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Global hypoellipticity of subelliptic operators on closed manifolds

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Abstract. We give a criterion of global hypoellipticity on closed manifolds for certain second order operators. Applying this criterion, global hypoellipticity of horizontal Laplacians and an example which has no infinitesimally transitive points are studied.

Key words: global hypoellipticity, horizontal Laplacian.

1. Introduction

A differential operator L on a C^∞ manifold M is *hypoelliptic*, if any distribution solution u of the equation $Lu = f$ is smooth at the place where f is smooth.

Similarly, L is called to be *globally hypoelliptic*, if $Lu \in C^\infty(M)$ for a distribution u implies $u \in C^\infty(M)$. Here $C^\infty(M)$ is the space of smooth functions on M.

It is obvious that if L is hypoelliptic, then L is globally hypoelliptic. In this paper, we concern with the global hypoellipticity of a differential operator on a closed (compact connected without boundary) manifold M.

Let $X_1, \dots, X_m$ be smooth (real) tangent vector fields on M. The differential operator L which we shall treat in this paper is given in the form

$$L := \sum_{i=1}^{m} {X_i}^* X_i, \tag{1.1}$$

where ${X_i}^*$ is the formal adjoint operator of X_i with respect to a fixed smooth Riemannian metric on M. Let $\mathcal{V}$ be the linear space spanned by X_i's

$$\mathcal{V} := \left\{ \sum_{i=1}^{m} f_i X_i : f_i \in C^\infty(M) \right\}, \tag{1.2}$$

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and put

$$\mathfrak{h} := \text{the Lie algebra generated by } \mathcal{V}. \tag{1.3}$$

Recall first the following theorem which is directly obtained by applying the theorem of Hörmander [6] to our closed manifold.

Theorem (Hörmander) *If $\mathfrak{h}$ is infinitesimally transitive at every $p \in M$, i.e., $\mathfrak{h}(p) = T_pM$ for all $p \in M$, then L given by* (1.1) *is hypoelliptic.*

Now, for every $Y \in \mathcal{V}$, we denote by $\exp tY$ the one parameter diffeomorphism group generated through integral curves by Y, and put

$$\mathscr{H} := \text{the closed subgroup generated by } \{\exp Y : Y \in \mathcal{V}\}. \tag{1.4}$$

In a sense $\mathscr{H}$ is a Lie group (generalized Lie group [11]), but in general its Lie algebra (see (2.4)) is much bigger than $\mathfrak{h}$ as it will be seen in Examples 1–3 below.

Remark that every C^∞ diffeomorphism φ of M onto itself acts as the adjoint action on the space of smooth vector fields, i.e., for a vector field Y, we define it by

$$(\mathrm{Ad}(\varphi)Y)f := \varphi^* Y {\varphi^{-1}}^* f \quad \text{for} \quad f \in C^\infty(M).$$

If $\mathfrak{h}$ is infinitesimally transitive at every $p \in M$, then the implicit function theorem gives that the group $\mathscr{H}$ acts transitively on M. However the converse does not hold in general. This otccurs in general if the Lie algebra $\mathfrak{h}$ has not the property that

$$\mathrm{Ad}(\exp Y)\mathfrak{h} = \mathfrak{h} \quad \text{for all} \quad Y \in \mathfrak{h}.$$

If a Lie algebra $\mathfrak{h}$ has not this property, then $\mathfrak{h}$ cannot be the Lie algebra of any *infinite dimensional Lie group* whatever the definition of infinite dimensional Lie groups is (cf. [11]).

On the other hand, if $\mathscr{H}$ is not transitive on M and the orbits give a smooth fiber bundle structure over some open subset of the orbit space $M/\mathscr{H}$, then L is not globally hypoelliptic (see [6]).

By the above observation, the case where the group $\mathscr{H}$ acts transitively on M, but the infinitesimal transitivity of $\mathfrak{h}$ fails on a subset of M or on the whole space M is the most interesting. There are several examples of such Lie algebra $\mathfrak{h}$:

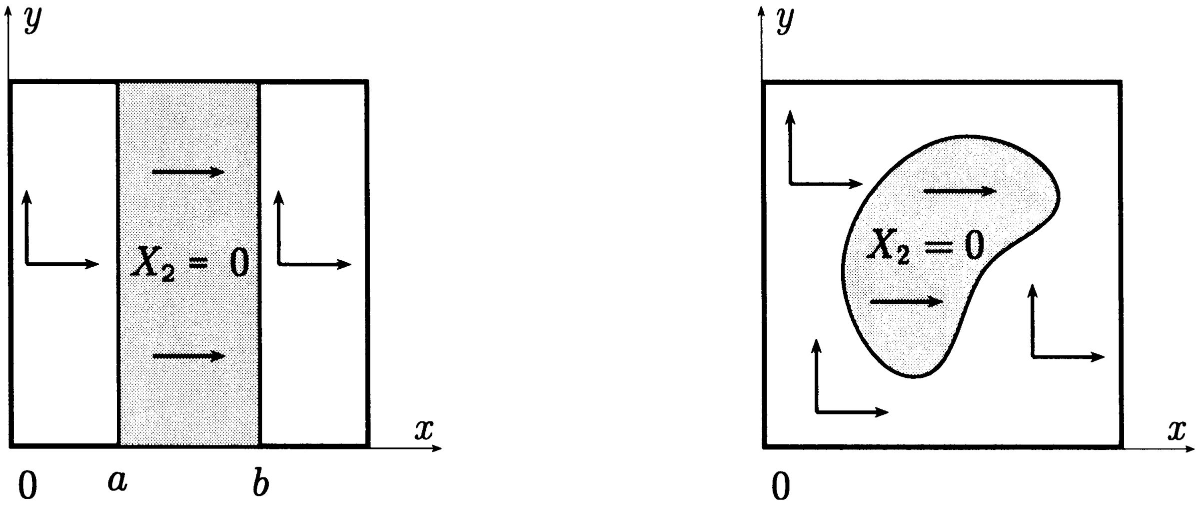

Fig. 1. Examples 1 and 2

Example 1. $M = \mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$ glued each opposite edges together: Let x, y be the coordinate functions on $\mathbb{T}^2$. We consider a differential operator

$$L = X_1{}^* X_1 + X_2{}^* X_2 = -\partial_x^2 - \zeta(x)^2 \partial_y^2, \tag{1.5}$$

where $X_1 = \partial_x$, $X_2 = \zeta\partial_y$ and $\zeta = \zeta(x)$ is a smooth function on $\mathbb{T}^2$ depending only on x, but $\zeta = 0$ on some interval $[a, b]$ such that $0 < a < b < 2\pi$ and $\zeta > 0$ on the complement.

