Hokkaido Mathematical Journal Vol. 27 (1998) p. 505–515

Higher Specht polynomials for the complex reflection group $G(r, p, n)$

(To Professor Takeshi Hirai on his sixtieth birthday)

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(Received December 16, 1996; Revised December 12, 1997)

Abstract. A basis of the quotient ring P/J_{+} is given, where P is the ring of polynomials and J_{+} is the ideal generated by the fundamental invariants under the action of the complex reflection group $G(r, p, n)$.

Key words: complex reflection groups, coinvariant rings, Clifford theory, tableaux.

1. Introduction

This note is concerned with a certain graded module over the imprimitive complex reflection group $G(r, p, n)$ [ST]. The group $G(r, p, n)$ $(r, p, n \geq 1,\ p|r)$ consists of the monomial matrices whose nonzero entries are of the form ζ^{j} $(0 \leq j < r)$ and such that the d-th power of the product of all nonzero entries is equal to 1, where we denote by ζ a primitive r-th root of 1, and $d = r/p$. In some special cases, $G(r, p, n)$ is isomorphic to the Weyl group:

$$
\begin{aligned}
G(1,1,n) &= W(A_{n-1}),\\
G(2,1,n) &= W(B_{n}) = W(C_{n}),\\
G(2,2,n) &= W(D_{n}),\\
G(6,6,2) &= W(G_{2}).
\end{aligned}
$$

Also it is naturally identified as a normal subgroup of the wreath product

$$G(r,n) = (\mathbf{Z}/r\mathbf{Z}) \wr S_{n} = \{(\zeta^{i_1}, \ldots, \zeta^{i_n}; \sigma) \mid i_k \in \mathbf{N},\ \sigma \in S_n\},$$

whose product is given by

$$(\zeta^{i_1}, \ldots, \zeta^{i_n}; \sigma)(\zeta^{j_1}, \ldots, \zeta^{j_n}; \tau) = (\zeta^{i_1 + j_{\sigma^{-1}(1)}}, \ldots, \zeta^{i_n + j_{\sigma^{-1}(n)}}; \sigma\tau).$$

1991 Mathematics Subject Classification : 05E10, 20C05.

Let $P = \mathbf{C}[x_1, \dots, x_n]$ be the polynomial ring of n indeterminates, on which the group $G(r,n)$ acts as follows:

$$((\zeta^{i_1}, \dots, \zeta^{i_n}; \sigma)f)(x_1, \dots, x_n) = f(\zeta^{i_{\sigma(1)}} x_{\sigma(1)}, \dots, \zeta^{i_{\sigma(n)}} x_{\sigma(n)}).$$

It is known that the fundamental invariants under this action are given by the elementary symmetric functions $e_j(x_1^r, \dots, x_n^r)$, $1 \leq j \leq n$. Let J'_+ be the ideal of P generated by these fundamental invariants and $R' = P/J'_+$ be the quotient ring, which is sometimes called the coinvariant algebra. It is also known that the $G(r,n)$-module R' is isomorphic to the group ring $\mathbf{C}G(r,n)$, which affords the left regular representation. A description of all the irreducible components of R' has been known in [ATY], in terms of what we call *higher Specht polynomials.* (See also [TY] for the case $r = 1$.) The irreducible representations of $G(r,n)$ are parameterized by the r-tuples of Young diagrams $(\lambda^0, \dots, \lambda^{r-1})$ with $|\lambda^0| + \cdots + |\lambda^{r-1}| = n$. In [ATY] (and [TY]) combinatorics of Young diagrams is used to determine a basis for each irreducible component of R'.

Now we consider the restriction of the above action of $G(r,n)$ on P to the subgroup $G(r,p,n)$. The fundamental invariants are $e_j(x_1^r, \dots, x_n^r)$ $(1 \leq j \leq n-1)$ and $e_n(x_1^d, \dots, x_n^d)$. Denote by J_+ the ideal generated by these polynomials and let $R = P/J_+$. The representation of $G(r,p,n)$ on R is again isomorphic to the left regular representation. Our problem is to describe the irreducible components of R as well as their bases. The key to our description is the Clifford theory [S] for a finite group G and its normal subgroup H.

