Hokkaido Mathematical Journal Vol. 27 (1998) p. 475–483

Orthogonal (g, f)-factorizations of bipartite graph

Yan GUIYING*, Jiaofeng PAN, Wang JIANFANG and C.K. WONG

(Received August 27, 1996; Revised April 14, 1997)

Abstract. We consider a simple graph. Let $g(x)$ and $f(x)$ be integer-valued functions defined on $V(G)$ with $f(x) \geq g(x) \geq 1$ for all $x \in V(G)$. A (g, f)-factor of a graph G is a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for each vertex x of F. In this paper, we mainly discuss the problem of orthogonal (g, f)-factorizations of bipartite graph. Furthermore, we generalized some predecessor's result.

Key words: bipartite graph, factor, factorization, orthogonal.

1. Introduction

All graphs under consideration are simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An edge joining vertices u and v is denoted by uv. For a vertex $v \in V(G)$, we denote by $d_G(v)$ the degree of v in G. Let $g(x)$ and $f(x)$ be integer-valued functions defined on $V(G)$ with $f(x) \geq g(x)$ for all $x \in V(G)$. A graph G is called a (g, f)-graph if $g(v) \leq d_G(v) \leq f(v)$ for each vertex $v \in V(G)$, and a (g, f)-factor of a graph G is a spanning (g, f)-subgraph of G. A (g, f)-factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ of a graph G is a partition of $E(G)$ into edge-disjoint spanning (g, f)-subgraphs. A subgraph H of G is orthogonal to $\mathcal{F}$ if $|E(H) \cap E(F_i)| = 1$ for all $1 \leq i \leq t$. A bipartite graph G with partite sets X and Y is denoted by $G = (X, Y; E(G))$ and its edge set is denoted by $E(G)$.

Now, we consider the following problem:

Given a graph G and its subgraph H, how many edge disjoint factors containing exactly one distinct edge of H are contained in G?

If G has a factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ such that $|E(H) \cap E(F_i)| = 1$ $(1 \leq i \leq t)$, we obtain a solution to the following problem [2]:

Given a subgraph H of G, does there exist a factorization $\mathcal{F}$ of G orthogonal to H?

1991 Mathematics Subject Classification : 05C75.

*Supported by Chuang-Xin Foundation of Academia Sinica, Croucher Foundation of Hong Kong and Shandong Youth Science Foundation.

In this paper, we will prove the following theorem:

Theorem *Let m and r be integers such that $1 \leq r < m$. Let G be a $(mg+m-r, mf-m+r)$-bipartite graph (i.e. $mg(x)+m-r \leq d_G(x) \leq mf(x)-m+r$) for all $x \in V(G)$), and H a subgraph of G with edge set $\{e_1, e_2, \dots, e_m\}$. Then G has $m-r+1$ edge disjoint (g,f)-factors containing $e_1, e_2, \dots, e_{m-r+1}$, respectively, and excluding $e_{m-r+2}, e_{m-r+3}, \dots, e_m$.*

When $r=1$, from the proof of the above theorem we have the following

Corollary *Every $(mg(x)+m-1, mf(x)-m+1)$-bipartite graph has a (g,f)-factorization orthogonal to a given subgraph with m edges.*

2. The Proof of the Theorem

Given a subset $X \subseteq V(G)$, we write $f(X) = \sum_{x \in X} f(x)$, $d_G(X) = \sum_{x \in X} d_G(x)$. $G[S]$ denotes the subgraph of G induced by S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. For $E' \subseteq E(G)$, $G[E']$ denotes the subgraph of G induced by E' and $G - E' = G[E - E']$. If S and T are disjoint subsets of $V(G)$, then $e_G(S,T)$ denotes the number of edges of G joining S and T. Other notation and definition in this paper can be found in [1].

