation in hyperbolic case and R. L. Bryant and P. A. Griffiths ([BG95]) did in parabolic case. They call these invariants Monge-Ampère invariants.

R. L. Bryant and P. A. Griffiths described the structure equation of non-Goursat system ([BG95, p. 556]) and showed that a non-Goursat parabolic system is locally equivalent to an equation of Monge-Ampère type if and only if the Monge-Ampère invariant $\Psi = S_0\,\pi_3 \wedge \pi_4$ vanishes.

In Section 2 we recall our notations and definitions, and describe the prolongation of Monge-Ampère systems. Separating hyperbolic and parabolic cases, in Section 3 we mention relations between the Monge characteristic systems of Monge-Ampère systems and those of the corresponding Monge-Ampère equations (Lemma 3.1, Theorem 3.3 and Corollary 3.4 in hyperbolic case and Lemma 3.9 and Theorem 3.11 in parabolic case). Additionally, we look at relations between numbers of independent first integrals of Monge characteristic systems of Monge-Ampère systems and those of the corresponding Monge-Ampère equations (Corollary 3.2 and 3.5 in hyperbolic case and Corollary 3.10 in parabolic case). Section 4 deals with characterizations of Monge-Ampère equations (Theorem 4.1 in hyperbolic case and Theorem 4.6 in parabolic case).

In this paper we assume all objects are of class $C^\infty$.

## 2. preliminaries

We recall some definitions and fix our notations, following [IL03] and [Yam82].

In order to consider partial differential equations as geometrical subjects, we utilize *differential systems* and *exterior differential systems*.

A *differential system* $D$ on a manifold $R$ is a subbundle of the tangent bundle $TR$ of $R$. A differential system $D$ is locally defined by linearly independent 1-forms $\omega^1, \dots, \omega^r$ as follows:

$$D = \{\omega^1 = \cdots = \omega^r = 0\},$$

where $r$ is the corank of $D$. The *Cauchy characteristic system* $\mathrm{Ch}(D)$ of $D$ is defined by

$$\mathrm{Ch}(D)(x) = \{X \in D(x) \mid X \lrcorner\, d\omega^i \equiv 0 \pmod{\omega^1_x, \dots, \omega^r_x}\}$$

at each point $x \in R$. For a point $x \in R$, $v$ is an *integral element* of the differential system $D$ if $v$ is a subspace of $T_xR$ such that $\omega^k|_v = 0$ and $d\omega^k|_v = 0$ for all $1 \le k \le r$. An *integral manifold* of the differential system $D$ is a submanifold $M$ of $R$ such that $\omega^k|_M = 0$ and hence $d\omega^k|_M = 0$ for all $1 \le k \le r$.

The *first derived system* $\partial D$ of a differential system $D$ is defined by, in terms of sections,

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$$

where $\mathcal{D}$ is the space of sections of $D$ and $[,]$ is Lie bracket for vector fields. Furthermore, the *k-th derived system* $\partial^k D$ is defined inductively as follows: if $\partial^{k-1} D$ is a differential system, then

$$\partial^k D = \partial(\partial^{k-1} D)$$

where we put $\partial^0 D = D$ for convention. A differential system $D$ is *completely integrable* if $\partial D$ coincides with $D$. A function $f$ on a domain of $R$ is a *first integral* of $D$ if $df \equiv 0 \pmod{D^\perp}$, where $D^\perp$ is the set of annihilators of $D$, and then we say that *$D$ has a first integral $f$.*

Let $J$ be a manifold of dimension $2n+1$ and $C$ a differential system on $J$ of corank 1, which means that, at each point $u \in J$, there exists a nonvanishing 1-form $\theta$ around $u$ such that $C = \{\theta = 0\}$. Then $(J, C)$ is called a *contact manifold* if $\theta \wedge (d\theta)^n$ is nonvanishing. The notation $(J, C)$ is often shortened to $J$ in this paper.

Starting from a $(2n+1)$-dimensional contact manifold $(J, C)$, we now construct *Lagrange-Grassmann bundle* $L(J)$ over $(J, C)$ and the *canonical system* $E$ on $L(J)$ as follows: let $L(J)$ be the space consisting of all $n$-dimensional integral elements of $C$, that is,

$$L(J) = \bigcup_{u \in J} L(J)_u \xrightarrow{\ \pi\ } J$$

where $L(J)_u$ is the Grassmannian of all Lagrangian subspaces of the symplectic vector space $(C(u), d\theta_u)$ and $\pi$ is the canonical projection. The canonical system $E$ on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \quad \text{for } v \in L(J).$$

We now take a coordinate system of $L(J)$ as follows: let us fix a point $v_o \in L(J)$. By Darboux's Theorem, there exists a canonical coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ around $u_o = \pi(v_o)$ such that $\theta = dz - \sum_{i=1}^n p_i dx^i$. We may assume that $dx^1 \wedge \cdots \wedge dx^n|_{v_o} \neq 0$. Taking a neighborhood $V$ of $v_o$ such that $dx^1 \wedge \cdots \wedge dx^n|_v \neq 0$ around $v \in V$, we define functions $p_{ij}$ on $V$ by

$$\begin{aligned} dp_1|_v &= p_{11}(v)dx^1|_v + \cdots + p_{1n}(v)dx^n|_v, \\ &\ \vdots \\ dp_n|_v &= p_{n1}(v)dx^1|_v + \cdots + p_{nn}(v)dx^n|_v. \end{aligned}$$

Since $d\theta|_v = 0$, we have $p_{ij} = p_{ji}$. Thus we have obtained the coordinate system $(x^i, z, p_i, p_{ij})$ $(1 \le i \le j \le n)$ of $L(J)$. Then $E$ is locally defined by

$$E = \{\varpi_0 = \varpi_1 = \cdots = \varpi_n = 0\}$$

where $\varpi_0 = dz - \sum_{i=1}^n p_i dx^i$ and $\varpi_i = dp_i - \sum_{j=1}^n p_{ij} dx^j$ for $i = 1, \ldots, n$.

Let us consider a single second order partial differential equation of one unknown function with two independent variables

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0. \tag{2.1}$$

Assuming that partial derivatives $\frac{\partial F}{\partial z_{xx}}$, $\frac{\partial F}{\partial z_{xy}}$ and $\frac{\partial F}{\partial z_{yy}}$ of $F$ are never simultaneously zero, we define a hypersurface $R$ of the Lagrange-Grassmann bundle $L(J)$ over a contact manifold $(J, C)$ and a differential system $D$ on $R$ as

$$\begin{aligned} R &= \{F(x, y, z, p, q, r, s, t) = 0\} \subset L(J), \\ D &= \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}, \end{aligned}$$

where $(x, y, z, p, q, r, s, t)$ is a coordinate system of $L(J)$ taken above and $\varpi_0 = (dz - pdx - qdy)|_R$, $\varpi_1 = (dp - rdx - sdy)|_R$, $\varpi_2 = (dq - sdx - tdy)|_R$. Generally, let $(R, D)$ be a single second order PDE and $\rho$ the canonical projection from $R$ to $J$. Namely $R$ is a hypersurface of $L(J)$ and $D$ is the differential system on $R$ obtained by restricting the canonical system $E$ to $R$. Then a 2-dimensional integral manifold of $D$ transverse to fibers of $\rho$ is locally the graph of a solution of the single second order PDE.

Let $(R, D)$ be a single second order PDE and we may write $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$. Let us assume $\rho$ is a submersion.

It is well-known that the structure equation of $D$ is expressed as follows: let us fix a point $v_o \in R$. If the equation $R$ is hyperbolic around $v_o$, the structure equation is

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22}\}$ is a coframe around $v_o \in R$ ([BCG$^+$91, p. 277]). If the equation $R$ is parabolic around $v_o$,

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \qquad\qquad\quad \omega^2 \wedge \pi_{12} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} + \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{12}, \pi_{22}\}$ is a coframe around $v_o \in R$ ([BCG$^+$91, p. 275]).

Then, if $R$ is hyperbolic or parabolic, the Monge characteristic system $\mathcal{M}_i$ of $(R, D)$ are defined as

$$\mathcal{M}_i = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \omega^i = \pi_{ii} = 0\right\} \quad \text{for } i = 1, 2,$$

or

$$\mathcal{M} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{12} = 0\right\},$$

respectively ([IL03, p. 213]).

Let $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$ be a Monge-Ampère system on $J$. For a point $u \in J$, if $\mathcal{I}_u$ has two, one or no independent decomposable 2-covector, modulo $\theta_u$, then $\mathcal{I}$ is called *hyperbolic*, *parabolic* or *elliptic* at $u$, respectively. Because the dimension of $J$ is five, a 2-covector $(\Psi + \lambda d\theta)_u$ is decomposable modulo $\theta_u$ if and only if $(\Psi + \lambda d\theta)_u^2 \equiv 0 \pmod{\theta_u}$. Then the relation

$$(\Psi + \lambda d\theta)_u^2 = \Psi_u \wedge \Psi_u + 2\lambda \Psi_u \wedge d\theta_u + \lambda^2 d\theta_u \wedge d\theta_u \equiv 0 \pmod{\theta_u} \quad (2.2)$$

yields a quadratic equation in a variable $\lambda$. Because a root of the quadric equation satisfies (2.2), $\mathcal{I}$ is hyperbolic, parabolic or elliptic at $u \in J$ if the quadratic equation has two, one or no real roots, respectively. If $\mathcal{I}$ has a decomposable 2-form $\omega \wedge \pi$, modulo $\theta$, then a *Monge characteristic system* $\mathcal{H}$ of $\mathcal{I}$ is defined as

$$\mathcal{H} = \{\theta = \omega = \pi = 0\},$$

which is a differential system of rank 2 on $J$.