2. Higher Specht polynomials for $G(r,n)$

Here we recall the results of [ATY] on an irreducible decomposition of the graded $G(r,n)$-module

$$R' = P/J'_+,$$

where $P = \mathbf{C}[x_1, \dots, x_n]$ and $J'_+ = (e_1(x_1^r, \dots, x_n^r), \dots, e_n(x_1^r, \dots, x_n^r))$. As is well-known the irreducible representations of $G(r,n)$ are parameterized by the set $\mathcal{P}_{r,n}$ of the r-tuples of Young diagrams $\lambda = (\lambda^0, \dots, \lambda^{r-1})$ with $|\lambda^0| + \cdots + |\lambda^{r-1}| = n$. By filling each cell with a positive integer in such a way that every k $(1 \leq k \leq n)$ occurs once, we obtain an *r-tableau* $T = (T^0, \dots, T^{r-1})$ of shape $\lambda = (\lambda^0, \dots, \lambda^{r-1})$. If the number k occurs in the

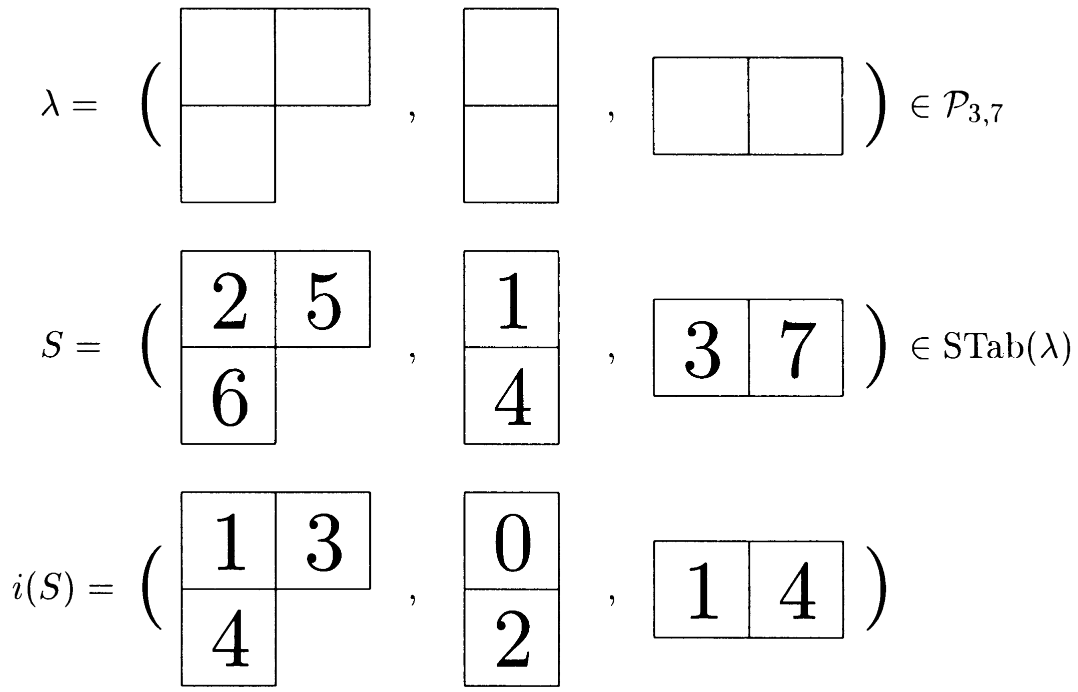


Fig. 1.

component T^ν, we may write $k \in T^\nu$. The set of the r-tableaux of shape λ is denoted by $\mathrm{Tab}(\lambda)$. An r-tableau $T = (T^0, \ldots, T^{r-1})$ is said to be *standard* if the numbers are increasing along each column and each row of T^ν $(0 \leq \nu < r)$. The set of the standard r-tableaux of shape λ is denoted by $\mathrm{STab}(\lambda)$.