In [3] Liu Guizhen got a necessary and sufficient condition for a bipartite graph to have a (g,f)-factor containing a given edge:

Lemma 2.1 [3]. *Let $G = (X, Y; E(G))$ be a bipartite graph and $g(x)$ and $f(x)$ be two positive integer-valued function defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then for any given edge e of G, G has a (g,f)-factor containing e if and only if for all $S \subseteq X$ and $T \subseteq Y$,*

$$\delta_G(S,T) = f(S) + d_G(T) - g(T) - e_G(S,T) \geq \varepsilon_1(S,T)$$

and

$$\delta_G(T,S) = f(T) + d_G(S) - g(S) - e_G(T,S) \geq \varepsilon_2(T,S)$$

where $\varepsilon_1(S,T) = 1$ if $e \in E_G(S, Y-T)$, otherwise, $\varepsilon_1(S,T) = 0$ and $\varepsilon_2(T,S) = 1$ if $e \in E_G(T, X-S)$, otherwise $\varepsilon_2(T,S) = 0$.

Let G be a graph. Hereafter m and r denote integers such that $1 \leq r < m$, and $g(x)$ and $f(x)$ denote two positive integer-valued functions defined on $V(G)$.

Lemma 2.2 *Let G be an (mg,mf)-bipartite graph. Then G has a (g,f)-factor containing any given edge e of G.*

Proof. The Claim clearly holds when $m=1$. In the following we assume that $m \geq 2$. Put

$$\begin{aligned} p(x) &= \max\{g(x), d_G(x) - ((m-1)f(x)+1)\} \\ q(x) &= \min\{f(x), d_G(x) - ((m-1)g(x)-1)\}. \end{aligned}$$

We shall prove that G has a (p,q)-factor containing e, which is obviously a required (g,f)-factor of G. Put

$$\Delta_1(x) = \frac{1}{m}d_G(x) - p(x), \quad \Delta_2(x) = q(x) - \frac{1}{m}d_G(x).$$

If $p(x) = g(x)$, then $\Delta_1(x) = \frac{d_G(x)}{m} - g(x) \geq \frac{mg(x)}{m} - g(x) \geq 0$; if $p(x) = d_G(x) - (m-1)f(x) - 1$, then $\Delta_1(x) = \frac{d_G(x)}{m} - d_G(x) + (m-1)f(x) + 1 \geq \frac{1-m}{m}[mf(x)] + (m-1)f(x) + 1 = 1$.
Thus

$$\Delta_1(x) \geq 0.$$

Similarly we have

$$\Delta_2(x) \geq 0$$

Now let $S \subseteq X$ and $T \subseteq Y$, we now prove that $\delta_G(S,T) \geq \varepsilon_1(S,T)$ for q and p. Since $d_G(T) - d_{G-S}(T) = d_G(S) - d_{G-T}(S) = e_G(T,S)$, we have

$$\begin{aligned} \delta_G(S,T) &= q(S) + d_G(T) - p(T) - e_G(S,T) \\ &= \left(\frac{d_G(T)}{m} - p(T)\right) + \left(q(S) - \frac{d_G(S)}{m}\right) \\ &\quad + \left(1 - \frac{1}{m}\right) d_{G-S}(T) + \frac{d_{G-T}(S)}{m} \\ &= \Delta_1(T) + \Delta_2(S) + \left(1 - \frac{1}{m}\right) d_{G-S}(T) + \frac{d_{G-T}(S)}{m}. \end{aligned}$$

If $e \in E_G(S, Y-T)$, then $d_{G-T}(S) \geq 1$, and $\delta_G(S,T) \geq \frac{1}{m}d_{G-T}(S) \geq \frac{1}{m}$, that is $\delta_G(S,T) \geq 1$ because $\delta_G(S,T)$ is a integer. Otherwise, we have $\delta_G(S,T) \geq \Delta_1(T) \geq 0$. Similarly, we have $\delta_G(T,S) \geq \varepsilon_2(T,S)$, therefore the proof is completed by lemma 2.1. □

Lemma 2.3 *Let G be an $(mg+m-r, mf-m+r)$-bipartite graph. Then for any given subgraph H with m edges $e_1, e_2, \ldots, e_m$ of G, the graph G has a (g,f)-factor containing e_1 and excluding $e_2, e_3, \ldots, e_m$.*