Example 2. $M = \mathbb{T}^2$: Let $X_1 = \partial_x$ and $X_2 = \zeta\partial_y$, where $\zeta = \zeta(x, y)$ is a smooth function such that ζ vanishes identically on a compact set $K \subsetneqq \mathbb{T}^2$, but $\zeta > 0$ otherwise. Consider the differential operator

$$L = X_1{}^* X_1 + X_2{}^* X_2 = -\partial_x^2 - \zeta^2 \partial_y^2 - 2\zeta\zeta_y\partial_y. \tag{1.6}$$

We assume that $\mathscr{H}$ generated by X_1 and X_2 is transitive on $\mathbb{T}^2$ (see Figure 1).

Example 3. $M = \mathbb{T}^3$: Let x, y, z be the coordinate functions on $\mathbb{T}^3$. Consider vector fields $X_1 = \partial_x$, $X_2 = \zeta(x)\partial_y$, $X_3 = \eta(x, y)\partial_z$, and the differential operator

$$L = X_1{}^* X_1 + X_2{}^* X_2 + X_3{}^* X_3 = -\partial_x^2 - \zeta(x)^2 \partial_y^2 - \eta(x, y)^2 \partial_z^2, \tag{1.7}$$

where $\zeta = \zeta(x)$ and $\eta = \eta(x, y)$ are smooth functions whose supports are

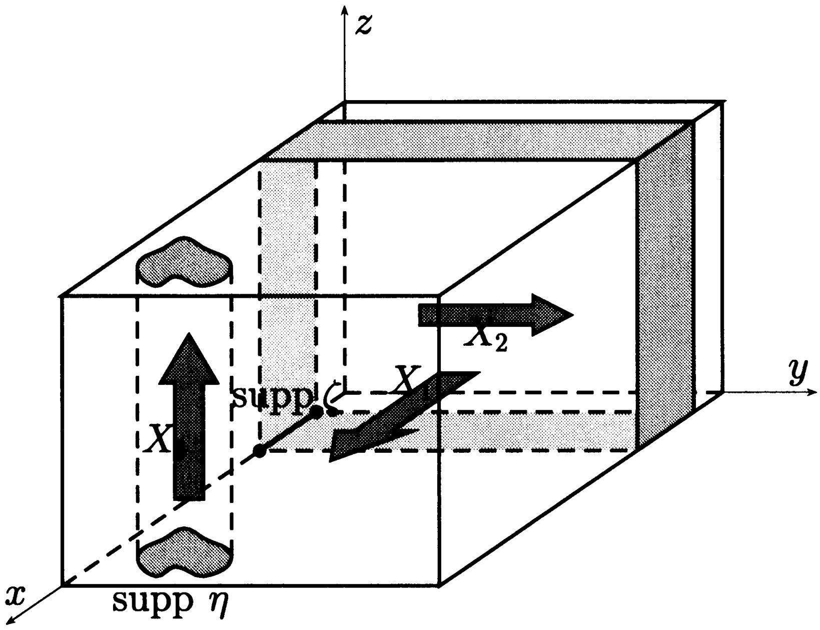

Fig. 2. Example 3

mutually disjoint and depend only on x and x, y, respectively. Hence the commutator $[X_2, X_3] = 0$. It is obvious that $\dim \mathfrak{h}(p) \leq 2$ at every $p \in \mathbb{T}^3$. Thus there is no infinitesimal transitive point. However, $\mathscr{H}$ acts transitively on $\mathbb{T}^3$ whenever ζ and η do not identically vanish (see Figure 2).

It is known in [4] that Example 1 is globally hypoelliptic. However, the proof is based on the feature that the operator L in (1.5) commutes with elliptic operators. Using this feature and the transitivity of $\mathscr{H}$, we obtain the L_2-regularity theorem, where L_2-regularity theorem means the following: The distribution solution u of $Lu = f$ is contained in L_2 whenever $f \in L_2$ (cf. Lemma 3.1).

In Example 1, L commutes with ∂_y and hence $[L, L - \partial_y^2] = 0$, where $L - \partial_y^2$ is elliptic. Thus, if $Lu \in C^\infty$, then $L(L - \partial_y^2)^m u \in C^\infty$, hence $(L - \partial_y^2)^m u \in L_2$ for any m, and then $u \in C^\infty$ by using elliptic regularity theorem.

Amano [2] proved the global hypoellipticity of operators involving Example 2. However, the proof was based essentially on the condition that there is a point where $\mathfrak{h}$ is infinitesimally transitive. Thus, the global hypoellipticity of operators of the type of Example 3 has not been proved.

In this paper, we first give a criterion of global hypoellipticity (cf. Theorem 3.3) in a slightly different shape from those given by Fediĭ [3] and

Amano [2], and by following several estimates in [2] that the operator L in Example 3 is globally hypoelliptic, if ζ and η are smooth nonnegative functions (cf. Section 4).

In [10], the first author extends Example 1 to the horizontal Laplacian (cf. Section 5) of a G-connection on a G-principal bundle over a closed manifold where G is a compact Lie group. In the case of horizontal Laplacian, the vertical part of $\mathfrak{h}(p)$ corresponds to the holonomy Lie algebra of the G-connection at the reference point p, and the transitivity of $\mathscr{H}$ corresponds to that the holonomy group is the total group G. Hence the horizontal Laplacian of a G-connection is globally hypoelliptic, if the holonomy group is G. This includes an example where $\mathfrak{h}$ is nowhere infinitesimally transitive.

Remark now the horizontal Laplacian $\Delta_{\mathcal{H}}$ is defined not only on G-principal bundle but also on every Riemannian fiber bundle (M, N, π). The feature of a G-connection is that every element of G acts on the total space M as a fiber preserving isometry which leaves $\Delta_{\mathcal{H}}$ invariant, and G acts transitively on each fiber.

Taking the criterion in Theorem 3.3 in Section 3 into account, we define here a notion of fiber preserving diffeomorphisms which leave $\Delta_{\mathcal{H}}$ almost invariant. That is, a volume preserving and fiber preserving diffeomorphism φ on M is called a *partially conformal transformation*, if there is a smooth function f_φ on M such that $\varphi^*\Delta_{\mathcal{H}}\varphi^{-1^*} = e^{f_\varphi}\Delta_{\mathcal{H}}$. Clearly, partially conformal transformations form a group, which could be infinite dimensional. The infinitesimal version of such transformations, that is, an *infinitesimal partially conformal transformation* is a divergence free vector field X on M, tangents to each fiber such that $[X, \Delta_{\mathcal{H}}] = f_X\Delta_{\mathcal{H}}$, where f_X is a smooth function depending on X. All infinitesimal partially conformal transformations form a Lie algebra.

We show in this paper that if the holonomy group acts transitively on M, and if infinitesimal partially conformal transformation group acts infinitesimally transitive along each fiber, then global hypoellipticity holds for the horizontal Laplacian of a (non-linear) connection on a fiber bundle with a compact fiber and a compact base space (Theorem 5.1).