Finally, we mention the relation between a Monge-Ampère system $\mathcal{I}$ and its prolongation $(R, D)$.

Let $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$ be a Monge-Ampère system on $J$ and let $L(J)$ denote the Lagrange-Grassmann bundle over $J$. We obtain the prolongation $(R, D)$

of $\mathcal{I}$ as follows: the *prolongation* of $\mathcal{I}$ (cf. [BCG+91], [IL03]) is the differential system $(R, D)$ that $R$ is the set of 2-dimensional integral elements of $\mathcal{I}$ and assume $R$ is a smooth manifold, and $D$ is the restriction of the canonical system $(Gr(2,TJ), \hat{E})$ to $R$. Here, $\pi : Gr(2,TJ) \longrightarrow J$ is the Grassmann bundle over $J$ consisting of all 2-dimensional subspaces of all tangent spaces to $J$ and $\hat{E}$ is defined by $\hat{E}(v) = \pi_*^{-1}(v)$ at each point $v \in Gr(2,TJ)$. In this case, by the definition of $R$, $L(J)$ contains $R$.

Let us fix a point $v_o \in L(J)$ and take a coframe $\{\theta, \omega^1, \omega^2, \pi_1, \pi_2\}$ around $u_o = \pi(v_o)$ such that

$$d\theta \equiv \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2 \pmod{\theta}.$$

We may assume $\omega^1 \wedge \omega^2|_{v_o} \neq 0$. Taking a neighborhood $V$ of $v_o$ such that $\omega^1 \wedge \omega^2|_v \neq 0$ at each point $v \in V$, we can take fiber coordinate functions $a, b, c$ on $V$ such that

$$\begin{aligned} \pi_1|_v &= a(v)\,\omega^1|_v + b(v)\,\omega^2|_v \\ \pi_2|_v &= b(v)\,\omega^1|_v + c(v)\,\omega^2|_v \end{aligned} \qquad v \in V.$$

Setting

$$\begin{aligned} \Psi = {} & A\pi_1 \wedge \omega^2 + B\,(-\pi_1 \wedge \omega^1 + \pi_2 \wedge \omega^2) \\ & - C\pi_2 \wedge \omega^1 + D\omega^1 \wedge \omega^2 + E\pi_1 \wedge \pi_2 \end{aligned} \tag{2.3}$$

where each capital letter indicates a function around $u_o$, we have

$$\Psi|_v = (Aa + 2Bb + Cc + D + E(ac - b^2))(v)\,\omega^1 \wedge \omega^2|_v.$$

Thus we set

$$\begin{aligned} R &= \{v \in V \mid \Psi|_v = 0\}, \\ &= \{Aa + 2Bb + Cc + D + E(ac - b^2) = 0\}, \end{aligned} \tag{2.4}$$

which is a subvariety of $L(J)$. Around each regular points of $R$, we may define $D$ as the restriction of $E$ to $R$.

For a Monge-Ampère system $\mathcal{I}$, the prolongation $(R, D)$ of $\mathcal{I}$ is called the *corresponding* Monge-Ampère equation in this article. In fact, as mentioned above, for a given Monge-Ampère system $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$, we can take a coordinate system $(x, y, z, p, q)$ such that $\theta = dz - pdx - qdy$ and set $\omega^1 = dx$, $\omega^2 = dy$, $\pi_1 = dp$, $\pi_2 = dq$, and then we set given $\Psi$ as in Equation (2.3). Therefore we obtain the coordinate description (2.4) of the Monge-Ampère equation $R$.

For a Monge characteristic system $\mathcal{H}$ of $\mathcal{I}$, the *corresponding* Monge characteristic system $\mathcal{M}$ means the Monge characteristic system of the corresponding Monge-Ampère equation. We show that

$$\mathcal{M} \subset \rho_*^{-1}(\mathcal{H}).$$

in the next section.

## 3. Relations Between the Monge Characteristic Systems of Monge-Ampère Equation and Those of Monge-Ampère System

We describe the structure equation of the corresponding Monge-Ampère equation and investigate relations between the Monge characteristic systems of Monge-Ampère systems and those of the corresponding Monge-Ampère equations. Furthermore, this observation gives us a guideline for the characterization of Monge-Ampère equations in Section 4.

### 3.1. Hyperbolic case

First, we choose a coframe adapted for a Monge-Ampère system: Let $\mathcal{I} = \{\theta, \Psi\}_{\text{diff}}$ be a Monge-Ampère system and let $(R, D)$ denote the corresponding Monge-Ampère equation. Let us fix a point $v_o \in R$. Assuming $\mathcal{I}$ is hyperbolic around $u_o = \pi(v_o)$, we can take different functions $\lambda_1$ and $\lambda_2$ around $u_o$ such that $\Psi + \lambda_1 d\theta$ and $\Psi + \lambda_2 d\theta$ are decomposable 2-forms, and take 1-forms $\omega^1, \omega^2, \pi_1', \pi_2'$ around $u_o$ such that

$$\omega^i \wedge \pi_i' \equiv \Psi + \lambda_i d\theta \pmod{\theta} \quad \text{for } i = 1, 2.$$

Since $\theta|_{v_o} = d\theta|_{v_o} = \Psi|_{v_o} = 0$, we have $\omega^1 \wedge \pi_1'|_{v_o} = \omega^2 \wedge \pi_2'|_{v_o} = 0$ and may thus assume $\omega^1|_{v_o} \neq 0$ and $\omega^2|_{v_o} \neq 0$. Hence $\pi_1|_{v_o}$ is a multiple of $\omega^1|_{v_o}$ and $\pi_2|_{v_o}$ is of $\omega^2|_{v_o}$. Since $\omega^1 \wedge \pi_1' - \omega^2 \wedge \pi_2' \equiv (\lambda_1 - \lambda_2) d\theta \pmod{\theta}$ and $\lambda_1 - \lambda_2 \neq 0$, we have

$$d\theta \equiv \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2 \pmod{\theta}$$

where $\pi_1 = \frac{1}{\lambda_1 - \lambda_2}\pi_1'$, $\pi_2 = -\frac{1}{\lambda_1 - \lambda_2}\pi_2'$. Since $\theta$ is a contact form, $\theta \wedge \omega^1 \wedge \omega^2 \wedge \pi_1 \wedge \pi_2 \neq 0$ around $u_o$. Hence $\{\theta, \omega^1, \omega^2, \pi_1, \pi_2\}$ is a coframe around $u_o$.

Secondly, let us take a neighborhood $V$ of $v_o$ such that $\omega^1 \wedge \omega^2 \,|_v \neq 0$ at each $v \in V$ and functions $a, b, c$ on $V$ such that

$$\begin{aligned} \pi_1|_v &= a(v)\, \omega^1|_v + b(v)\, \omega^2|_v \\ \pi_2|_v &= b(v)\, \omega^1|_v + c(v)\, \omega^2|_v \end{aligned} \qquad v \in V.$$

Since $\omega^1 \wedge \pi_1|_v = 0$, we get $b(v) = 0$. Thus

$$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\},$$

where $\varpi_0 = \rho^*\theta$, $\varpi_1 = \rho^*\pi_1 - a\rho^*\omega^1$, $\varpi_2 = \rho^*\pi_2 - c\rho^*\omega^2$.

For $i = 1, 2$, we have

$$\begin{aligned} d\pi_i &\equiv \pi_1 \wedge (A_i\pi_2 + B_i\omega^1 + C_i\omega^2) + \pi_2 \wedge (E_i\omega^1 + F_i\omega^2) + G_i\omega^1 \wedge \omega^2, \\ d\omega^i &\equiv \pi_1 \wedge (H_i\pi_2 + I_i\omega^1 + J_i\omega^2) + \pi_2 \wedge (K_i\omega^1 + L_i\omega^2) + N_i\omega^1 \wedge \omega^2, \end{aligned} \tag{3.1}$$

modulo $\theta$, where each capital letter with an additional character indicates smooth functions around $u_o$ on $J$. Let us omit the pullback $\rho^*$ in what follows. Then on $R$, we have

$$\begin{aligned} &d\pi_1 - ad\omega^1 \\ &\quad\equiv (A_1ac + C_1a - E_1c + G_1 - H_1a^2c - J_1a^2 + K_1ac - N_1a)\omega^1 \wedge \omega^2 \\ &d\pi_2 - cd\omega^2 \\ &\quad\equiv (A_2ac + C_2a - E_2c + G_2 - H_2ac^2 - J_2ac + K_2c^2 - N_2c)\omega^1 \wedge \omega^2 \end{aligned}$$

modulo $\varpi_0, \varpi_1, \varpi_2$. Namely, from $d\varpi_0 = d\theta$, $d\varpi_1 = d\pi_1 - ad\omega^1 - da \wedge \omega^1$, $d\varpi_2 = d\pi_2 - bd\omega^2 - db \wedge \omega^2$, we obtain the structure equation:

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \qquad\qquad \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases} \tag{3.2}$$

where

$$\pi_{11} = da + (A_1 ac + C_1 a - E_1 c + G_1 - H_1 a^2 c - J_1 a^2 + K_1 ac - N_1 a)\omega^2,$$

$$\pi_{22} = dc - (A_2 ac + C_2 a - E_2 c + G_2 - H_2 ac^2 - J_2 ac + K_2 c^2 - N_2 c)\omega^1.$$

**Lemma 3.1**

$$\mathcal{M}_i \subset \rho_*^{-1}(\mathcal{H}_i) \quad \textit{and} \quad \partial\mathcal{M}_i \subset \rho_*^{-1}(\mathcal{H}_i) \qquad \textit{for } i = 1, 2.$$

*Proof.* As we use the coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22}\}$ taken above,

$$\begin{aligned} \mathcal{M}_i &= \left\{\varpi_0 = \varpi_1 = \varpi_2 = \omega^i = \pi_{ii} = 0\right\}, \\ \rho_*^{-1}(\mathcal{H}_i) &= \left\{\rho^*\theta = \rho^*\omega^i = \rho^*\pi_i = 0\right\}, \\ &= \left\{\varpi_0 = \varpi_i = \omega^i = 0\right\}. \end{aligned}$$

By (3.1) and (3.2), we have $d\varpi_0 \equiv d\varpi_i \equiv d\omega^i \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^i, \pi_{ii}}$. Thus

$$\partial\mathcal{M}_i \subset \left\{\varpi_0 = \varpi_i = \omega^i = 0\right\} = \rho_*^{-1}(\mathcal{H}_i). \qquad \square$$

**Corollary 3.2** *If $\mathcal{H}_i$ has two independent first integrals, then $\mathcal{M}_i$ also has at least two.*

Here, "independent" means independence as function, that is, there exists two first integrals $f_1$, $f_2$ of $\mathcal{H}_i$ such that $df_1 \wedge df_2 \neq 0$.

Though we obtain this corollary from the structure equation (3.2), to obtain more information, we need to analyze the structure equation in more detail:

**Theorem 3.3** *Let $\mathcal{I}$ be a hyperbolic Monge-Ampère system on a 5-dimensional contact manifold $J$ and let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the Monge characteristic systems of $\mathcal{I}$, and let $(R, D)$ denote the corresponding Monge-Ampère equation and $\mathcal{M}_1$ and $\mathcal{M}_2$ the corresponding Monge characteristic* *systems. Then, for* $i = 1,\ 2$, $\partial\mathcal{M}_i$, $\partial^2\mathcal{M}_i$ *and* $\partial\mathcal{H}_i$ *are differential systems, and it follows that* $\operatorname{codim}\partial^2\mathcal{M}_i = 3$ *and*

$$\partial^2\mathcal{M}_i \subset \rho_*^{-1}(\partial\mathcal{H}_i).$$

*Proof.* Let us choose the coframe $\{\theta, \omega^1, \omega^2, \pi_1, \pi_2\}$ taken above. For $i = 1, 2$, putting

$$d\pi_i \equiv \pi_1 \wedge (A_i\pi_2 + B_i\omega^1 + C_i\omega^2) + \pi_2 \wedge (E_i\omega^1 + F_i\omega^2) + G_i\omega^1 \wedge \omega^2$$
$$d\omega^i \equiv \pi_1 \wedge (H_i\pi_2 + I_i\omega^1 + J_i\omega^2) + \pi_2 \wedge (K_i\omega^1 + L_i\omega^2) + N_i\omega^1 \wedge \omega^2$$

modulo $\theta$, where each capital letter with an additional character indicates smooth functions on $J$, we have

$$\left.\begin{aligned} d\pi_i \equiv A_i\varpi_1 \wedge \varpi_2 &+\varpi_1 \wedge \big(B_i\omega^1 + (A_ic + C_i)\,\omega^2\big) \\ &+\varpi_2 \wedge \big((-A_ia + E_i)\,\omega^1 + F_i\omega^2\big) \\ &+(A_iac + C_ia - E_ic + G_i)\,\omega^1 \wedge \omega^2 \\ d\omega^i \equiv H_i\varpi_1 \wedge \varpi_2 &+\varpi_1 \wedge \big(I_i\omega^1 + (H_ic + J_i)\,\omega^2\big) \\ &+\varpi_2 \wedge \big((-H_ia + K_i)\,\omega^1 + L_i\omega^2\big) \\ &+(H_iac + J_ia - K_ic + N_i)\,\omega^1 \wedge \omega^2 \end{aligned}\right\} \pmod{\varpi_0}$$

and hence

$$\begin{aligned} d\varpi_1 \equiv \omega^1 \wedge \pi_{11} &+\varpi_1 \wedge \big((B_1 - I_1a)\,\omega^1 + (A_1c + C_1 - H_1ac - J_1a)\,\omega^2\big) \\ &+\varpi_2 \wedge \big((-A_1a + E_1 + H_1a^2 - K_1a)\,\omega^1 + (F_1 - L_1a)\,\omega^2\big) \\ &+(A_1 - H_1a)\,\varpi_1 \wedge \varpi_2 \qquad \pmod{\varpi_0}, \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} &+\varpi_1 \wedge \big((B_2 - I_2c)\,\omega^1 + (A_2c + C_2 - H_2c^2 - J_2c)\,\omega^2\big) \\ &+\varpi_2 \wedge \big((-A_2a + E_2 + H_2ac - K_2c)\,\omega^1 + (F_2 - L_2c)\,\omega^2\big) \\ &+(A_2 - H_2c)\,\varpi_1 \wedge \varpi_2 \qquad \pmod{\varpi_0}, \end{aligned}$$

where

$$\pi_{11} = da + (A_1ac + C_1a - E_1c + G_1 - H_1a^2c - J_1a^2 + K_1ac - N_1a)\,\omega^2,$$
$$\pi_{22} = dc - (A_2ac + C_2a - E_2c + G_2 - H_2ac^2 - J_2ac + K_2c^2 - N_2c)\,\omega^1.$$

By definition, one Monge characteristic system is

$$\mathcal{H}_1 = \left\{\theta = \omega^1 = \pi_1 = 0\right\}.$$

Since the structure equation of $\mathcal{H}_1$ is

$$\begin{cases} d\theta \equiv \omega^2 \wedge \pi_2 \\ d\omega^1 \equiv -L_1\, \omega^2 \wedge \pi_2 \\ d\pi_1 \equiv -F_1\, \omega^2 \wedge \pi_2 \end{cases} \quad (\text{mod } \theta, \omega^1, \pi_1),$$

the first derived system of $\mathcal{H}_1$ is

$$\partial\mathcal{H}_1 = \left\{\widetilde{\omega}^1 = \widetilde{\pi}_1 = 0\right\},$$

where $\widetilde{\omega}^1 = \omega^1 + L_1\theta$, $\widetilde{\pi}_1 = \pi_1 + F_1\theta$, and hence $\partial\mathcal{H}_1$ is a differential system on $J$.

On the other hand, let us recall the corresponding Monge characteristic system

$$\mathcal{M}_1 = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \omega^1 = \pi_{11} = 0\right\}.$$

Since the structure equation of $\mathcal{M}_1$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv 0 \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} \\ d\omega^1 \equiv 0 \\ d\pi_{11} \equiv -(A_1 a - E_1 - H_1 a^2 + K_1 a)\, \omega^2 \wedge \pi_{22} \end{cases} \qquad (\text{mod } \varpi_0, \varpi_1, \varpi_2, \omega^1, \pi_{11}),$$

the first derived system of $\mathcal{M}_1$ is

$$\partial\mathcal{M}_1 = \left\{\varpi_0 = \varpi_1 = \omega^1 = \overline{\pi}_{11} = 0\right\},$$

where $\overline{\pi}_{11} = \pi_{11} + (A_1 a - E_1 - H_1 a^2 + K_1 a)\, \varpi_2$, and hence $\partial\mathcal{M}_1$ is a differential system on $R$. Since

$$\begin{cases} d\varpi_0 \equiv \omega^2 \wedge \varpi_2 \\ d\varpi_1 \equiv -(F_1 - L_1 a)\,\omega^2 \wedge \varpi_2 \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1), \\ d\omega^1 \equiv -L_1\,\omega^2 \wedge \varpi_2 \end{cases} \tag{3.3}$$

the second derived system of $\mathcal{M}_1$ is

$$\partial^2 \mathcal{M}_1 \subset \left\{ \widehat{\varpi}_1 = \widehat{\omega}^1 = 0 \right\}, \tag{3.4}$$

where $\widehat{\varpi}_1 = \varpi_1 + (F_1 - L_1 a)\,\varpi_0$, $\widehat{\omega}^1 = \omega^1 + L_1 \varpi_0$. We have

$$\begin{aligned} \rho^*(\widetilde{\omega}^1) &= \omega^1 + L_1 \varpi_0 = \widehat{\omega}^1, \\ \rho^*(\widetilde{\pi}_1) &= \varpi_1 + a\omega^1 + F_1 \varpi_0 = \widehat{\varpi}_1 + a\widehat{\omega}^1, \end{aligned} \tag{3.5}$$

and hence $\partial^2 \mathcal{M}_1$ satisfies the inclusion

$$\partial^2 \mathcal{M}_1 \subset \rho_*^{-1}(\partial \mathcal{H}_1) = \left\{ \rho^* \widetilde{\omega}^1 = \rho^* \widetilde{\pi}_1 = 0 \right\}. \tag{3.6}$$