Proof. Put

$$\begin{aligned} p(x) &= \max\{g(x), d_G(x) - ((m-1)f(x) - m + r + 1)\} \\ q(x) &= \min\{f(x), d_G(x) - ((m-1)g(x) + m - (r+1))\}. \end{aligned}$$

Set $G' = G - \{e_2, e_3, \ldots, e_m\}$ and $e_1 = uv$. Then G' is a $(mg - r + 1, mf - m + r)$-graph. Since $mg(x) + m - r \leq mf(x) - m + r$, we have $f(x) \geq g(x) + 2 - \frac{2r}{m}$ and thus $g(x) \leq p(x) < q(x) \leq f(x)$. We shall prove that G' has a (p,q)-factor containing e_1, which is obviously a required (g,f)-factor of G. Put

$$\Delta_1(x) = \frac{1}{m} d_{G'}(x) - p(x), \quad \Delta_2(x) = q(x) - \frac{1}{m} d_{G'}(x).$$

1. If $p(x) = g(x)$ and $x \in \{u, v\}$, then $\Delta_1(x) = \frac{d_{G'}(x)}{m} - g(x) \geq \frac{mg(x)+m-r-d_H(x)+1}{m} - g(x) \geq 1 - \frac{r+d_H(x)-1}{m}$; and if $p(x) = g(x)$ and $x \notin \{u, v\}$, then $\Delta_1(x) \geq \frac{d_{G'}(x)}{m} - g(x) \geq \frac{d_G(x)-d_H(x)}{m} - g(x) \geq \frac{mg(x)-m+r-d_H(x)}{m} - g(x) = 1 - \frac{r+d_H(x)}{m}$.

We next assume that $p(x) = d_G(x) - (m-1)f(x) + m - r - 1$.

(i) If $d_G(x) = mf(x) - m + r$, then $p(x) = f(x) - 1$.

Thus $\Delta_1(x) = \frac{d_G(x)-d_H(x)}{m} - f(x) + 1 = \frac{r-d_H(x)}{m}$ or $\Delta_1(x) \geq \frac{r-d_H(x)+1}{m}$ according to $x \notin \{u, v\}$ or $x \in \{u, v\}$.

(ii) If $d_G(x) \leq mf(x) - m + r - 1$, and $x \notin \{u, v\}$, then $\Delta_1(x) = \frac{d_G(x)-d_H(x)}{m} - d_G(x) + (m-1)f(x) - m + r + 1 \geq \frac{1-m}{m}(mf(x) - m + r - 1) + (m-1)f(x) - m + r + 1 - \frac{d_H(x)}{m} = 1 + \frac{r-d_H(x)-1}{m}$, and if $d_G(x) \leq mf(x) - m + r - 1$ and $x \in \{u, v\}$, then $\Delta_1(x) \geq 1 + \frac{r-d_H(x)}{m}$.

Thus

$$\Delta_1(x) \geq \begin{cases} \dfrac{r - d_H(x)}{m} & \text{if } m \geq 2r \\[2ex] 1 - \dfrac{r + d_H(x)}{m} & \text{if } m < 2r. \end{cases}$$

2. If $q(x) = f(x)$, and $d_G(x) = mf(x) - m + r$, then $\Delta_2(x) = q(x) - \frac{d_{G'}(x)}{m} \geq q(x) - \frac{d_G(x)}{m} \geq f(x) - \frac{1}{m}(mf(x) - m + r) = 1 - \frac{r}{m}$; otherwise,

$d_G(x) \le mf(x)-m+r-1$, then $\Delta_2(x) = q(x) - \frac{d_{G'}(x)}{m} \ge f(x) - \frac{1}{m}(mf(x) - m + r - 1) = 1 - \frac{r-1}{m}$.

Next we assume that $q(x) = d_G(x) - (m-1)g(x) - m + r + 1$, then $\Delta_2(x) = d_G(x) - (m-1)g(x) - m + r + 1 - \frac{d_{G'}(x)}{m} \ge (1-\frac{1}{m})d_G(x) - (m-1)g(x) - m + r + 1 \ge \frac{r}{m}$.