See Section 5 for the precise notion of infinitesimal transitivity along each fiber. Note that there are examples of connections such that the infinitesimal holonomy is nowhere transitive but the holonomy group is transitive on the total space.

2. Positivity on the Orthogonal Space to Constant Functions

We fix a Riemannian metric on M. We put

$$\mathcal{E}_{\perp}(M) := \left\{ f \in C^{\infty}(M) : \int_M f\, d\mu = 0 \right\},$$

where $d\mu$ is the volume element induced by the fixed Riemannian metric. The space $\mathcal{E}_{\perp}(M)$ is the one of all smooth functions which are orthogonal to constants functions. We denote by $\langle f, h\rangle_0 := \int_M f \cdot h\, d\mu$ the usual L_2-inner product (we consider real valued functions) and by $\|f\|_0$ the L_2-norm. Let $H^k(M)$ be the usual Sobolev space with the norm $\|f\|_k$ given by $\|(1+\Delta)^{k/2} f\|_0$ by using the Laplacian with respect to an arbitrarily fixed C^{∞} Riemannian metric. It is known that the space $H^k(M)$ is independent of the choice of Riemannian metric and $H^{-k}(M)$ is the dual space of $H^k(M)$.

We put

$$H^k_{\perp}(M) := \left\{ f \in H^k(M) : \langle f, 1\rangle_0 = 0 \right\}.$$

We remark first the following:

Theorem 2.1 *Let L be the differential operator* (1.1) *and $\mathscr{H}$ the closed Lie group defined by* (1.4). *If the $\mathscr{H}$ acts transitively on M, then L is positive definite on $\mathcal{E}_{\perp}(M)$, that is, there is a constant $c > 0$ such that*

$$\int_M Lf \cdot f\, d\mu \geq c\|f\|_0^2 \quad \textit{for any} \quad f \in \mathcal{E}_{\perp}(M). \tag{2.1}$$

For the proof of Theorem 2.1, suppose there were no such a posoitive constant c. The assumption gives an existence of a sequence $\{f_n\}_n$ in $\mathcal{E}_{\perp}(M)$ such that $\|f_n\|_0 = 1$ and that

$$\lim_{n\to\infty} \langle Lf_n, f_n\rangle_0 = 0. \tag{2.2}$$

So to obtain a contradiction, it is enough to show that (2.2) implies

$$\lim_{n\to\infty} \|f_n\|_0 = 0.$$

Since $\langle Lf_n, f_n\rangle_0 = \sum_{i=1}^m \|X_i f_n\|_0^2$, the definition (1.2) of $\mathcal{V}$ and (2.2) give

$$\lim_{n\to\infty} \|Xf_n\|_0 = 0 \quad \text{for all} \quad X \in \mathcal{V}. \tag{2.3}$$

Next we introduce another Lie algebra $\tilde{\mathfrak{h}}$ as

$$\tilde{\mathfrak{h}} := \text{the Lie algebra spaned by } \{\mathrm{Ad}(\varphi)X : X \in \mathcal{V},\ \varphi \in \mathscr{H}\}, \quad (2.4)$$

where $\mathscr{H}$ is the group (1.4). This $\tilde{\mathfrak{h}}$ can be viewed as the Lie algebra of $\mathscr{H}$ (see Lemma 2.3).

Lemma 2.2 *For every $Y \in \tilde{\mathfrak{h}}$, we have*

$$\lim_{n\to\infty} \|Y f_n\|_{-1} = 0. \quad (2.5)$$

Proof. Every $Y \in \tilde{\mathfrak{h}}$ is a linear combination of elements written in the form

$$\mathrm{Ad}(\exp Y_\ell \cdots \exp Y_2 \exp Y_1)X,$$

with $Y_1, Y_2, \ldots, Y_\ell, X \in \mathcal{V}$. Thus it is enough to prove (2.5) for the vector field

$$Z_\ell = \mathrm{Ad}(\exp Y_\ell \cdots \exp Y_2 \exp Y_1)X.$$

For $\ell = 0$, it is trivial that $\lim_{n\to\infty} \|X f_n\|_{-1} = 0$ by (2.3) and $\|X f_n\|_{-1} \leq \|X f_n\|_0$.

Now suppose (2.5) holds for $Z_{\ell-1}$. This implies $\lim_{n\to\infty} \|Z_{\ell-1} f_n\|_{-1} = 0$ and we have only to prove that

$$\lim_{n\to\infty} \|\mathrm{Ad}(\exp Y_\ell) Z_{\ell-1} f_n\|_{-1} = 0.$$

Remark that $\mathrm{Ad}(\exp Y_\ell) Z_{\ell-1} f_n = (\exp Y_\ell)^* Z_{\ell-1} (\exp -Y_\ell)^* f_n$, and $(\exp Y_\ell)^*$ on a closed manifold M leaves the Sobolev space $H_\perp^{-1}(M)$ invariant. Hence we have only to show that

$$\lim_{n\to\infty} \|Z_{\ell-1}(\exp -Y_\ell)^* f_n\|_{-1} = 0.$$

Now, remark that

$$Z_{\ell-1}(\exp -Y_\ell)^* f_n = Z_{\ell-1} f_n - \int_0^1 Z_{\ell-1}(\exp -tY_\ell)^* Y_\ell f_n \, dt.$$

Since $\lim_{n\to\infty} \|Y_\ell f_n\|_0 = 0$ by (2.3), we see

$$\lim_{n\to\infty} \|(\exp -tY_\ell)^* Y_\ell f_n\|_0 = 0.$$

This implies

$$\lim_{n\to\infty} \|Z_{\ell-1}(\exp -tY_\ell)^* Y_\ell f_n\|_{-1} = 0,$$

since $Z_{\ell-1}$ is a differential operator of order 1. Therefore,

$$\begin{aligned}\|Z_{\ell-1}&(\exp -Y_\ell)^* Y_\ell f_n\|_{-1}\\ &\le \|Z_{\ell-1} f_n\|_{-1} + \int_0^1 \|Z_{\ell-1}(\exp -tY_\ell)^* Y_\ell f_n\|_{-1}\, dt \to 0,\end{aligned}$$

by using the uniform continuity in t. □

The following is due to Sussmann [14].

Lemma 2.3 (Sussmann) *The group $\mathscr{H}$ acts transitively on M, if and only if $\tilde{\mathfrak{h}}$ is infinitesimal transitive at every $p \in M$.*

Proof of Theorem 2.1. Suppose $\mathscr{H}$ acts transitively on M. By Lemma 2.3, we see at every $p \in M$ that $T_pM = \tilde{\mathfrak{h}}(p)$. Since M is compact, there are finite number of elements $Y_1, \dots, Y_\ell \in \tilde{\mathfrak{h}}$ such that $\{Y_1(p), \dots, Y_\ell(p)\}$ spans the tangent space T_pM at every $p \in M$. By Lemma 2.2, we see that

$$\lim_{n\to\infty} \|Y_i f_n\|_{-1} = 0 \quad \text{for every} \quad 1 \le i \le \ell.$$

Hence, the inequality

$$\|f_n\|_0 \le C(\|Y_1 f_n\|_{-1} + \dots + \|Y_\ell f_n\|_{-1})$$

gives $\lim_{n\to\infty} \|f_n\|_0 = 0$, which completes the proof of Theorem 2.1. □

By Theorem 2.1, we have the following apriori estimate.