Let $S = (S^0, \ldots, S^{r-1}) \in \mathrm{STab}(\lambda)$. We associate a word $w(S)$ in the following way. First we read each column of the component S^0 from the bottom to the top starting from the left. We continue this procedure for the components S^1 and so on. For the word $w(S)$ we define the index $i(w(S))$ inductively as follows. The number 1 in the word $w(S)$ has index $i(1) = 0$. If the number k has index $i(k) = p$ and the number $k+1$ is sitting to the left (resp. right) of k, then $k+1$ has index $p+1$ (resp. p). Finally, assigning the indices to the corresponding cells, we get a shape $\lambda = (\lambda^0, \ldots, \lambda^{r-1})$, with each cell filled with a nonnegative integer, which is denoted by $i(S) = (i(S)^0, \ldots, i(S)^{r-1})$. An example of standard 3-tableaux and the indices is given in Figure 1.

Let $T = (T^0, \ldots, T^{r-1})$ be an r-tableau of shape λ. For each component T^ν $(0 \leq \nu < r)$, the Young symmetrizer e_{T^ν} of T^ν is defined by

$$e_{T^\nu} = \frac{1}{\alpha_{T^\nu}} \sum_{\sigma \in R(T^\nu), \tau \in C(T^\nu)} \mathrm{sgn}(\tau)\tau\sigma,$$

where $R(T^\nu)$ and $C(T^\nu)$ are the row stabilizer and the column stabilizer of T^ν, respectively, and α_{T^ν} is the product of hook lengths for the shape λ^ν. To state the definition of higher Specht polynomials, we regard a tableau T on a Young diagram λ as a map

$$T : \{\text{cells of } \lambda\} \longrightarrow \mathbf{Z}_{\geq 0},$$

which assigns to a cell ξ of λ the number $T(\xi)$ written in the cell ξ in T. For $S \in \mathrm{STab}(\lambda)$ and $T \in \mathrm{Tab}(\lambda)$, define the higher Specht polynomial $\Delta_{S,T}(x)$ by

$$\Delta_{S,T}(x) = \prod_{\nu=0}^{r-1} \left\{ e_{T^\nu}(x_{T^\nu}^{r\,i(S)^\nu}) \prod_{k \in T^\nu} x_k^\nu \right\},$$

where

$$x_{T^\nu}^{r\,i(S)^\nu} = \prod_{\xi \in \lambda^\nu} x_{T^\nu(\xi)}^{r\,i(S)^\nu(\xi)}.$$

The following is a fundamental result of [ATY] on the higher Specht polynomials for $G(r,n)$.

Theorem 1

1. *The subspace $V_S(\lambda) = \sum_{T \in \mathrm{Tab}(\lambda)} \mathbf{C}\Delta_{S,T}(x)$ of P affords an irreducible representation of the complex reflection group $G(r,n)$.*
2. *The set $\{\Delta_{S,T}(x) \mid T \in \mathrm{STab}(\lambda)\}$ gives a basis for $V_S(\lambda)$.*
3. *For $S_1 \in \mathrm{STab}(\lambda)$ and $S_2 \in \mathrm{STab}(\mu)$, the representations afforded by $V_{S_1}(\lambda)$ and $V_{S_2}(\mu)$ are isomorphic if and only if S_1 and S_2 have the same shape,* i.e., *$\lambda = \mu$. The isomorphism is given by*

$$\Delta_{S_1,T}(x) \mapsto \Delta_{S_2,T}(x) \ (T \in \mathrm{STab}(\lambda)).$$

4. *The coinvariant algebra $R' = P/J'_+$ admits an irreducible decomposition*

$$R' = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \bigoplus_{S \in \mathrm{STab}(\lambda)} (V_S(\lambda) \bmod J'_+)$$

as a $G(r,n)$-module.