So we have

$$\Delta_2(x) \ge \begin{cases} \dfrac{r}{m} & \text{if } m \ge 2r \\ 1 - \dfrac{r}{m} & \text{if } m < 2r,\ d_G(x) = mf(x) - m + r \\ 1 - \dfrac{r-1}{m} & \text{otherwise.} \end{cases}$$

Now let $S \subseteq X$ and $T \subseteq Y$, we now prove that $\delta_{G'}(S,T) \ge \varepsilon_1(S,T)$ for q and p. Similarly, we have

$$\delta_{G'}(S,T) = \Delta_1(T) + \Delta_2(S) + \left(1 - \frac{1}{m}\right) d_{G'-S}(T) + \frac{d_{G'-T}(S)}{m}$$

Case 1: $T = \emptyset$.

In this case $\delta_{G'}(S,T) = q(S)$. Thus $\delta_{G'}(S,T) = 0$ if $S = \emptyset$; $\delta_{G'}(S,T) \ge 1$ if $S \ne \emptyset$. So $\delta(S,T) \ge \varepsilon_1(S,T)$.

Case 2: $T \ne \emptyset$ and $e_1 = uv \in E(S, Y-T)$, in particular $S \ne \emptyset$.

In this case we have $\varepsilon_1(S,T) = 1$. We consider two subcases.

Subcase 2.1 $r < m < 2r$.

(i) $d_{G'-S}(T) \ge 1$.

$$\begin{aligned} \delta_{G'}(S,T) &\ge \Delta_1(T) + \Delta_2(S) + \left(1 - \frac{1}{m}\right) d_{G'-S}(T) + \frac{1}{m} d_{G'-T}(S) \\ &\ge \sum_{x \in T} \left(1 - \frac{r + d_H(x)}{m}\right) + 1 - \frac{r}{m} + 1 - \frac{1}{m} + \frac{1}{m} \\ &> -\frac{(r-1)}{m} + 2 - \frac{r}{m} > 0. \end{aligned}$$

(ii) $d_{G'-S}(T) = 0$.

(a) If $|T| = 1$, then $d_{G'-T}(S) \ge mg(x) + m - r - 1$ for $x \in T$, and thus.

$$\delta_{G'}(S,T) \ge \Delta_1(T) + \Delta_2(S) + \frac{1}{m} d_{G'-T}(S)$$

$$\geq 1 - \frac{r+(m-1)}{m} + 1 - \frac{r}{m} + \frac{1}{m}(mg(x)+m-r-1)$$
$$= g(x) + 2 - \frac{3r}{m} > 0.$$

(b) $|T| \geq 2$. Set $B = \{x \,|\, d_H(x) > 0,\ x \in T\}$. If $|B| \leq 1$, then $|S| \geq mg(x) + m - r$ for $x \in T - B$, and thus

$$\begin{aligned}
\delta_{G'}(S,T) &\geq \Delta_1(T) + \Delta_2(S) + \frac{1}{m} d_{G'-T}(S) \\
&\geq -1 + \frac{1}{m} + \left(1 - \frac{r}{m}\right)(mg(x)+m-r) + \frac{1}{m} \\
&\geq -1 + \frac{1}{m} + \left(1 - \frac{r}{r+1}\right)[(r+1)g(x)+1] + \frac{1}{m} \\
&\qquad\qquad (m \geq r+1) \\
&= g(x) + \frac{1}{r+1} + \frac{3}{m} - 1 > 0.
\end{aligned}$$

If $|B| \geq 2$ and there exists vertex $x \in T$ such that $d_G(x) = mf(x) - m + r$, then $|S| \geq mf(y) - m + r - \frac{m-1}{2}$ for some $y \in T$.