Corollary 2.4 *For any integer $N > 0$, the following estimate holds for every $u \in H^2(M)$,*

$$\|u\|_0 \le C\|Lu\|_0 + D_N\|u\|_{-N}, \tag{2.6}$$

where C and D_N are positive constants, and D_N may depend on N but C is independent of N.

The global hypoellipticity follows from (2.6), if L satisfies several supplementary conditions.

3. Sufficient Conditions for Global Hypoellipticity

The inequality (2.6) yields the L_2-regularity theorem, if L commutes with an elliptic operator $\tilde{\Lambda}$ of order $m > 0$. Namely, we have the following:

Lemma 3.1 *Suppose* (2.6) *holds for* L. *If* L *commutes with an elliptic operator* $\tilde{\Lambda}$ *of order* $m > 0$, *then* $Lu \in H^0_\perp(M)$ *for some distribution* u *implies* $u \in H^0_\perp(M)$.

Proof. We assume $m = 1$ without loss of generality and considering $cI + \tilde{\Lambda}$ if necessary, we may assume that $\tilde{\Lambda}_\epsilon = I + \epsilon\tilde{\Lambda}$ gives for every $\epsilon > 0$ an isomorphism of $H^{s+1}(M)$ onto $H^s(M)$ for every s.

If $u \in H^{-N}(M)$, then we have $\tilde{\Lambda}_\epsilon^{-(N+2)} u \in H^2(M)$, and by (2.6), we have

$$\|\tilde{\Lambda}_\epsilon^{-(N+2)} u\|_0 \leq C\|L\tilde{\Lambda}_\epsilon^{-(N+2)} u\|_0 + D_N\|\tilde{\Lambda}_\epsilon^{-(N+2)} u\|_{-N}.$$

Since $u \in H^{-N}(M)$, we see that $\lim_{\epsilon\to 0}\|\tilde{\Lambda}_\epsilon^{-(N+2)} u\|_{-N} = \|u\|_{-N} < \infty$. Suppose $Lu \in H^0(M)$. Since $L\tilde{\Lambda}_\epsilon^{-(N+2)} u = \tilde{\Lambda}_\epsilon^{-(N+2)} Lu$, we have

$$\lim_{\epsilon\to 0}\|L\tilde{\Lambda}_\epsilon^{-(N+2)} u\|_0 = \lim_{\epsilon\to 0}\|\tilde{\Lambda}_\epsilon^{-(N+2)} Lu\|_0 = \|Lu\|_0 < \infty.$$

This implies that

$$\|u\|_0 = \lim_{\epsilon\to 0}\|\tilde{\Lambda}_\epsilon^{-(N+2)} u\|_0 < \infty,$$

and $u \in H^0(M)$. Thus we obtain the L_2-regularity. □

Since L commutes with an elliptic operator $\tilde{\Lambda}$, we see also that L is globally hypoelliptic by the remark in the introduction.

By the above observation we see that inequality (2.6) and the norm estimate of the commutator with an elliptic operator plays a crucial role.

Indeed, even if there is no elliptic operator which commutes with L, several useful sufficient conditions for global hypoellipticity has been known by Amano [2], Fediĭ [3] and Morimoto [9]. In what follows we give a short summary of their results by following mainly [2] and [3], but the statement of the result is given in a little different shape.

We call a pseudo-differential operator Λ a regulator, if Λ gives an isomorphism $H^{s+1}(M)$ onto $H^s(M)$ for every $s \in \mathbb{R}$. The pseudo-differential operator with the total symbol $(|\xi|^2 + 1)^{1/2}$ is an example of regulators, where $|\xi|$ is the norm with respect to the Riemannian metric on M (cf. [12],

[16]). We first remark the following:

Proposition 3.2 *If L satisfies the inequality* (2.6) *and the following inequality* (3.1):

For every $\delta > 0$ and for every integer $N > 0$, there is a constant $C_{N,\delta} > 0$ such that

$$\|((\mathrm{ad}\,\Lambda)^{\ell} L)\, u\|_{-\ell} \leq \delta \|Lu\|_0 + C_{N,\delta}\|u\|_{-N} \quad \ell = 1, 2, \tag{3.1}$$

for all $u \in C^\infty(M)$, where $(\mathrm{ad}\,\Lambda)L = [\Lambda, L]$.

Then for every integers k and $N > 0$, and for every $\delta > 0$, there exist positive constants C_k, $D_{N,k}$ and $C_{N,\delta,k}$ such that the inequalities

$$\|u\|_k \leq C_k\|Lu\|_k + D_{N,k}\|u\|_{-N}, \tag{3.2a}$$

$$\|((\mathrm{ad}\,\Lambda)L)\, u\|_{k-1} \leq \delta\|Lu\|_k + C_{N,\delta,k}\|u\|_{-N}, \tag{3.2b}$$

$$\|((\mathrm{ad}\,\Lambda)^2 L)\, u\|_{k-2} \leq \delta\|Lu\|_k + C_{N,\delta,k}\|u\|_{-N} \tag{3.2c}$$

hold for all $u \in C^\infty(M)$.

Proof. First we prove inequalities (3.2a) by induction for positive integers k. From (2.6) and (3.1), inequalities (3.2a) hold for $k = 0$. Suppose that inequalities (3.2a) hold for $k \geq 0$. Then we have

$$\begin{aligned}
&\|((\mathrm{ad}\,\Lambda)L)\, u\|_k \\
&\quad = \|\Lambda((\mathrm{ad}\,\Lambda)L)\, u\|_{k-1} \\
&\quad \leq \|((\mathrm{ad}\,\Lambda)L)\Lambda u\|_{k-1} + \|((\mathrm{ad}\,\Lambda)^2 L)\, u\|_{k-1} \\
&\quad \leq \delta\|L\Lambda u\|_k + C_{N+1,\delta,k}\|u\|_{-N} + \|((\mathrm{ad}\,\Lambda)^2 L)\, u\|_{k-1}.
\end{aligned} \tag{3.3}$$