3. Review of the Clifford theory

We briefly review the Clifford theory following [S, pp. 380–381]. Let H be a normal subgroup of a finite group G such that the quotient group G/H is cyclic. We have in mind the case $G = G(r,n)$ and $H = G(r,p,n)$. Let C denote the group of 1-dimensional representations, or characters, $(\delta, \mathbf{C}_\delta)$ of G such that $H \subset \operatorname{Ker}\delta$. In other words, C is the group of the characters of G/H, which is isomorphic to G/H. Two irreducible representations (ϕ, V) and (ψ, W) of G are said to be *associates* if there exists $\delta \in C$ such that $\psi = \delta \otimes \phi$. For a fixed irreducible representation (ϕ, V) of G, let

$$C_\phi = \{\delta \in C \mid \phi \cong \delta \otimes \phi\}$$

be the stabilizer of ϕ and let $(\delta, \mathbf{C}_\delta)$ be a generator of C_ϕ. There exists a G-module isomorphism $V \longrightarrow \mathbf{C}_\delta \otimes V$. Composing this with the H-module isomorphism $\mathbf{C}_\delta \otimes V \longrightarrow V$, $1_\delta \otimes v \mapsto v$ (where 1_δ is a fixed basis element of $\mathbf{C}_\delta$), we obtain an H-module isomorphism $A : V \longrightarrow V$ satisfying $A(\phi(g)v) = \delta(g)\phi(g)A(v)$ for all $g \in G$ and $v \in V$. If $|C_\phi| = e$, then A^e commutes with G and, by Schur's lemma, A^e is a nonzero scalar. By normalizing the constant, we assume that $A^e = \mathbf{1}_V$ and call such A the *associator* of (ϕ, V). Choose an associator A for (ϕ, V) and let

$$V = \bigoplus_{\ell=0}^{e-1} E^{(\ell)}$$

denote the eigenspace decomposition of V with respect to A, where $E^{(\ell)}$ is the eigenspace with eigenvalue $e^{\frac{2\pi i\ell}{e}}$. Since $H \subset \operatorname{Ker}\delta$, each $E^{(\ell)}$ is an H-module. Moreover the $E^{(\ell)}$'s are inequivalent irreducible H-modules of the same dimension $(\dim V)/e$. The Frobenius reciprocity tells us that $\operatorname{Ind}_H^G E^{(\ell)}$ is the multiplicity free direct sum of all the associates of (ϕ, V). From these results, we can conclude that the irreducible representations of H are parameterized by the pairs $(\mathcal{O}, \varepsilon)$ consisting of a C-orbit $\mathcal{O}$ through an irreducible representation of G and a character $\varepsilon \in C$ that stabilizes $\mathcal{O}$.

4. Higher Specht polynomials for $G(r,p,n)$

We now apply the Clifford theory to the case $G = G(r,n)$ and $H = G(r,p,n)$. Define the linear character δ of $G(r,n)$ by $\delta(\zeta^{i_1}, \ldots, \zeta^{i_n}; \sigma) = \zeta^{i_1+\cdots+i_n}$ so that our cyclic group is $C = \langle \delta^d \rangle \cong \mathbf{Z}/p\mathbf{Z}$. Define the *shift* *operator* sh on $\mathcal{P}_{r,n}$ (resp. on $\mathrm{Tab}(\lambda)$) by

$$\mathrm{sh}(\lambda^0,\ldots,\lambda^{r-1}) = (\lambda^{r-1},\lambda^0,\ldots,\lambda^{r-2})$$
$$(\text{resp. } \mathrm{sh}(T^0,\ldots,T^{r-1}) = (T^{r-1},T^0,\ldots,T^{r-2})).$$

By the realization of the irreducible representations of $G(r,n)$ described in Section 2, one sees that

$$\mathbf{C}_\delta \otimes V_S(\lambda) \xrightarrow{\sim} V_{\mathrm{sh}(S)}(\mathrm{sh}(\lambda)) : 1_\delta \otimes \Delta_{S,T}(x) \mapsto \Delta_{\mathrm{sh}(S),\mathrm{sh}(T)}(x),$$