$$\begin{aligned}
\delta_{G'}(S,T) &\geq \Delta_1(T) + \Delta_2(S) + \frac{1}{m} d_{G'-T}(S) \\
&\geq -1 + \frac{1}{m} + \left(1 - \frac{r}{m}\right)\left(mf(y) - \frac{3m}{2} + r + \frac{1}{2}\right) + \frac{1}{m} \\
&\geq -1 + \frac{1}{m} + \frac{1}{m}\left(mf(y) - \frac{3m}{2} + r + \frac{1}{2}\right) + \frac{1}{m} \\
&\qquad\qquad (r \leq m-1) \\
&= -1 + \frac{1}{m} + f(y) - \frac{3}{2} + \frac{r}{m} + \frac{3}{2m} \\
&\qquad\qquad \left(f(x) \geq g(x) + 2 - \frac{2r}{m} > 1\right) \\
&\geq -1 + \frac{1}{m} + \frac{1}{2} + \frac{2r+3}{2m} \\
&= -\frac{1}{2} + \frac{2r+5}{2m} \\
&> -\frac{1}{2} + \frac{1}{2} + \frac{5}{2m} \\
&\quad 0. \qquad\qquad (m < 2r)
\end{aligned}$$

Otherwise, $d_G(x) \leq mf(x) - m + r - 1$ for all $x \in V(S)$. $|S| \geq mg(y) +$

$m - r - \frac{m-1}{2} = mg(y) + \frac{m}{2} - r + \frac{1}{2}$ for some $y \in T$ and thus

$$\begin{aligned}
\delta_{G'}(S,T) &\geq \Delta_1(T) + \Delta_2(S) + \frac{1}{m} d_{G'-T}(S) \\
&= -1 + \frac{1}{m} + \left(1 - \frac{r}{m} + \frac{1}{m}\right)\left(mg(y) + \frac{m}{2} - r + \frac{1}{2}\right) + \frac{1}{m} \\
&\geq -1 + \frac{1}{m} + \frac{2}{m}\left(mg(y) + \frac{m}{2} - r + \frac{1}{2}\right) + \frac{1}{m} \quad (m > r) \\
&= 2g(y) - \frac{2r}{m} + \frac{3}{m} > 0.
\end{aligned}$$

Subcase 2.2 $\quad m > 2r$.

(i) $d_{G'-S}(T) \geq 1$.

$$\begin{aligned}
\delta_{G'}(S,T) &= \Delta_1(T) + \Delta_2(S) + \left(1 - \frac{1}{m}\right) d_{G'-S}(T) + \frac{1}{m} d_{G'-T}(S) \\
&\geq \sum_{x \in T} \frac{r - d_H(x)}{m} + \frac{r}{m} + 1 - \frac{1}{m} + \frac{1}{m} \\
&= \frac{r - (m-1)}{m} + \frac{r}{m} + 1 \\
&= \frac{2r+1}{m} > 0.
\end{aligned}$$

(ii) $d_{G'-S}(T) = 0$.

(a) If $|T| \leq m - 1$, then we have the following inequalities by $d_{G'-T}(S) \geq mg(x) + m - r - (m-1) = mg(x) + 1 - r$ for $x \in T$.

$$\begin{aligned}
\delta_{G'}(S,T) &= \Delta_1(T) + \Delta_2(S) + \frac{1}{m} d_{G'-T}(S) \\
&\geq \sum_{x \in T} \frac{r - d_H(x)}{m} + \frac{r}{m} + \frac{1}{m}(mg(x) + 1 - r) \\
&\geq -1 + \frac{r+1}{m} + \frac{r}{m} + g(x) + \frac{1}{m} - \frac{r}{m} \\
&= g(x) - 1 + \frac{r+1}{m} > 0.
\end{aligned}$$

(b) If $|T| \geq m$, then

$$\delta_{G'}(S,T) \geq \Delta_1(T) + \Delta_2(S) + \frac{1}{m} d_{G'-T}(S)$$

$$\geq \sum_{x\in T} \frac{r - d_H(x)}{m} + \frac{r}{m} + \frac{1}{m}$$
$$\geq \frac{r|T| - (m-1)}{m} + \frac{r}{m} + \frac{1}{m}$$
$$= r - 1 + \frac{r+2}{m} > 0.$$

So, $\delta_{G'}(S,T) \geq 1$ because $\delta_{G'}$ is an integer. Namely, $\delta_{G'}(S,T) \geq \varepsilon_1(S,T)$.