Remark that

$$\begin{aligned}
&\|((\mathrm{ad}\,\Lambda)^2 L)\, u\|_{k-1} \\
&\quad \leq \|((\mathrm{ad}\,\Lambda)^2 L)\Lambda u\|_{k-2} + \|((\mathrm{ad}\,\Lambda)^3 L)\, u\|_{k-2} \\
&\quad \leq \|((\mathrm{ad}\,\Lambda)^2 L)\Lambda u\|_{k-2} + B_k\|u\|_k \\
&\quad \leq \|((\mathrm{ad}\,\Lambda)^2 L)\Lambda u\|_{k-2} + B_k(C_k\|Lu\|_k + D_{N,k}\|u\|_{-N}),
\end{aligned}$$

with some constant B_k, because $(\mathrm{ad}\,\Lambda)^3 L$ is an operator of order 2. Applying induction hypothesis (3.2c) to the first term in the right hand side and interpolation inequality to $\|Lu\|_k$, we have

$$
\begin{aligned}
&\|((\operatorname{ad}\Lambda)^2 L)\, u\|_{k-1} \\
&\quad \le \delta\|L\Lambda u\|_k + \delta\|Lu\|_{k+1} + C'_{N,\delta,k}\|u\|_{-N} \\
&\quad \le 2\delta\|Lu\|_{k+1} + \delta\|((\operatorname{ad}\Lambda)L)\, u\|_k + C'_{N,\delta,k}\|u\|_{-N}.
\end{aligned} \tag{3.4}
$$

Substituting (3.4) into (3.3), we obtain

$$
\|((\operatorname{ad}\Lambda)L)\, u\|_k \le \frac{3\delta}{1-2\delta}\|Lu\|_{k+1} + C''_{N+1,\delta,k}\|u\|_{-N}.
$$

Hence we have the desired inequality

$$
\|((\operatorname{ad}\Lambda)L)\, u\|_k \le \delta\|Lu\|_{k+1} + C_{N,\delta,k+1}\|u\|_{-N}. \tag{3.5}
$$

Inserting (3.5) to (3.4), we have the desired one for $\|((\operatorname{ad}\Lambda)^2 L)\, u\|_{k-1}$. Finally, we compute

$$
\begin{aligned}
\|u\|_{k+1} &= \|\Lambda u\|_k \\
&\le C_k\|L\Lambda u\|_k + D_{N+1,k}\|u\|_{-N} \\
&\le C_k\|Lu\|_{k+1} + C_k\|((\operatorname{ad}\Lambda)L)\, u\|_k + D_{N+1,k}\|u\|_{-N}.
\end{aligned}
$$

Applying (3.5) to the second term, we have the desired inequality for $\|u\|_{k+1}$.

For negative integer we use another induction. Suppose that inequalities (3.2a) hold for $k \le 0$. Using $[\Lambda^{-1}, L] = -\Lambda^{-1}((\operatorname{ad}\Lambda)L)\Lambda^{-1}$, from (3.2b) we have

$$
\begin{aligned}
\|[\Lambda^{-1}, L]u\|_k &= \|((\operatorname{ad}\Lambda)L)\Lambda^{-1}u\|_{k-1} \\
&\le \delta\|L\Lambda^{-1}u\|_k + C_{N-1,\delta,k}\|u\|_{-N} \\
&\le \delta\|Lu\|_{k-1} + \delta\|[\Lambda^{-1}, L]u\|_k + C_{N-1,\delta,k}\|u\|_{-N}.
\end{aligned}
$$

Taking $\delta = 1/2$, we have

$$
\|[\Lambda^{-1}, L]u\|_k \le \|Lu\|_{k-1} + D'_{N,k}\|u\|_{-N}. \tag{3.6}
$$

Using (3.2a) to $\Lambda^{-1}u$, we have

$$
\begin{aligned}
\|u\|_{k-1} &\le C_k\|L\Lambda^{-1}u\|_k + D_{N-1,k}\|u\|_{-N} \\
&\le C_k\|Lu\|_{k-1} + C_k\|[\Lambda^{-1}, L]u\|_k + D_{N-1,k}\|u\|_{-N}.
\end{aligned}
$$

Substituting (3.6) to the second term, we see

$$
\|u\|_{k-1} \le C_{k-1}\|Lu\|_{k-1} + D_{N,k-1}\|u\|_{-N}. \tag{3.7}
$$

Next using $[\Lambda^{-1}, (\operatorname{ad}\Lambda)L] = -\Lambda^{-1}((\operatorname{ad}\Lambda)^2 L)\Lambda^{-1}$ we have

$$\begin{aligned}\|((\operatorname{ad}\Lambda)L)u\|_{k-2} &\le \|((\operatorname{ad}\Lambda)L)\Lambda^{-1}u\|_{k-1} + \|[\Lambda^{-1}, (\operatorname{ad}\Lambda)L]u\|_{k-1} \\ &\le \|((\operatorname{ad}\Lambda)L)\Lambda^{-1}u\|_{k-1} + \|((\operatorname{ad}\Lambda)^2 L)\Lambda^{-1}u\|_{k-2}.\end{aligned}$$

Applying induction hypothesis (3.2b), (3.2c) and then (3.6) to the right hand side, we obtain the desired one for $\|((\operatorname{ad}\Lambda)L)u\|_{k-2}$:

$$\|((\operatorname{ad}\Lambda)L)u\|_{k-2} \le \delta\|Lu\|_{k-1} + C_{N,\delta,k-1}\|u\|_{-N}. \tag{3.8}$$

Finally we compute $\|((\operatorname{ad}\Lambda)^2 L)u\|_{k-3}$ as follow:

$$\begin{aligned}&\|((\operatorname{ad}\Lambda)^2 L)u\|_{k-3} \\ &\quad\le \|((\operatorname{ad}\Lambda)^2 L)\Lambda^{-1}u\|_{k-2} + \|\Lambda^{-1}((\operatorname{ad}\Lambda)^3 L)\Lambda^{-1}u\|_{k-2} \\ &\quad\le \delta\|L\Lambda^{-1}u\|_k + C_{N-1,\delta,k}\|u\|_{-N} + B_k\|u\|_{k-2},\end{aligned}$$

because $\Lambda^{-1}((\operatorname{ad}\Lambda)^3 L)\Lambda^{-1}$ is an operator of order 0. Using (3.6) and the general interpolation inequality

$$\|u\|_{k-2} \le \epsilon\|u\|_{k-1} + C'_{N,\epsilon,k}\|u\|_{-N} \quad \text{for any} \ \ \epsilon > 0, \tag{3.9}$$

we see that

$$\|((\operatorname{ad}\Lambda)^2 L)u\|_{k-3} \le \delta\|Lu\|_{k-1} + C_{N,\delta,k-1}\|u\|_{-N}.$$

This completes the induction for negative integers. □

The trick of obtaining global hypoellipticity is as follows: Setting $\Lambda_\epsilon = I + \epsilon\Lambda$, we use the operator Λ_ϵ^{-n} for a positive integer n as a mollifier. To compute the commutator $[\Lambda_\epsilon^{-n}, L]$, it is useful to remark that

$$[\Lambda_\epsilon^{-n}, L] = \sum_{\ell=1}^{n}(-1)^\ell \binom{n}{\ell} \Lambda_\epsilon^{-\ell}((\operatorname{ad}\Lambda_\epsilon)^\ell L)\Lambda_\epsilon^{-n} \tag{3.10}$$

holds for every positive integer n, and that $(\operatorname{ad}\Lambda_\epsilon)L = \epsilon\,(\operatorname{ad}\Lambda)L$. Using this we show the following theorem.