Furthermore, since

$$d\overline{\pi}_{11} \equiv (acdA_1 + adC_1 - cdE_1 + dG_1 - a^2 cdH_1 - a^2 dJ_1 + acdK_1 - adM_1) \wedge \omega^2$$

modulo $\varpi_0$, $\varpi_1$, $\omega^1$, $\overline{\pi}_{11}$, $\omega^2 \wedge \varpi_2$, and

$$\begin{aligned} dA_1 \wedge \omega^2 &\equiv dC_1 \wedge \omega^2 \equiv dE_1 \wedge \omega^2 \equiv dG_1 \wedge \omega^2 \\ &\equiv dH_1 \wedge \omega^2 \equiv dJ_1 \wedge \omega^2 \equiv dK_1 \wedge \omega^2 \equiv dM_1 \wedge \omega^2 \equiv 0 \end{aligned}$$

modulo $\varpi_0$, $\varpi_1$, $\varpi_2$, $\omega^1$, $\overline{\pi}_{11}$, we have

$$d\overline{\pi}_{11} \equiv 0 \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1, \overline{\pi}_{11}, \omega^2 \wedge \varpi_2).$$

Thus $\partial^2 \mathcal{M}_1$ is a differential system and $\operatorname{codim} \partial^2 \mathcal{M}_1 = 3$.

Similarly, we can prove the claims in the case of $\mathcal{H}_2$ and $\mathcal{M}_2$. □

The following Corollary is a key of characterization of Monge-Ampère equation (see Theorem 4.1)

**Corollary 3.4**

$$\rho_*^{-1}(\mathcal{H}_i) = \partial\mathcal{M}_i + \mathrm{Ch}(\partial D) \quad \textit{for } i = 1, 2.$$

From (3.3), (3.4), (3.5) and (3.6), we get the following corollary:

**Corollary 3.5** *If $\mathcal{M}_i$ has three independent first integrals, then $\mathcal{H}_i$ also has two.*

**Remark 3.6** As it is seen in Corollary 3.2, if $\mathcal{H}_i$ has two independent first integrals, then $\mathcal{M}_i$ also has at least two. However, it is not always true that $\mathcal{H}_i$ also has two independent first integrals if $\mathcal{M}_i$ has two independent first integrals. For example, let us consider the hyperbolic Monge-Ampère equation ([Boo59], [Gou90], [For06])

$$r - t - \frac{np}{x} = 0,$$

where $n$ is an integer. The Monge-Ampère system is

$$\left\{\theta = dz - pdx - qdy,\ \Psi = dp \wedge dy + dq \wedge dx - \frac{np}{x} dx \wedge dy\right\}_{\text{diff}}$$

and decomposable 2-forms are

$$\Psi \pm d\theta = \left(dp \mp dq - \frac{np}{x} dx\right) \wedge (dy \mp dx).$$

Then we have

$$d\theta = \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2,$$

where $\omega_1 = \frac{1}{2}(dx - dy)$, $\omega_2 = \frac{1}{2}(dy + dx)$, $\pi_1 = dp - dq - \frac{np}{x} dx$, $\pi_2 = dq + dp - \frac{np}{x} dx$.

We obtain the derived systems $\partial^k \mathcal{H}_i$ for each $i = 1, 2$ as follows: Since the structure equation of $\mathcal{H}_1 = \{\theta = \omega^1 = \pi_1 = 0\}$ is

$$\begin{cases} d\theta \equiv \omega^2 \wedge \pi_2 \\ d\omega^1 = 0 \\ d\pi_1 \equiv \dfrac{n}{2x} \omega^2 \wedge \pi_2 \end{cases} \qquad (\mathrm{mod}\ \theta, \omega^1, \pi_1),$$

the first derived system is

$$\partial\mathcal{H}_1 = \left\{\omega^1 = \pi_1' = 0\right\},$$

where $\pi_1' = \pi_1 - \frac{n}{2x}\theta$. Since the structure equation of $\partial\mathcal{H}_1$ is

$$\begin{cases} d\omega^1 = 0 \\ d\pi_1' \equiv \dfrac{n(n+2)}{4x^2}\,\omega^2 \wedge \theta \end{cases} \qquad (\mathrm{mod}\ \omega^1, \pi_1'),$$

$\partial\mathcal{H}_1$ is completely integrable if and only if $n = 0$ or $-2$.

On the other hand, let us recall the corresponding Monge characteristic system

$$\mathcal{M}_1 = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \omega^1 = \pi_{11} = 0\right\},$$

where $\varpi_0 = \rho^*\theta$, $\varpi_1 = \rho^*\pi_1 - a\rho^*\omega^1$, $\varpi_2 = \rho^*\pi_2 - c\rho^*\omega^2$ and let us omit the pullback $\rho^*$ in what follows. Then we have

$$\begin{aligned} d\varpi_0 &= \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2, \\ d\varpi_1 &= \omega^1 \wedge \pi_{11} - \frac{n}{2x}\varpi_1 \wedge (\omega^1 + \omega^2) - \frac{n}{2x}\varpi_2 \wedge (\omega^1 + \omega^2), \\ d\varpi_2 &= \omega^2 \wedge \pi_{22} - \frac{n}{2x}\varpi_1 \wedge (\omega^1 + \omega^2) - \frac{n}{2x}\varpi_2 \wedge (\omega^1 + \omega^2), \end{aligned}$$

where $\pi_{11} = da - \frac{n(a-c)}{2x}\omega^2$, $\pi_{22} = dc + \frac{n(a-c)}{2x}\omega^1$. Since the structure equation of $\mathcal{M}_1$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv 0 \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} \\ d\omega^1 = 0 \\ d\pi_{11} \equiv -\dfrac{n}{2x}\,\omega^2 \wedge \pi_{22} \end{cases} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1, \pi_{11}),$$

the first derived system is

$$\partial\mathcal{M}_1 = \left\{\varpi_0 = \varpi_1 = \omega^1 = \widetilde{\pi}_{11} = 0\right\},$$

where $\widetilde{\pi}_{11} = \pi_{11} + \frac{n}{2x}\varpi_2$. Since the structure equation of $\partial\mathcal{M}_1$ is

$$\begin{cases} d\varpi_0 \equiv \omega^2 \wedge \varpi_{22} \\ d\varpi_1 \equiv \dfrac{n}{2x}\,\omega^2 \wedge \varpi_2 \\ d\omega^1 = 0 \\ d\widetilde{\pi}_{11} \equiv -\dfrac{n}{2x^2}\,\omega^2 \wedge \varpi_2 \end{cases} \qquad (\text{mod } \varpi_0, \varpi_1, \omega^1, \widetilde{\pi}_{11}),$$

the second derived system is

$$\partial^2\mathcal{M}_1 = \left\{\varpi_1' = \omega^1 = \pi_{11}' = 0\right\},$$

where $\varpi_1' = \varpi_1 - \frac{n}{2x}\varpi_0$, $\pi_{11}' = \widetilde{\pi}_{11} + \frac{n}{2x^2}\varpi_0 = \pi_{11} + \frac{n}{2x^2}\varpi_0 + \frac{n}{2x}\varpi_2$. Since the structure equation of $\partial^2\mathcal{M}_1$ is

$$\begin{cases} d\varpi_1' \equiv \dfrac{n(n+2)}{4x^2}\,\omega^2 \wedge \varpi_0 \\ d\omega^1 = 0 \\ d\pi_{11}' \equiv \dfrac{n(n+2)(n-4)}{8x^3}\,\omega^2 \wedge \varpi_0 \end{cases} \qquad (\text{mod } \varpi_1', \omega^1, \pi_{11}'), \tag{3.7}$$

$\partial^2\mathcal{M}_1$ is completely integrable if and only if $n = -2$ or $0$.

Let us continue the calculation except for the case of $n = -2$ or $0$. Equation (3.7) implies

$$\partial^3\mathcal{M}_1 = \left\{\omega^1 = \widehat{\pi}_{11} = 0\right\},$$

where $\widehat{\pi}_{11} = \pi_{11}' - \frac{n-4}{2x}\varpi_1' = \pi_{11} + \frac{n(n-2)}{4x^2}\varpi_0 - \frac{n-4}{2x}\varpi_1 + \frac{n}{2x}\varpi_2$. Since we have

$$d\widehat{\pi}_{11} \equiv -\frac{n(n+4)(n-2)}{8x^3}\,\omega^2 \wedge \varpi_0 + \frac{(n+4)(n-2)}{4x^2}\,\omega^2 \wedge \varpi_1 \quad (\text{mod } \omega^1),$$

$\partial^3\mathcal{M}_1$ is completely integrable if and only if $n = -4$ or $2$. In the other cases, $\partial^4\mathcal{M}_1 = \left\{\omega^1 = 0\right\}$.