Case 3: $\varepsilon_1(S,T) = 0$, then $\delta_{G'}(S,T) \geq \Delta_1(T) \geq \sum_{x\in T} \frac{r-d_H(x)}{m} \geq -1 + \frac{r+1}{m} > -1$ if $r < m < 2r$ and then $\delta_{G'}(S,T) \geq \sum_{x\in T}(1 - \frac{r+d_H(x)}{m}) \geq \sum_{x\in T} \frac{d_H(x)}{m} \geq -1 + \frac{1}{m} > -1$ if $m \geq 2r$.
So, $\delta_{G'}(S,T) \geq 0$.
Similarly, we can show that $\delta_{G'}(S,T) \geq \varepsilon_2(T,S)$.

By Lemma 2.1, G has a (p,q)-factor F_0 containing e_1 but not containing $e_2, e_3, \ldots, e_m$. The proof is completed. □

From the above proof, we see that if $p(x) = g(x)$, clearly, $p(x) \geq d_G(x) - (m-1)f(x) + m - r - 1$, then $d_G(x) - d_{F_0}(x) \leq d_G(x) - (d_G(x) - (m-1)f(x) + m - r - 1) = (m-1)f(x) - (m-1) + r$; if $p(x) = d_G(x) - (m-1)f(x) + m - r - 1$, then $d_G(x) - d_{F_0}(x) \leq d_G(x) - (d_G(x) - (m-1)f(x) + m - r - 1) = (m-1)f(x) - (m-1) + r$.
Similarly, we have $d_G(x) - d_{F_0}(x) \geq (m-1)g(x) + (m-1) - r$. Therefore, $G - E(F_0)$ is a $((m-1)g + (m-1) - r, (m-1)f - (m-1) + r)$-graph.
Finally, we give the proof of the theorem.

Proof. Set $G_0 = G$, $G_1 = G_0 - E(F_0)$, G_1 is a $((m-1)g + (m-1) - r, (m-1)f - (m-1) + r)$-graph. If $m - 1 > r$, then G_1 has a (g,f)-factor F_2 containing e_2 and excluding $e_1, e_3, \ldots, e_m$ whose proof is similar to Lemma 2.3. On the analogy of this, set $G_i = G_{i-1} - E(F_{i-1})$, $(2 \leq i \leq m - r)$, then G_i has a (g,f)-factor F_i containing e_i and excluding $e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_m$, therefore, G has $m - r$ (g,f)-factors $F_1, F_2, \ldots, F_{m-r}$ containing $e_1, e_2, \ldots, e_{m-r}$, respectively, and excluding $e_{m-r+1}, e_{m-r+2}, \ldots, e_m$. Now, G_{m-r} is a (rg, rf)-graph, by lemma 2.2, G_{m-r} has a (g,f)-factor F_{m-r+1} containing e_{m-r+1}.
Thus $F_1, F_2, \ldots, F_{m-r+1}$ are the required factors. The proof is completed. □

Acknowledgment The authors would like to thank the referees for their helpful comments and suggestions.

References

[1] Bondy J.A. and Murty U.R., *Graph Theory with Applications.* The Macmillan Press LTD, 1976.

[2] Heinrich B.A. and Guizhen L., *Contemporary design theory A collection of surveys.* John Wiley and Sons. Inc. (1992), 13–37.

[3] Guizhen L., *(g, f)-factorizations of graphs orthogonal to m-star.* Science in China A1 (1995).

Yan Guiying
Institute of Applied Mathematics
Chinese Academy of Sciences
P.O.Box 2734, Beijing 100080, P.R. China
E-mail: yangy@amath3.amt.ac.cn

Jiaofeng Pan
Department of Computer Science and Engineering
The Chinese University of Hong Kong
Shatin, N., T.
E-mail: jfpan@cse.cuhk.edu.hk

Wang Jianfang
Institute of Applied Mathematics
Chinese Academy of Sciences
P.O.Box 2734, Beijing 100080, P.R. China
E-mail: jfwang@amath3.amt.ac.cn

C.K. Wong
Department of Computer Science and Engineering
The Chinese University of Hong Kong
Shatin, N., T.
on leave from IBM T.J. Watson Research Center
P.O.Box 218, Yorktown Heights, N.Y., U.S.A.
E-mail: wongck@cse.cuhk.edu.hk