Theorem 3.3 *Let L be a differential operator on a closed manifold M of order 2. If there is a regulator Λ such that a system of inequalities* (2.6) *and* (3.1) *holds for L, then L is globally hypoelliptic on M.*

Proof. First we show that for every $\epsilon > 0$, every positive integers k, n and a $u \in H^{k+2-n}(M)$ there exists a positive constant $D_{N,k,n}$ independent of ϵ such that

$$\|[\Lambda_\epsilon^{-n}, L]u\|_k \le \frac{1}{2}\|L\Lambda_\epsilon^{-n}u\|_k + D_{N,k,n}\|\Lambda_\epsilon^{-n}u\|_{-N}. \tag{3.11}$$

To see this, we remark first that (3.10) gives for $u \in H^{k+2-n}(M)$

$$\| \left[\Lambda_\epsilon^{-n}, L\right] u\|_k \le \sum_{\ell=1}^{n} \binom{n}{\ell} \|\Lambda_\epsilon^{-\ell}\epsilon^\ell((\mathrm{ad}\,\Lambda)^\ell L)\Lambda_\epsilon^{-n}u\|_k. \tag{3.12}$$

Remark that for a fixed $v \in H^k(M)$, there is a constant B_v depending on v but independent of ϵ such that

$$\|\Lambda_\epsilon^{-\ell}v\|_k \le B_v\|v\|_{k-\ell}. \tag{3.13}$$

Using (3.13), we see if $u \in H^{k+2-n}(M)$ is fixed, then there is a constant C_k independent of ϵ such that

$$\| \left[\Lambda_\epsilon^{-n}, L\right] u\|_k \le C_k \sum_{\ell=1}^{n} \epsilon^\ell \|((\mathrm{ad}\,\Lambda)^\ell L)\Lambda_\epsilon^{-n}u\|_{k-\ell}.$$

Using (3.2b) and (3.2c) for $\ell = 1, 2$, and using that $(\mathrm{ad}\,\Lambda_\epsilon)^\ell L$ is an operator of order 2 for any $\ell \ge 3$, we have the following: For every $\delta > 0$ and for sufficiently large N, there is a constant $D_{N,k,n}$ independent of ϵ such that

$$\| \left[\Lambda_\epsilon^{-n}, L\right] u\|_k \le \delta\|L\Lambda_\epsilon^{-n}u\|_k + D_{N,k,n}\|\Lambda_\epsilon^{-n}u\|_{-N}.$$

Here we have used the interpolation inequality (3.9). This implies (3.11). Now the theorem follows from the next Lemma 3.4. $\square$

Lemma 3.4 *Suppose that L satisfies for every integer $k > 0$ and for a sufficiently large N,*

$$\|u\|_k \le C_k\|Lu\|_k + D_{N,k}\|u\|_{-N} \quad \textit{for} \quad u \in H^{k+2}(M), \tag{3.14}$$

and suppose that for every positive integers k, n and for $u \in H^{k+2-n}(M)$, the following inequality holds:

$$\|[\Lambda_\epsilon^{-n}, L]u\|_k \le \frac{1}{2}\|L\Lambda_\epsilon^{-n}u\|_k + D_{N,k,n}\|\Lambda_\epsilon^{-n}u\|_{-N}, \tag{3.15}$$

where $D_{N,k,n}$ does not depend on ϵ but may depend on u. If $u \in H^{-N}(M)$ and $Lu \in C^\infty(M)$, then $u \in C^\infty(M)$.

Proof. Suppose $u \in H^{-N}(M)$. By setting $n = k + N + 2$, we have $\Lambda_\epsilon^{-n}u \in H^{k+2}(M)$. Note that applying (3.15) to the right hand side of the following inequality

$$\|L\Lambda_\epsilon^{-n}u\|_k \le \|\Lambda_\epsilon^{-n}Lu\|_k + \|\left[\Lambda_\epsilon^{-n}, L\right] u\|_k,$$

we have

$$\|L\Lambda_\epsilon^{-n}u\|_k \le 2\|\Lambda_\epsilon^{-n}Lu\|_k + 2D_{N,k,n}\|\Lambda_\epsilon^{-n}u\|_{-N}, \tag{3.16a}$$

$$\|\left[\Lambda_\epsilon^{-n}, L\right] u\|_k \le \|\Lambda_\epsilon^{-n}Lu\|_k + 2D_{N,k,n}\|\Lambda_\epsilon^{-n}u\|_{-N}. \tag{3.16b}$$

Using (3.14) for $\Lambda_\epsilon^{-n}u \in H^{k+2}(M)$, we see

$$\begin{aligned}
&\|\Lambda_\epsilon^{-n}u\|_k \\
&\le C_k\|L\Lambda_\epsilon^{-n}u\|_k + D_{N,k}\|\Lambda_\epsilon^{-N}u\|_{-N} \\
&\le C_k\|\Lambda_\epsilon^{-n}Lu\|_k + C_k\|\left[\Lambda_\epsilon^{-n}, L\right] u\|_k + D_{N,k}\|\Lambda_\epsilon^{-N}u\|_{-N}.
\end{aligned} \tag{3.17}$$

Substituting (3.16b) into the right hand side of (3.17), we obtain

$$\|\Lambda_\epsilon^{-n}u\|_k \le 2C_k\|\Lambda_\epsilon^{-n}Lu\|_k + D'_{N,k,n}\|\Lambda_\epsilon^{-n}u\|_{-N}. \tag{3.18}$$

Note that C_k and $D'_{N,k,n}$ do not depend on ϵ. Since $u \in H^{-N}(M)$, we see

$$\lim_{\epsilon\to 0}\|\Lambda_\epsilon^{-n}u\|_{-N} = \|u\|_{-N} < \infty.$$

If $Lu \in C^\infty(M)$ in addition, then we have $\lim_{\epsilon\to 0}\|\Lambda_\epsilon^{-n}Lu\|_k = \|Lu\|_k < \infty$. From (3.18) these inequalities imply that $\|u\|_k = \lim_{\epsilon\to 0}\|\Lambda_\epsilon^{-n}u\|_k < \infty$, which shows $u \in H^k(M)$. Since k is an arbitrary integer, we have $u \in C^\infty(M)$. □

4. Global Hypoellipticity for Example 3

Since general theory for the global hypoellipticity is still hard to construct, we restrict our concern in this section to the case Example 3. We shall show in this section that (3.1) is obtained under the assumption that

$$\zeta(x) \ge 0, \qquad \eta(x,y) \ge 0. \tag{4.1}$$

The proof is given by showing the inequalities requested in Theorem 3.3.