The case of $\mathcal{H}_2$ and $\mathcal{M}_2$ are as follows: $\partial\mathcal{H}_2 = \left\{\omega^2 = \pi_2' = 0\right\}$, where $\pi_2' = \pi_2 - \frac{n}{2x}\,\theta$, and

$$d\pi_2' = \frac{n(n+2)}{4x^2}\,\omega^1 \wedge \theta.$$

On the other hand, we can obtain

$$\partial^2\mathcal{M}_2 = \left\{\varpi_2' = \omega^2 = \pi_{22}' = 0\right\},$$

where $\varpi_2' = \varpi_2 - \frac{n}{2x}\,\varpi_0$, $\pi_{22}' = \pi_{22} + \frac{n}{2x}\,\varpi_1 + \frac{n}{2x^2}\,\varpi_0$, and

$$\begin{aligned} d\varpi_2' &\equiv \frac{n(n+2)}{4x^2}\,\omega^1 \wedge \varpi_0 \\ d\pi_{22}' &\equiv \frac{n(n+2)(n-4)}{8x^3}\,\omega^1 \wedge \varpi_0 \end{aligned} \qquad (\mathrm{mod}\ \varpi_2', \omega^2, \pi_{22}').$$

If $n \neq -2$ and $0$, we have

$$\partial^3\mathcal{M}_2 = \left\{\omega^2 = \overline{\pi}_{22} = 0\right\},$$

where $\overline{\pi}_{22} = \pi_{22}' - \frac{n-4}{2x}\varpi_2' = \pi_{22} + \frac{n(n-2)}{4x^2}\varpi_0 + \frac{n}{2x}\varpi_1 - \frac{n-4}{2x}\varpi_2$. Then

$$d\overline{\pi}_{22} \equiv -\frac{n(n+4)(n-2)}{8x^3}\,\omega^1 \wedge \varpi_0 + \frac{(n+4)(n-2)}{4x^2}\,\omega^1 \wedge \varpi_2 \quad (\mathrm{mod}\ \omega^2).$$

For $i = 1, 2$, we have obtained

Table 1. The Number of Independent First Integrals of Each Monge Characteristic System

| $n$ | the number of independent first integrals of $\mathcal{M}_i$ | the number of independent first integrals of $\mathcal{H}_i$ |
|---|---|---|
| $-2$, 0 | 3 | 2 |
| $-4$, 2 | 2 | 1 |
| the others | 1 | 1 |

### 3.2. Parabolic case

First, we choose a coframe adapted for a Monge-Ampère system: Let $\mathcal{I} = \{\theta, \Psi\}_{\mathrm{diff}}$ be a Monge-Ampère system and let $(R, D)$ denote the corresponding Monge-Ampère equation. Let us fix a point $v_o \in R$. Assuming that $\mathcal{I}$ is a parabolic system around $u_o = \pi(v_o)$, we can take a function $\lambda$ around $u_o$ such that $\Psi + \lambda d\theta$ is a decomposable 2-form. Hence we may suppose that $\Psi = \omega \wedge \pi$ is a decomposable 2-form. By definition, since the quadratic equation in a variable $\lambda$ given by

$$(\Psi + \lambda d\theta)^2 = 2\lambda \Psi \wedge d\theta + \lambda^2 d\theta \wedge d\theta = 0$$

has the multiple root $\lambda = 0$, we get

$$\Psi \wedge d\theta = \omega \wedge \pi \wedge d\theta = 0.$$

This implies

$$d\theta \equiv \omega^1 \wedge \pi + \omega \wedge \pi_2 \pmod{\theta},$$

where $\omega^1$ and $\pi_2$ are 1-forms around $u_o$. Because $\theta$ is a contact form, $\theta \wedge \omega^1 \wedge \pi \wedge \omega \wedge \pi_2 \neq 0$. Hence $\{\theta, \omega^1, \omega, \pi, \pi_2\}$ is a coframe around $u_o$. If $\omega|_{v_o}$ and $\pi|_{v_o}$ are simultaneously never zero, we may assume $\omega|_{v_o} \neq 0$. Since $d\theta|_{v_o} = 0$, it follows $\omega^1 \wedge \omega|_{v_o}$ must be non-zero.

Namely, we may suppose $\omega^1 \wedge \omega|_{v_o} \neq 0$ except for the case that both $\omega|_{v_o}$ and $\pi|_{v_o}$ vanish (see Remark 3.8 below).

Secondly, let us take a neighborhood $V$ of $v_o$ such that $\omega^1 \wedge \omega\,|_v \neq 0$ at each $v \in V$. Since $\Psi|_v = 0$ for any $v \in V$, we can take fiber coordinates $a, b, c$ on $V$ such that

$$\begin{aligned} \pi|_v &= a(v)\,\omega^1|_v + b(v)\,\omega|_v \\ \pi_2|_v &= b(v)\,\omega^1|_v + c(v)\,\omega|_v \end{aligned} \qquad v \in V.$$

Since $\omega \wedge \pi|_v = 0$, we get $a(v) = 0$. Thus

$$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\},$$

where $\varpi_0 = \rho^*\theta$, $\varpi_1 = \rho^*\pi - b\rho^*\omega$, $\varpi_2 = \rho^*\pi_2 - b\rho^*\omega^1 - c\rho^*\omega$ and let us omit the pullback $\rho^*$ in what follows.

Putting

$$\begin{aligned} d\pi &\equiv \pi\wedge(A\pi_2+B\omega^1+C\omega)+\pi_2\wedge(E\omega^1+F\omega)+G\omega^1\wedge\omega \\ d\omega &\equiv \pi\wedge(H\pi_2+I\omega^1+J\omega)+\pi_2\wedge(K\omega^1+L\omega)+N\omega^1\wedge\omega \end{aligned} \quad (\mathrm{mod}\ \theta),$$

where each capital letter indicates smooth functions on $J$, we have

$$d\pi - b d\omega \equiv -(Ab^2+Bb+Ec-Fb-G-Hb^3-Ib^2-Kbc+Lb^2+Nb)\omega^1\wedge\omega,$$

modulo $\varpi_0, \varpi_1, \varpi_2$. Hence we obtain the structure equation:

**Lemma 3.7**

$$\begin{cases} d\varpi_0 \equiv \omega^1\wedge\pi \quad + \omega\wedge\varpi_2 & (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 \equiv \qquad\qquad\quad \omega\wedge\pi_{12} & (\mathrm{mod}\ \varpi_0,\varpi_1,\varpi_2), \\ d\varpi_2 \equiv \omega^1\wedge\pi_{12}+\omega\wedge\pi_{22} & (\mathrm{mod}\ \varpi_0,\varpi_1,\varpi_2), \end{cases}$$

*where* $\pi_{12} = db + (Ab^2+Bb+Ec-Fb-G-Hb^3-Ib^2-Kbc+Lb^2+Nb)\,\omega^1$.

**Remark 3.8** If both $\omega|_{v_o}$ and $\pi|_{v_o}$ vanish, it must satisfy $\omega^1\wedge\pi_2|_{v_o}\neq 0$. We consider a neighborhood $V$ of $v_o$ such that $\omega^1\wedge\pi_2\,|_v\neq 0$ at each $v\in V$.

$$D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\},$$

where $\varpi_0 = \rho^*\theta$, $\varpi_1 = \rho^*\pi - a\rho^*\omega^1 - b\rho^*\pi_2$, $\varpi_2 = \rho^*\omega - b\rho^*\omega^1 - c\rho^*\pi_2$. Since $\omega\wedge\pi|_v = 0$ for all $v\in V$, $R\cap V = \{ac-b^2=0\}$ and hence $v_o$ is a singular point of $R\cap V$. Thus we omit a point $v_o$ such that both $\omega|_{v_o}$ and $\pi|_{v_o}$ vanish.

**Lemma 3.9**

$$\mathcal{M}\subset\rho_*^{-1}(\mathcal{H}).$$

*Proof.* As we use the coframe taken above,

$$\begin{aligned} \mathcal{M} &= \{\varpi_0=\varpi_1=\varpi_2=\omega=\pi_{12}\}, \\ \rho_*^{-1}(\mathcal{H}) &= \{\rho^*\theta=\rho^*\omega=\rho^*\pi=0\}, \\ &= \{\varpi_0=\varpi_1=\omega=0\}, \end{aligned}$$

and hence our assertion follows. □

**Corollary 3.10** *If $\mathcal{H}$ has two independent first integrals, then $\mathcal{M}$ also has at least two.*

In the same way as in the case of hyperbolic system, let us analyze the structure equation in more detail:

**Theorem 3.11** *Let $\mathcal{I}$ be a parabolic Monge-Ampère system on a 5-dimensional contact manifold $J$ and let $(R, D)$ denote the corresponding Monge-Ampère equation. Then it follows that*

$$\rho_*^{-1}(\mathcal{H}) = \partial(\mathcal{M} + \mathrm{Ch}(\partial D)) \tag{3.8}$$

*and the Monge characteristic system $\mathcal{H}$ of $\mathcal{I}$ is completely integrable if and only if the Monge characteristic $\mathcal{M}$ of $D$ is completely integrable.*

*Moreover, if $\mathcal{M}$ does not coincide with $\partial\mathcal{M}$, and $\partial\mathcal{M}$ is a differential system on $R$, then it follows that*

$$\partial^2\mathcal{M} = \rho_*^{-1}(\mathcal{H}).$$

*Proof.* Let us choose a coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega, \pi_{12}, \pi_{22}\}$ taken above. By definition,

$$\mathcal{H} = \{\theta = \omega = \pi = 0\}.$$

Since

$$\begin{cases} d\theta \equiv 0 \\ d\omega \equiv -E\omega^1 \wedge \pi_2 \qquad (\mathrm{mod}\ \theta, \omega, \pi), \\ d\pi \equiv -K\omega^1 \wedge \pi_2 \end{cases}$$

$\mathcal{H}$ is completely integrable if and only if $E$ and $K$ vanish locally.