is a G-module isomorphism for any $S \in \mathrm{STab}(\lambda)$, $\lambda \in \mathcal{P}_{r,n}$. Hence the C-orbits are characterized by $\mathcal{P}_{r,n}/\sim$, where we denote $\lambda \sim \mu$ if $\mu = \mathrm{sh}^{dj}\lambda$ for some $j = 0,1,\ldots,p-1$. For convenience we will denote $\mathrm{Sh} = \mathrm{sh}^d$. For $\lambda \in \mathcal{P}_{r,n}$, let $b(\lambda)$ be the minimal j such that $\mathrm{Sh}^j\lambda = \lambda$, i.e., $b(\lambda) = |\{\mu \in \mathcal{P}_{r,n} \mid \lambda \sim \mu\}|$ and put $e(\lambda) = p/b(\lambda)$. The stabilizer C_λ of λ is a subgroup of C generated by $\delta^{b(\lambda)d}$, so that $|C_\lambda| = e(\lambda)$ and $|C/C_\lambda| = b(\lambda)$. The corresponding associator is denoted by A_λ. In other words, the associator A_λ is realized on $V_S(\lambda)$ by

$$A_\lambda\left(\Delta_{S,T}(x)\right) = \Delta_{S,\mathrm{Sh}^{-b(\lambda)}(T)}(x) \qquad (T \in \mathrm{Tab}(\lambda)).$$

For $h = 1,2,\ldots,r$, let

$$\mathrm{STab}(\lambda)_h = \{T = (T^0,\ldots,T^{r-1}) \in \mathrm{STab}(\lambda) \mid 1 \in T^\nu,\ 0 \le \nu < h\}.$$

Note that, if $T \in \mathrm{STab}(\lambda)_{db(\lambda)}$, then the standard r-tableaux

$$T, \mathrm{Sh}^{b(\lambda)}(T), \mathrm{Sh}^{2b(\lambda)}(T), \ldots, \mathrm{Sh}^{(e(\lambda)-1)b(\lambda)}(T)$$

are all distinct. Let $\lambda = (\lambda^0,\ldots,\lambda^{r-1})$ be an element of $\mathcal{P}_{r,n}$. Fix $S \in \mathrm{STab}(\lambda)$ and $\ell = 0,1,\ldots,e(\lambda)-1$. For each $T \in \mathrm{STab}(\lambda)$, we define a polynomial

$$\Delta^{(\ell)}_{S,T}(x) := \sum_{m=0}^{e(\lambda)-1} \zeta^{\ell m d b(\lambda)} \Delta_{S,\mathrm{Sh}^{mb(\lambda)}(T)}(x),$$

as an element of $R' = P/J'_+$. Since $\Delta^{(\ell)}_{S,T_1}(x)$ coincides with $\Delta^{(\ell)}_{S,T_2}(x)$ up to constant if T_1 and T_2 are in the same $\langle \mathrm{Sh}^{b(\lambda)}\rangle$-orbit in $\mathrm{STab}(\lambda)$, we only have to consider the polynomials associated with $T \in \mathrm{STab}(\lambda)_{db(\lambda)}$.

Let $\mathcal{D}_S(T)$ ($S,T \in \mathrm{STab}(\lambda)$) denote the set $\{\Delta_{S,\mathrm{Sh}^{mb(\lambda)}T}(x) \mid m = 0,\ldots,e(\lambda)-1\}$. Then, for each $S \in \mathrm{STab}(\lambda)$, we have a partition of the polynomials $\Delta_{S,T}(x)$, $T \in \mathrm{Stab}(\lambda)$ as follows:

$$\{\Delta_{S,T}(x) \mid T \in \mathrm{STab}(\lambda)\} = \coprod_{T \in \mathrm{STab}(\lambda)_{db(\lambda)}} \mathcal{D}_S(T).$$

Since $\{\Delta_{S,T}(x) \mid T \in \mathrm{STab}(\lambda)\}$ is linearly independent over $\mathbf{C}$, the polynomials $\{\Delta^{(\ell)}_{S,T}(x) \mid T \in \mathrm{STab}(\lambda)_{db(\lambda)}\}$ is also linearly independent for fixed S and ℓ.