The condition (4.1) gives for $|\beta| \le 2$ that

$$\partial^\beta\zeta^2(p) = 0 \ \text{ where } \ \zeta^2(p) = 0, \quad \partial^\beta\eta^2(p) = 0 \ \text{ where } \ \eta^2(p) = 0.$$

Let $Z_{\zeta,\eta} = \{p \in \mathbb{T}^3; \zeta(p) = \eta(p) = 0\}$ and W be an arbitrary open subset of $\mathbb{T}^3$ such that $\overline{W} \cap Z_{\zeta,\eta} = \emptyset$. Using the assumption that the supports of

ζ and η are mutually disjoint, we see easily that for every multi-index β, there exists $C_\beta > 0$ such that

$$|\partial^\beta \zeta^2| \le C_\beta |\zeta^2|, \quad |\partial^\beta \eta^2| \le C_\beta |\eta^2| \quad \text{on } W.$$

It follows for every β, $|\beta| \le 2$, that

$$\|L_{(\beta)} u\|_0 \le C \|Lu\|_0 \quad \text{for } u \in C_0^\infty(W), \tag{4.2}$$

where

$$L_{(\beta)} u = -(\partial^\beta \zeta^2)\partial_y^2 u - (\partial^\beta \eta^2)\partial_z^2 u.$$

Let Λ be the pseudo-differential operator with symbol $(|\xi|^2 + 1)^{1/2}$. We use Λ as a regulator. Since the symbol does not contain the variables of the base manifold, the product formula on $\mathbb{T}^3$ gives:

$$[L, \Lambda^s] - \sum_{1 \le |\gamma| \le N+2} C_\gamma (\Lambda^s)^{(\gamma)} L_{(\gamma)} \in \psi\mathrm{DO}^{s-N}, \tag{4.3}$$

where $\psi\mathrm{DO}^r$ is the space of all pseudo differential operators of order r and $(\Lambda^s)^{(\gamma)}$ is the one with symbol $\partial^\gamma((|\xi|^2+1)^{s/2})$. Remark that $(\Lambda^s)^{(\gamma)}$ is an operator of order $s - |\gamma|$. Using (4.2) and the interpolation inequality

$$\|v\|_{-1} \le \delta \|v\|_0 + C_N \|v\|_{-N},$$

we have for every positive δ and for β with $1 \le |\beta| \le 2$, that

$$\|L_{(\beta)} u\|_{-|\beta|} \le \delta \|Lu\|_0 + C_{\delta,N} \|u\|_{-N} \quad \text{for } u \in C_0^\infty(W). \tag{4.4}$$

On the other hand, following a little delicate estimate according to Amano [2] from (1.3) to (1.9), we have that for every $\delta > 0$ there exists a neighborhood U of $Z_{\zeta,\eta}$ such that

$$\sum_{|\beta|=1} \|L_{(\beta)} u\|_0 \le \delta(\|Lu\|_1 + \|u\|_1) + D\|u\|_{-N} \quad \text{for } u \in C_0^\infty(U). \tag{4.5}$$

For the case $|\beta| = 2$, since U is a small neighborhood of $Z_{\zeta,\eta}$, we can assume that the coefficients of $L_{(\beta)}$ are less than δ on U. Hence we see that

$$\|L_{(\beta)} u\|_0 \le \delta \|u\|_2 \quad \text{for } u \in C_0^\infty(U), \tag{4.6}$$

and since $[\Lambda^{-2}, L_{(\beta)}]$ is an operator of order -1, inequality (4.6) yields easily

$$\|L_{(\beta)} u\|_{-2} \le \delta \|u\|_0 + D\|u\|_{-N} \quad \text{for } u \in C_0^\infty(U). \tag{4.7}$$

On the other hand, we have

$$\|L_{(\beta)}u\|_{-1} \le \|L_{(\beta)}\Lambda^{-1}u\|_0 + \|[\Lambda^{-1}, L_{(\beta)}]\, u\|_0.$$

Using (4.5) to the first term and using (4.3) and (4.7) to the second term, we have that

$$\|L_{(\beta)}u\|_{-1} \le \delta(\|Lu\|_0 + \|u\|_0) + D\|u\|_{-N} \quad \text{for} \quad u \in C_0^\infty(U) \tag{4.8}$$

holds for every β such that $|\beta| = 1$. Here we have used the fact that $[L, \Lambda^{-1}]$ and $[L_{(\beta)}, \Lambda^{-1}]$ are operators of order 0. Hence we see that for every multi-index β such that $1 \le |\beta| \le 2$ and $\delta > 0$, there are estimates

$$\|L_{(\beta)}u\|_{-|\beta|} \le \delta(\|Lu\|_0 + \|u\|_0) + D\|u\|_{-N} \quad \text{for} \quad u \in C_0^\infty(U). \tag{4.9}$$

Suppose now that $U \cup W = \mathbb{T}^3$. For a smooth nonnegative function ϕ with $\operatorname{supp}\phi \subset U$ and $\phi = 1$ on $Z_{\zeta,\eta}$, we have from Lemma 1.5 of Amano [2]

$$\|\, [\phi, L]\, u\|_0 \le C(\|Lu\|_0 + \|u\|_0). \tag{4.10}$$

Using (4.2) and (4.10), we obtain from (4.4) and (4.9) for β, $1 \le |\beta| \le 2$

$$\|L_{(\beta)}u\|_{-|\beta|} \le \delta(\|Lu\|_0 + \|u\|_0) + D\|u\|_{-N} \quad \text{for} \quad u \in C^\infty(\mathbb{T}^3).$$

Now (2.6) implies

$$\|L_{(\beta)}u\|_{-|\beta|} \le 2\delta\|Lu\|_0 + D\|u\|_{-N} \qquad 1 \le |\beta| \le 2.$$

By the product formula (4.3), we have

$$\begin{aligned} \|((\operatorname{ad}\Lambda)L)u\|_{-1} &\le C \sum_{|\beta|=1} \|L_{(\beta)}u\|_{-1} + C'\|u\|_{-1} + D\|u\|_{-N}, \\ \|((\operatorname{ad}\Lambda)^{\ell}L)u\|_{-2} &\le C \sum_{|\beta|=2} \|L_{(\beta)}u\|_{-2} + C'\|u\|_{-1} + D\|u\|_{-N}. \end{aligned}$$

By these we obtain (3.1) and hence we have

Theorem 4.1 *Subelliptic operator L in Example 3 is globally hypoelliptic, if $\zeta \ge 0$ and $\eta \ge 0$.*

Though the proof of Theorem 4.1 depends essentially on the nonnegativity of the coefficients ζ and η, we have the following

Conjecture The differential operator L defined by (1.1) is globally hypoelliptic, if the group $\mathscr{H}$ acts transitively on M.