On the other hand,

$$\mathcal{M} = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega = \pi_{12} = 0\}.$$

Since

$$dA \wedge \omega^1 \equiv dB \wedge \omega^1 \equiv dF \wedge \omega^1 \equiv dE \wedge \omega^1 \equiv dG \wedge \omega^1 \equiv dH \wedge \omega^1$$
$$\equiv dI \wedge \omega^1 \equiv dL \wedge \omega^1 \equiv dK \wedge \omega^1 \equiv dM \wedge \omega^1 \equiv db \wedge \omega^1 \equiv 0,$$

modulo $\varpi_0, \varpi_1, \varpi_2, \omega^1, \overline{\pi}_{11}$, we get

$$d\pi_{12} \equiv (Kb - E)\,\omega^1 \wedge \pi_{22} \quad (\text{mod } \varpi_0, \varpi_1, \varpi_2, \omega, \pi_{12}).$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d\varpi_2 \equiv d\omega \equiv 0 \ \ (\text{mod } \varpi_0, \varpi_1, \varpi_2, \omega, \pi_{12})$ and $b$ is one of the fiber coordinates, $\mathcal{M}$ is completely integrable if and only if $E$ and $K$ vanish locally. Hence second assertion follows.

Since $d\varpi_0 \equiv d\varpi_1 \equiv d\omega \equiv 0$ and $d\varpi_2 \equiv \omega^1 \wedge \pi_{12} \ \ (\text{mod } \varpi_0, \varpi_1, \varpi_2, \omega)$, first assertion follows.

Moreover, let us suppose that $\mathcal{M}$ does not coincide with $\partial\mathcal{M}$ and $\partial\mathcal{M}$ is a differential system on $R$. Then

$$\partial\mathcal{M} = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega = 0\}.$$

Since the structure equation of $\partial\mathcal{M}$ is

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv 0 \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} \\ d\omega \equiv 0 \end{cases} \quad (\text{mod } \varpi_0, \varpi_1, \varpi_2, \omega),$$

$$\partial^2\mathcal{M} = \{\varpi_0 = \varpi_1 = \omega = 0\}.$$

Consequently, we have obtained

$$\rho_*^{-1}(\mathcal{H}) = \{\rho^*\theta = \rho^*\omega = \rho^*\pi = 0\} = \partial^2\mathcal{M}. \qquad \square$$

## 4. Geometric Characterization of Monge-Ampère Equations

The results in the previous section guide us to consider the geometric characterization of Monge-Ampère equations. In fact, let $\mathcal{M}_i$ be Monge characteristic systems of a hyperbolic PDE $(R, D)$. The corank 3 differential systems $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ are candidate for Monge characteristic systems of a hyperbolic Monge-Ampère system. On the other hand, in parabolic case, the corank 3 differential system $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is candidate for Monge characteristic system of a parabolic Monge-Ampère system.

We use the notation in Section 2 through this section. In this paper we say that *a differential system $P$ on $L(J)$ drops down to $J$* if there exists a differential system $Q$ such that the pullback of $Q$ by the canonical projection $\rho : R \longrightarrow L(J)$ coincides with $P$.

### 4.1. Hyperbolic case

Let $(R, D)$ be a hyperbolic PDE and set $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic system of $(R, D)$.

First of all, let us describe the structure equation of $\mathcal{M}_i$ for each $i = 1, 2$. We recall the structure equation of $D$

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\mathrm{mod}\ \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \qquad\qquad \omega^2 \wedge \pi_{22} & (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

and Monge characteristic systems $\mathcal{M}_i = \left\{\varpi_0 = \varpi_1 = \varpi_3 = \omega^i = \pi_{ii} = 0\right\}$, $i = 1, 2$. Since $\mathcal{M}_1$ is of rank 2 and $d\varpi_0 \equiv d\varpi_1 \equiv 0$, $d\varpi_2 \equiv \omega^2 \wedge \pi_{22}$ $(\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1, \pi_{11})$, $\partial\mathcal{M}_1$ is of constant rank 3. Similarly, $\partial\mathcal{M}_2$ is so. We can write $d\omega^1 \equiv \omega^2 \wedge (h_1\pi_{11} + k_1\pi_{22})$ $(\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1)$, and $d\omega^2 \equiv \omega^1 \wedge (h_2\pi_{11} + k_2\pi_{22})$ $(\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2)$. Then since

$$\begin{aligned} 0 &= d^2\varpi_0 \\ &\equiv -\omega^1 \wedge (d\varpi_1 + \varpi_2 \wedge (h_2\pi_{11} + k_2\pi_{22})) \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2), \\ 0 &= d^2\varpi_0 \\ &\equiv -\omega^2 \wedge (d\varpi_2 + \varpi_1 \wedge (h_1\pi_{11} + k_1\pi_{22})) \quad (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1), \end{aligned}$$

we have

$$d\varpi_1 \equiv \omega^1 \wedge \pi_{11} - \varpi_2 \wedge (h_2\pi_{11} + k_2\pi_{22}) \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2),$$

$$d\varpi_2 \equiv \omega^2 \wedge \pi_{22} - \varpi_1 \wedge (h_1\pi_{11} + k_1\pi_{22}) \quad (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2 \wedge \varpi_1).$$

Furthermore, since

$$
\begin{aligned}
0 &= d^2\varpi_1 \\
&\equiv (-k_1 + h_2)\,\omega^2 \wedge \pi_{11} \wedge \pi_{22} \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1),
\end{aligned}
$$

we have $k_1 = h_2$. Replacing $\omega^1 - k_1\varpi_2$ and $\omega^2 - h_2\varpi_1$ with $\omega^1$ and $\omega^2$ respectively, we get

$$
\begin{aligned}
d\varpi_0 &\equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 && (\mathrm{mod}\ \varpi_0), \\
d\varpi_1 &\equiv \omega^1 \wedge \pi_{11} - k_2\,\varpi_2 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2), \\
d\varpi_2 &\equiv \omega^2 \wedge \pi_{22} - h_1\,\varpi_1 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2 \wedge \varpi_2), \\
d\omega^1 &\equiv h_1\,\omega^2 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1), \\
d\omega^2 &\equiv k_2\,\omega^1 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2).
\end{aligned}
$$

Then $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D) = \{\varpi_0 = \varpi_i = \omega^i = 0\}$ and we can write those structure equations as follows:

$$
\begin{cases}
d\varpi_0 \equiv \omega^2 \wedge \varpi_2 \\
d\varpi_1 \equiv -k_2\,\varpi_2 \wedge \pi_{22} \\
d\omega^1 \equiv h_1\,\omega^2 \wedge \pi_{11} + \varpi_2 \wedge (A_1\pi_{11} + B_1\pi_{22})
\end{cases}
\quad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1) \tag{4.1}
$$

$$
\begin{cases}
d\varpi_0 \equiv \omega^1 \wedge \varpi_1 \\
d\varpi_2 \equiv -h_1\,\varpi_1 \wedge \pi_{11} \\
d\omega^2 \equiv k_2\,\omega^1 \wedge \pi_{22} + \varpi_2 \wedge (A_2\pi_{11} + B_2\pi_{22})
\end{cases}
\quad (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^2) \tag{4.2}
$$

As it is seen in Corollary 3.4, for a hyperbolic Monge-Ampère system, pullbacks of Monge characteristic systems $\mathcal{H}_i$ coincide with $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$, where $\mathcal{M}_i$ is the corresponding Monge characteristic system of the corresponding Monge-Ampère equation $(R, D)$. Conversely, we obtain the next theorem:

**Theorem 4.1** *Let $(R, D)$ be a hyperbolic PDE and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of $(R, D)$. If $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ drops down to $J$, or equivalently, $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ drops down to $J$, then there exists a Monge-Ampère system $\mathcal{I}$ such that $(R, D)$ coincides with the corresponding Monge-Ampère equation locally. Moreover, $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ and $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ are then pullbacks of the Monge characteristic systems of the system*

$\mathcal{I}$.