Lemma 2 *Let S and T be standard r-tableaux of shape λ and $\ell = 0, 1, \ldots, e(\lambda) - 1$. Then the polynomial $\Delta_{S,T}(x)$ is a nonzero element in $R = P/J_+$ if and only if $S \in \mathrm{STab}(\lambda)_d$.*

Proof. Suppose that $S \in \mathrm{STab}(\lambda) \setminus \mathrm{STab}(\lambda)_d$. Then the number 0 does not appear in $i(S)^0, \ldots, i(S)^{d-1}$. Hence the partial product $\prod_{\nu=0}^{d-1} \{e_{T^\nu}(x_{T^\nu}^{ri(S)^\nu}) \prod_{k \in T^\nu} x_k^\nu\}$ of $\Delta_{S,T}(x)$ has the factor $\prod_{\nu=0}^{d-1}(\prod_{k\in T^\nu} x_k^r)$. On the other hand, the remaining product $\prod_{\nu=d}^{r-1}\{e_{T^\nu}(x_{T^\nu}^{ri(S)^\nu}) \prod_{k\in T^\nu} x_k^\nu\}$ has the factor $\prod_{\nu=d}^{r-1}(\prod_{k\in T^\nu} x_k^d)$. Since $d \mid r$, $\Delta_{S,T}(x)$ is divisible by $(x_1 \cdots x_n)^d$ in P, i.e., $V_S(\lambda) \subset J_+$.

To prove that $V_S(\lambda)$ survives in $R = P/J_+$ for $S \in \mathrm{STab}(\lambda)_d$, it is enough to see that $m(S)$ equals the multiplicity of the irreducible $G(r,p,n)$-module which is isomorphic to $V_S^{(\ell)}(\lambda)$, where

$$m(S) := \sum_{\mu} \sharp\{S' \in \mathrm{STab}(\mu)_d \mid V_{S'}^{(\ell')}(\mu) \cong V_S^{(\ell)}(\lambda), \quad \text{for some } \ell' = 0, 1, \ldots, e(\mu) - 1\},$$

and the sum is taken over the set $\{\mu \in \mathcal{P}_{r,n} \mid \mu \sim \lambda\}$. Indeed, it is easily seen that

$$\begin{aligned} m(S) &= |\mathrm{STab}(\lambda)_d| \times \sharp\{\mu \in \mathcal{P}_{r,n} \mid \mu \sim \lambda\} \\ &= \frac{|\mathrm{Stab}(\lambda)|}{p} \times b(\lambda) \\ &= \frac{|\mathrm{Stab}(\lambda)|}{e(\lambda)} \\ &= \frac{\dim V_S(\lambda)}{e(\lambda)} \\ &= \dim V_S^{(\ell)}(\lambda). \end{aligned}$$

Since R is isomorphic to the regular representation of $G(r,p,n)$, the proof completes. □

We now have a family of polynomials

$$\{\Delta^{(\ell)}_{S,T}(x) \in R \mid S \in \mathrm{STab}(\lambda)_d,\ T \in \mathrm{STab}(\lambda)_{db(\lambda)},\\ \ell = 0, 1, \ldots, e(\lambda) - 1\}.$$

It is shown in Theorem 3 below that they are linearly independent. We call these polynomials the higher Specht polynomials for the complex reflection group $G(r, p, n)$.

Theorem 3 *Let* $\lambda = (\lambda^0, \ldots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$, *and for each* $S \in \mathrm{STab}(\lambda)$ *and* $0 \le \ell \le e(\lambda) - 1$, *put* $V^{(\ell)}_S = \bigoplus_{T \in \mathrm{STab}(\lambda)} \mathbf{C}\Delta^{(\ell)}_{S,T}(x)$ *as a subspace of* R'.