First, we prove the following lemma:

**Lemma 4.2** *Let $(R, D)$ be a hyperbolic PDE and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of $(R, D)$. If $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ drops down to $J$ for each $i = 1, 2$, there exists a Monge-Ampère system $\mathcal{I}$ such that $(R, D)$ coincides with the corresponding Monge-Ampère equation locally. Moreover, $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ and $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ are then pullbacks of the Monge characteristic systems of the system $\mathcal{I}$.*

*Proof.* We show that, for each $v \in R$, there exists a Monge-Ampère system $\mathcal{I}$ around $\rho(v) \in J$ and a neighborhood $V$ of $v$ such that $(R, D)$ coincides with the prolongation of $\mathcal{I}$ on $V$.

Let us fix a point $v_o \in R$.

Since $\partial\mathcal{M}_i + Ch(\partial D)$ drops down to $J$ for each $i = 1, 2$, there exists 1-forms $\widehat{\pi}_1$, $\widehat{\omega}^1$, $\widehat{\pi}_2$, $\widehat{\omega}^2$ around $u_o = \rho(v_o)$ such that $\widehat{\omega}^1 \wedge \widehat{\omega}^2|_{v_o} \neq 0$ and

$$\partial\mathcal{M}_i + \mathrm{Ch}(\partial D) = \left\{\rho^*\theta = \rho^*\widehat{\pi}_i = \rho^*\widehat{\omega}^i = 0\right\}, \quad \text{for each } i = 1, 2.$$

Let $V$ be a neighborhood of $v_o$ such that $\widehat{\omega}^1 \wedge \widehat{\omega}^2|_v \neq 0$ for each $v \in V$.

Let denote $A_1, A_2$ and $B$ non-zero functions on $V$ such that for $i = 1, 2$, $\omega^i \wedge \varpi_i \equiv A_i \rho^*\widehat{\omega}^i \wedge \rho^*\widehat{\pi}_i \pmod{\varpi_0}$ and $\varpi_0 = B\rho^*\theta$. Then we get

$$\rho^* d\theta \equiv \frac{A_1}{B}\rho^*\widehat{\omega}^1 \wedge \rho^*\widehat{\pi}_1 + \frac{A_2}{B}\rho^*\widehat{\omega}^2 \wedge \rho^*\widehat{\pi}_2 \pmod{\rho^*\theta}.$$

This implies that there exists functions $\widehat{K}_1, \widehat{K}_2$ around $u_o$ such that $\rho^*\widehat{K}_i = \frac{A_i}{B}$ for $i = 1, 2$, and hence we have

$$d\theta \equiv (\widehat{K}_1\widehat{\omega}^1) \wedge \widehat{\pi}_1 + (\widehat{K}_2\widehat{\omega}^2) \wedge \widehat{\pi}_2 \pmod{\theta}.$$

Now let us consider the following hyperbolic Monge-Ampère system

$$\mathcal{I} = \left\{\theta,\, \widehat{\omega}^1 \wedge \widehat{\pi}_1\right\}_{\mathrm{diff}} = \left\{\theta,\, \widehat{\omega}^2 \wedge \widehat{\pi}_2\right\}_{\mathrm{diff}}$$

and its Monge characteristic systems are $\mathcal{H}_i = \left\{\theta = \widehat{\pi}_i = \widehat{\omega}^i = 0\right\}$. From the definition of $\mathcal{I}$, each point of $V$ is an integral element of $\mathcal{I}$. That is, $(R, D)$ coincides with the corresponding Monge-Ampère equation locally. □

Next, we show that, for a hyperbolic PDE $(R, D)$, $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$ if and only if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$:

**Lemma 4.3** *Let $(R, D)$ be a hyperbolic PDE and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of $(R, D)$. Then, $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$ if and only if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$.*

*Proof.* From the structure equation (4.1), if $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, we have $h_1 = k_2 = 0$. From the structure equation (4.2), if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, we have $h_1 = k_2 = 0$.

As we assume $h_1 = k_2 = 0$, Equation (4.1) is

$$\begin{aligned}
d\varpi_1 &\equiv \omega^1 \wedge \pi_{11} && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2),\\
d\omega^1 &\equiv \varpi_2 \wedge (A_1\pi_{11} + B_1\pi_{22}) && (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^1),\\
d\varpi_2 &\equiv \omega^2 \wedge \pi_{22} && (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2 \wedge \varpi_2),\\
d\omega^2 &\equiv \varpi_1 \wedge (A_2\pi_{11} + B_2\pi_{22}) && (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^2).
\end{aligned}$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d(\omega^1 \wedge \varpi_2) \equiv d(\omega^2 \wedge \varpi_2) \equiv 0 \pmod{\varpi_0, \varpi_1, \omega^2 \wedge \varpi_2, \omega^1}$,

$$\begin{aligned}
0 &= d^2\varpi_1\\
&\equiv -B_1\varpi_2 \wedge \pi_{11} \wedge \pi_{22} \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2 \wedge \varpi_2, \omega^1).
\end{aligned}$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d\omega^1 \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^1}$,

$$\begin{aligned}
0 &= d^2\omega^1\\
&\equiv -A_1\omega^2 \wedge \pi_{11} \wedge \pi_{22} \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^1).
\end{aligned}$$

Therefore, we have $A_1 = B_1 = 0$.

On the other hand, since $d\varpi_0 \equiv d\varpi_2 \equiv d(\omega^1 \wedge \varpi_1) \equiv d(\omega^2 \wedge \varpi_1) \equiv 0 \pmod{\varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2}$,

$$\begin{aligned}
0 &= d^2\omega^2\\
&\equiv A_2\varpi_1 \wedge \pi_{11} \wedge \pi_{22} \quad (\mathrm{mod}\ \varpi_0, \varpi_2, \omega^1 \wedge \varpi_1, \omega^2).
\end{aligned}$$

Since $d\varpi_0 \equiv d\varpi_2 \equiv d\omega^2 \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^2}$,

$$\begin{aligned} 0 &= d^2\omega^2 \\ &\equiv B_2\omega^1 \wedge \pi_{11} \wedge \pi_{22} \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^2}. \end{aligned}$$

Therefore, we have $A_2 = B_2 = 0$.

Consequently, our assertion follows. □

*Proof of Theorem 4.1.* From Equation (4.1), if $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ drops down to $J = R/\mathrm{Ch}(\partial D)$, $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ must coincides with $\mathrm{Ch}(\partial D)$. Similarly, from Equation 4.2, if $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ drops down to $J$, $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ must coincides with $\mathrm{Ch}(\partial D)$. Conversely, if $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$, or equivalently, if $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)) = \mathrm{Ch}(\partial D)$, then $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ drops down to $J$ for each $i = 1, 2$. Thus $\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D)$ drops down to $J$ if and only if $\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D)$ drops down to $J$. Consequently, our assertion follows from Theorem 4.2 and this argument. □

**Remark 4.4** $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ and $h_1$, $k_2$ in the above proof are corresponding to the $M_i$-characteristic vector field systems $\mathrm{Char}(I_F, dM_i)$ and Monge-Ampère invariants introduced in [GK93]. They characterize Monge-Ampère equation by the invariants. On the other hand, we characterize Monge-Ampère equation by the property that $\partial\mathcal{M}_i + \mathrm{Ch}(\partial D)$ should satisfy and find that these differential systems coincides with pullbacks of the Monge characteristic systems of the corresponding Monge-Ampère system if $(R, D)$ is a hyperbolic Monge-Ampère system.

From Lemma 4.3, we can translate Theorem 4.1 into the following corollary:

**Corollary 4.5** *Let $(R, D)$ be a hyperbolic PDE and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the Monge characteristic systems of $(R, D)$. If $\mathrm{Ch}(\partial\mathcal{M}_1 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, or equivalently, $\mathrm{Ch}(\partial\mathcal{M}_2 + \mathrm{Ch}(\partial D))$ coincides with $\mathrm{Ch}(\partial D)$, there exists a Monge-Ampère system $\mathcal{I}$ such that $(R, D)$ coincides with the corresponding Monge-Ampère equation locally.*

### 4.2. Parabolic case

In parabolic case, we obtain similar results to hyperbolic case. Unlike hyperbolic case, as it is seen in Theorem 3.11, the regularity of the first derived system of the Monge characteristic system of a parabolic PDE does not follow from the regularity of the Monge characteristic system. In order to obtain a similar result to Theorem 4.1, we need a further assumption of the regularity (see Theorem 4.7). The result for *Goursat equation*, i.e. a parabolic equation whose Monge characteristic system is completely integrable (hence the assumption is naturally satisfied) is particularly important.