1. *We have the eigenspace decomposition* $V_S(x) = \bigoplus_{\ell=0}^{e(\lambda)-1} V^{(\ell)}_S(x)$ *for the associator* A_λ.
2. *The space* $V^{(\ell)}_S(\lambda)$ *affords an irreducible representation of* $G(r, p, n)$.
3. *The* $G(r, p, n)$*-module* $R = P/J_+$ *admits an irreducible decomposition*

$$R = \bigoplus_{\lambda} \bigoplus_{S \in \mathrm{STab}(\lambda)_d} \bigoplus_{\ell=0}^{e(\lambda)-1} V^{(\ell)}_S(\lambda),$$

where λ *runs over a system of complete representatives of* $\mathcal{P}_{r,n}/\sim$.

Proof.

1. For a standard r-tableau $S \in \mathrm{STab}(\lambda)$, a subspace $V^{(\ell)}_S(\lambda)$ of $V_S(\lambda)$ is defined by

$$V^{(\ell)}_S(\lambda) := \bigoplus_{T \in \mathrm{STab}(\lambda)_{db(\lambda)}} \mathbf{C}\Delta^{(\ell)}_{S,T}(x),$$

for each $\ell = 0, 1, \ldots, e(\lambda) - 1$. Recall that the associator A_λ of $V_S(\lambda)$ is defined by $A_\lambda(\Delta_{S,T}(x)) = \Delta_{S,\mathrm{Sh}^{-b(\lambda)}T}(x)$. Since $A_\lambda(\Delta^{(\ell)}_{S,T}(x)) = \zeta^{\ell db(\lambda)}\Delta^{(\ell)}_{S,T}(x)$, the subspaces $V^{(\ell)}_S(\lambda)$ are contained in distinct eigenspaces of A_λ. Hence we have

$$\bigoplus_{\ell=0}^{e(\lambda)-1} V^{(\ell)}_S(\lambda) \subset V_S(\lambda).$$

Since the dimension of $V_S^{(\ell)}(\lambda)$ is

$$|\mathrm{STab}(\lambda)_{db(\lambda)}| = \frac{1}{e(\lambda)}|\mathrm{STab}(\lambda)| = \frac{1}{e(\lambda)} \dim V_S(\lambda)$$

for each $\ell = 0, 1, \ldots, e(\lambda) - 1$, the dimensions of the both side of the above inclusion coincide. Therefore we have the direct sum decomposition

$$\bigoplus_{\ell=0}^{e(\lambda)-1} V_S^{(\ell)}(\lambda) = V_S(\lambda).$$

This also gives the eigenspace decomposition of $V_S(\lambda)$ with respect to the associator A_λ.

2. This follows directly from 1 and the Clifford theory in Section 3.
3. Let π be the $G(r,n)$-module epimorphism

$$\pi : R' = P/J'_+ \to R = P/J_+; f \bmod J'_+ \mapsto f \bmod J_+.$$

By Lemma 2, we have $\pi(V_S(\lambda)) = 0$ if $S \in \mathrm{STab}(\lambda) \setminus \mathrm{STab}(\lambda)_d$, and $\pi(V_S(\lambda)) \cong V_S(\lambda)$ if $S \in \mathrm{STab}(\lambda)_d$. This implies that $\{\Delta_{S,T}(x) \in R \mid S \in \mathrm{STab}(\lambda)_d,\ T \in \mathrm{STab}(\lambda)\}$ are linearly independent in R. Hence the higher Specht polynomials

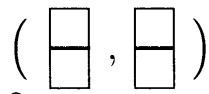
$$\{\Delta_{S,T}^{(\ell)}(x) \in R \mid S \in \mathrm{STab}(\lambda)_d,\ T \in \mathrm{STab}(\lambda)_{db(\lambda)},\ \ell = 0, 1, \ldots, e(\lambda) - 1\},$$

are also linearly independent. Therefore we have the direct sum decomposition

$$\begin{aligned} R = \pi(R') &= \pi\left(\bigoplus_{\lambda\in\mathcal{P}_{r,n}} \bigoplus_{S\in\mathrm{STab}(\lambda)} V_S(\lambda)\right) \\ &\cong \bigoplus_{\lambda\in\mathcal{P}_{r,n}/\sim} \bigoplus_{S\in\mathrm{STab}(\lambda)_d} V_S(\lambda) \\ &= \bigoplus_{\lambda\in\mathcal{P}_{r,n}/\sim} \bigoplus_{S\in\mathrm{STab}(\lambda)_d} \bigoplus_{\ell=0}^{e(\lambda)-1} V_S^{(\ell)}(\lambda). \end{aligned}$$

This is an irreducible decomposition of the left regular representation R of $G(r,p,n)$. □

5. Examples

In this section, we give some examples of higher Specht polynomials. First we consider $G(2,1,4) = W(B_4)$. Let $\lambda =$ , $T_1 = \left(\begin{matrix} 1 \\ 2 \end{matrix} , \begin{matrix} 3 \\ 4 \end{matrix} \right)$, $T_2 = \mathrm{sh}(T_1) = \left(\begin{matrix} 3 \\ 4 \end{matrix} , \begin{matrix} 1 \\ 2 \end{matrix} \right)$ and $S = \left(\begin{matrix} 1 \\ 4 \end{matrix} , \begin{matrix} 2 \\ 3 \end{matrix} \right)$, so that $i(S) = \left(\begin{matrix} 0 \\ 2 \end{matrix} , \begin{matrix} 0 \\ 1 \end{matrix} \right)$. The higher Specht polynomials associated with (S, T_1) and (S, T_2) are, respectively,

$$\begin{aligned} \Delta_{S,T_1}(x) &= \left\{ \frac{1}{2}(\mathrm{id} - s_1)x_2^4 \right\} \left\{ \frac{1}{2}(\mathrm{id} - s_3)x_4^2 \right\} x_3 x_4 \\ &= \frac{1}{4}(x_2^4 - x_1^4)(x_4^2 - x_3^2)x_3 x_4, \end{aligned}$$

$$\begin{aligned} \Delta_{S,T_2}(x) &= \left\{ \frac{1}{2}(\mathrm{id} - s_3)x_4^4 \right\} \left\{ \frac{1}{2}(\mathrm{id} - s_1)x_2^2 \right\} x_1 x_2 \\ &= \frac{1}{4}(x_2^2 - x_1^2)(x_4^4 - x_3^4)x_1 x_2. \end{aligned}$$

Here $s_1 = (12)$ and $s_3 = (34)$ are transpositions and id stands for the identity. Next consider the case $G(2,2,4) = W(D_4)$, where $d = 1$. For the above λ, we see that $b(\lambda) = 1$ and $e(\lambda) = 2$. Therefore the 6-dimensional representation $V_S(\lambda)$ of $G(2,1,4)$ decomposes into 2 irreducible components $V_S^{(0)}(\lambda)$ and $V_S^{(1)}(\lambda)$ under $G(2,2,4)$, each of which is 3-dimensional. Accordingly the higher Specht polynomial associated with (S, T_1) decomposes to

$$\Delta_{S,T_1}^{(0)}(x) = \Delta_{S,T_1}(x) + \Delta_{S,T_2}(x),$$

and

$$\Delta_{S,T_1}^{(1)}(x) = \Delta_{S,T_1}(x) - \Delta_{S,T_2}(x).$$

If we take $S_1 = \left(\begin{matrix} 2 \\ 3 \end{matrix} , \begin{matrix} 1 \\ 4 \end{matrix} \right)$ so that $i(S_1) = \left(\begin{matrix} 1 \\ 2 \end{matrix} , \begin{matrix} 0 \\ 2 \end{matrix} \right)$, then

$$\begin{aligned} \Delta_{S_1,T_1}(x) &= \left\{ \frac{1}{2}(\mathrm{id} - s_1)x_1^2 x_2^4 \right\} \left\{ \frac{1}{2}(\mathrm{id} - s_3)x_4^4 \right\} x_3 x_4 \\ &= \frac{1}{4}(x_1^2 x_2^4 - x_1^4 x_2^2)(x_4^4 - x_3^4)x_3 x_4 \end{aligned}$$

$$= \frac{1}{4}(x_1x_3^3 - x_1^3x_2)(x_4^4 - x_3^4)x_1x_2x_3x_4,$$

which does not survive in R.

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