First of all, let us describe the structure equation of $\mathcal{M}$. We recall the structure equation of $D$

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \qquad\qquad\quad \omega^2 \wedge \pi_{12} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} + \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

and the Monge characteristic system $\mathcal{M} = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{12} = 0\}$. As we write $d\omega^2 \equiv \omega^1 \wedge (h\pi_{12} + k\pi_{22}) \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^2}$,

$$\begin{aligned} 0 &= d^2\varpi_0 \\ &\equiv -\omega^1 \wedge (d\varpi_1 + \varpi_2 \wedge (h\pi_{12} + k\pi_{22})) \pmod{\varpi_0, \varpi_1, \omega^2}. \end{aligned}$$

Thus we have

$$d\varpi_1 \equiv \omega^2 \wedge \pi_{12} - \varpi_2 \wedge (h\pi_{12} + k\pi_{22}) \pmod{\varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2}.$$

Since $d\varpi_0 \equiv d\varpi_1 \equiv d(\omega^1 \wedge \varpi_2) \equiv d(\omega^2 \wedge \varpi_2) \equiv 0 \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^1 \wedge \omega^2, \omega^2 \wedge \pi_{12}}$, we have

$$\begin{aligned} 0 &= d^2\varpi_1 \\ &\equiv -2k\omega^1 \wedge \pi_{12} \wedge \pi_{22} - \omega^2 \wedge d\pi_{12} \pmod{\varpi_0, \varpi_1, \varpi_2, \omega^1 \wedge \omega^2, \omega^2 \wedge \pi_{12}} \end{aligned}$$

and hence $k = 0$. Thus replacing $\omega^2 - h\varpi_2$ with $\omega^2$, we get

$$\begin{aligned} d\varpi_0 &\equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 && (\text{mod } \varpi_0), \\ d\varpi_1 &\equiv \omega^2 \wedge \pi_{12} && (\text{mod } \varpi_0, \varpi_1, \omega^1 \wedge \varpi_2, \omega^2 \wedge \varpi_2), \\ d\omega^2 &\equiv 0 && (\text{mod } \varpi_0, \varpi_1, \varpi_2, \omega^2). \end{aligned}$$

Then $\partial(\mathcal{M}+\mathrm{Ch}(\partial D)) = \left\{\varpi_0 = \varpi_1 = \omega^2 = 0\right\}$ and we can write its structure equation as follows:

$$\begin{cases} d\varpi_0 \equiv 0 \\ d\varpi_1 \equiv E\omega^1 \wedge \varpi_2 \\ d\omega^2 \equiv \varpi_2 \wedge (A\pi_{12} + B\pi_{22} + C\omega^1) \end{cases} \qquad (\mathrm{mod}\ \varpi_0, \varpi_1, \omega^2) \tag{4.3}$$

Furthermore, since

$$\begin{aligned} 0 &= d^2\varpi_1 \\ &\equiv -\omega^2 \wedge (d\pi_{12} - E\,\omega^1 \wedge \pi_{22}) \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2 \wedge \pi_{12}), \end{aligned}$$

we have

$$d\pi_{12} \equiv E\omega^1 \wedge \pi_{22} \quad (\mathrm{mod}\ \varpi_0, \varpi_1, \varpi_2, \omega^2, \pi_{12}). \tag{4.4}$$

Therefore the regularity of $\partial\mathcal{M}$ correspond to the regularity of the function $E$.

As it is seen in Theorem 4.7, for a parabolic Monge-Ampère system, the pullback of the Monge characteristic system $\mathcal{H}$ coincides with $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$, where $\mathcal{M}$ is the corresponding Monge characteristic system of the corresponding Monge-Ampère equation $(R, D)$. Conversely, we obtain the next theorem:

**Theorem 4.6** *Let $(R, D)$ be a parabolic PDE. Let $\mathcal{M}$ denote the Monge characteristic system of $(R, D)$. If $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ drops down to $J$, there exists a Monge-Ampère system $\mathcal{I}$ such that $(R, D)$ coincides with the corresponding Monge-Ampère equation locally. Moreover, $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is then the pullback of the Monge characteristic system of the system $\mathcal{I}$.*

*Proof.* We show that, for each $v \in R$, there exists a Monge-Ampère system $\mathcal{I}$ around $\rho(v)$ and a neighborhood $V$ of $v$ such that $(R, D)$ coincides with the prolongation of $\mathcal{I}$ on $V$.

Let us fix a point $v_o \in R$. Since $\partial(\mathcal{M}+\mathrm{Ch}(\partial D))$ drops down to $J$, there exists 1-forms $\widehat{\pi}$, $\widehat{\omega}$ around $u_o = \rho(v_o)$ such that $\widehat{\omega}|_{v_o} \neq 0$ and

$$\partial(\mathcal{M} + \mathrm{Ch}(\partial D)) = \{\rho^*\theta = \rho^*\widehat{\pi} = \rho^*\widehat{\omega} = 0\}\,.$$

Let denote $A$ and $B$ non-zero functions around $v_o$ such that $\omega^2 \wedge \varpi_1 \equiv A\rho^*\widehat{\omega} \wedge \rho^*\widehat{\pi} \pmod{\varpi_0}$ and $\varpi_0 = B\rho^*\theta$, then we have $d\varpi_0 \wedge \omega^2 \wedge \varpi_1 \equiv AB\rho^*(d\theta \wedge \widehat{\omega} \wedge \widehat{\pi}) \pmod{\varpi_0}$. Since $d\varpi_0 \wedge \omega^2 \wedge \varpi_1 \equiv 0 \pmod{\varpi_0}$, we have $d\theta \wedge \widehat{\omega} \wedge \widehat{\pi} \equiv 0 \pmod{\theta}$. Therefore, there exists 1-forms $\widehat{\omega}^1$, $\widehat{\pi}_2$ around $u_o$ such that

$$d\theta \equiv \widehat{\omega}^1 \wedge \widehat{\pi} + \widehat{\omega} \wedge \widehat{\pi}_2 \pmod{\theta}.$$

Since $\theta$ is a contact form, $\theta \wedge \widehat{\omega}^1 \wedge \widehat{\omega} \wedge \widehat{\pi} \wedge \widehat{\pi}_2 \neq 0$, which means $\theta$, $\widehat{\omega}^1$, $\widehat{\omega}$, $\widehat{\pi}$, $\widehat{\pi}_2$ are linearly independent. We may assume that $\widehat{\omega}^1 \wedge \widehat{\omega}|_{v_o} \neq 0$.

Let $V$ be a neighborhood of $v_o$ such that $\widehat{\omega}^1 \wedge \widehat{\omega}|_v \neq 0$ for each $v \in V$.

Now let us consider the following parabolic Monge-Ampère system

$$\mathcal{I} = \{\theta, \widehat{\omega} \wedge \widehat{\pi}\}_{\text{diff}}$$

and its Monge characteristic system is $\mathcal{H} = \{\theta = \widehat{\pi} = \widehat{\omega} = 0\}$. From the definition of $\mathcal{I}$, each point of $V$ is an integral element of $\mathcal{I}$. That is, $(R, D)$ coincides with the corresponding Monge-Ampère equation locally. □

**Theorem 4.7** *Let $(R, D)$ be a parabolic PDE. Let $\mathcal{M}$ denote the Monge characteristic system of $(R, D)$ and assume the first derived system $\partial\mathcal{M}$ of $\mathcal{M}$ is also a differential system. If $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$, there exists a Monge-Ampère system $\mathcal{I}$ such that $(R, D)$ coincides with the corresponding Monge-Ampère equation locally.*

*Proof.* It is sufficient to show that $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ drops down to $J$ if $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$.

From Equation (4.4) and the assumption of the regularity of $\partial\mathcal{M}$, $E$ uniformly vanishes or is not zero at each point of $R$. In the former case, $\mathcal{M}$ is completely integrable, that is, $R$ is a Goursat equation. In the latter case, $\partial\mathcal{M}$ is of constant rank 3.

As $E$ uniformly vanishes, from $d\varpi_1 \equiv \omega^2 \wedge \pi_{12}$, modulo $\varpi_0, \varpi_1, \omega^2 \wedge \varpi_2$,0

$$\begin{aligned} 0 &= d^2\varpi_1 \\ &\equiv \varpi_2 \wedge (-B\pi_{12} \wedge \pi_{22} + C\omega^1 \wedge \pi_{12}) \pmod{\varpi_0, \varpi_1, \omega^2}. \end{aligned}$$

Therefore $B$ and $C$ vanish on $R$. Additionally, because the structure equation (4.3) is satisfied and $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$, $A$ is zero at each point of $R$. Consequently, $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is completely integrable and hence drops down to $J$.

As $E$ is not zero at each point of $R$, because the structure equation (4.3) is satisfied and $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D)))(v)$ contains $\mathrm{Ch}(\partial D)(v)$ at each point $v \in R$, we get $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D))) = \mathrm{Ch}(\partial D)$. Consequently, $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ drops down to $J$. □

Particularly we note that

**Corollary 4.8** *Let $(R, D)$ be a Goursat equation and $\mathcal{M}$ the Monge characteristic system of $(R, D)$. That is, $\mathcal{M}$ is completely integrable. Then, $(R, D)$ is a Monge-Ampère equation if and only if $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is completely integrable.*

*Proof.* If a Goursat equation $(R, D)$ is a Monge-Ampère equation, from Theorem 3.11, the Monge characteristic system $\mathcal{H}$ of the corresponding Monge-Ampère system $\mathcal{I}$ is completely integrable, and equivalently $\rho_*^{-1}(\mathcal{H}) = \partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is completely integrable. Conversely, if $\partial(\mathcal{M} + \mathrm{Ch}(\partial D))$ is completely integrable, $\mathrm{Ch}(\partial(\mathcal{M} + \mathrm{Ch}(\partial D))) = \partial(\mathcal{M} + \mathrm{Ch}(\partial D)) = \{\varpi_0 = \varpi_1 = \omega^2 = 0\}$ contains $\mathrm{Ch}(\partial D)$. Hence, from Theorem 4.7, $(R, D)$ is a Monge-Ampère equation. □

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