5. Global Hypoellipticity of Horizontal Laplacians

In this section we show that the criterion given in Section 3 can be applied to non-linear connections. Let (M, N, π) be a smooth fiber bundle with the projection $\pi : M \to N$ such that M is a closed manifold. Fix a smooth Riemannian metric on M. Let $\mathcal{H}_p$ be the orthogonal complement of the tangent space T_pF_p of the fiber F_p through p in the tangent space T_pM of M. The distribution $\{\mathcal{H}_p : p \in M\}$ gives a horizontal one of a (non-linear) connection on M.

Let $(\tilde{x}^1, \dots, \tilde{x}^m)$ be a local coordinate system of N on a neighborhood $\tilde{U}$ of $\pi(p)$ and let $(x^1, \dots, x^m, y^1, \dots, y^r)$ be one on a neighborhood U of p such that $x^i = \pi^*(\tilde{x}^i)$ and $(y^1, \dots, y^r)$ is one along fiber. Let $\partial_1, \dots, \partial_m$ be the coordinate frame field on a neighborhood of $\pi(p)$ in N. Let X_i be the horizontal lift of the vector field ∂_i. Using the local frame field

$$X_1, \dots, X_m,\ \partial_{y^1}, \dots, \partial_{y^r} \quad \text{on } U,$$

we construct an orthonormal local frame field

$$e_1, \dots, e_m,\ e_{m+1}, \dots, e_{m+r} \quad \text{on } U,$$

with $e_i(q) \in \mathcal{H}_q$, $1 \le i \le m$ and $e_{m+j}(q) \in T_qF_q$, $1 \le j \le r$ at every $q \in U$.

We define a differential operator $\Delta_{\mathcal{H}}$ on M by

$$\Delta_{\mathcal{H}} f := \sum_{i=1}^{m} {e_i}^* e_i\, f, \tag{5.1}$$

where ${e_i}^*$ is the formal adjoint operator of the vector field e_i viewed as a differential operator of order 1. For $f, g \in C_0^\infty(U)$, we have

$$\langle \Delta_{\mathcal{H}} f,\, g\rangle_0 = \sum_{i=1}^{m} \int_U (e_i f)(e_i g)\, d\mu.$$

Thus, we see $\Delta_{\mathcal{H}}$ is independent of the choice of orthonormal local frame fields and that $\Delta_{\mathcal{H}}$ is globally defined differential operator on M.

Let $g_{ij} = (X_i, X_j)$ and $h_{kl} = (\partial_{y^k}, \partial_{y^l})$, then

$$d\mu = \sqrt{g}\, dx_1 \cdots dx_m dy_1 \cdots dy_r, \qquad g = \det(g_{ij}) \det(h_{kl}).$$

Since e_i is written as a linear combination of $X_1, \dots, X_m$, we see that

$$\Delta_{\mathcal{H}} = \frac{1}{\sqrt{g}} \sum_{i,j} X_i \sqrt{g}\, g^{ij} X_j, \tag{5.2}$$

where g^{ij} is the inverse matrix of g_{ij}. We call $\Delta_{\mathcal{H}}$ the *horizontal Laplacian* on M.

Let $\mathfrak{h}$ be the Lie algebra generated by

$$C^\infty(U)X_1 \oplus \ldots \oplus C^\infty(U)X_m.$$

Let $\mathfrak{h}_p$ be the germ of $\mathfrak{h}$ at the point p. It is called the *infinitesimal holonomy Lie algebra* at p.

Let $\mathscr{H}$ be the minimal closed subgroup of the group of all C^∞ diffeomorphisms on M containing diffeomorphisms

$$\{\exp(\phi_U^1 X_1), \ldots, \exp(\phi_U^m X_m)\},$$

where U moves among all coordinate neighborhood on M and ϕ_U^i moves among all smooth functions such that $\operatorname{supp}\phi_U^i \subset U$. We call $\mathscr{H}$ the holonomy group of the horizontal distribution.

It can occur that although $\dim \mathfrak{h}_p(p)$ is less than $\dim M$ everywhere, the group $\mathscr{H}$ acts transitively on M.

Let $\mathfrak{C}$ be the Lie algebra of all infinitesimal partially conformal transformations on M. $\mathfrak{C}$ is called *infinitesimally transitive along fibers*, if $\dim \mathfrak{C}(p) = \dim F_p$ holds at every $p \in M$.

In this section we prove the following theorem:

Theorem 5.1 *Suppose that holonomy group $\mathscr{H}$ acts transitively on M and and that $\dim \mathfrak{C}(p) = \dim F_p$ holds at every $p \in M$. Then the horizontal Laplacian $\Delta_{\mathcal{H}}$ is globally hypoelliptic on M.*

Proof. Since $Y \in \mathfrak{C}$ is divergent free, we have $Y^* = -Y$. Since M is compact, there is a finite number of elements $Y_1, \ldots, Y_s \in \mathfrak{C}$ such that $Y_1(p), \ldots, Y_s(p)$ span F_p at every $p \in M$. It follows that $\Delta_{\mathcal{H}} - \sum Y_i^2$ is an elliptic operator, which is positive semi-definite, and hence we can use $\Lambda = (\Delta_{\mathcal{H}} - \sum Y_i^2 + 1)^{1/2}$ as a regulator. Since the holonomy group acts transitively on M, Corollary 2.4 gives (2.6). Thus, we have only to show the inequalities (3.1).

To obtain (3.1), we remark first that

$$\|\,[Y_i, \Delta_{\mathcal{H}}]\,u\|_s = \|f_{Y_i}\Delta_{\mathcal{H}}u\|_s \leq C_s\|\Delta_{\mathcal{H}}u\|_s \quad s \in \mathbb{R}. \tag{5.3}$$

Hence, we have for each i

$$\|[Y_i^2, \Delta_{\mathcal{H}}]u\|_{s-1} \leq C\|\Delta_{\mathcal{H}}u\|_s,$$

and hence for $A = \Delta_{\mathcal{H}} - \sum Y_i^2 + 1$ that

$$\| [A, \Delta_{\mathcal{H}}]\, u\|_{s-1} \le C\|\Delta_{\mathcal{H}} u\|_s.$$

Using induction, we have for every non-negative integer k that

$$\|[A^k, \Delta_{\mathcal{H}}]u\|_s \le C\|\Delta_{\mathcal{H}} u\|_{s+2k-1}. \tag{5.4}$$

We apply the following identity to the case $A = \Delta_{\mathcal{H}} - \sum Y_i^2 + 1$, and $B = \Delta_{\mathcal{H}}$:

$$\begin{aligned}[\sqrt{A}, B] = \sum_{k=0}^{N+1} \left(\frac{1}{2\sqrt{A}}\mathrm{ad}(\sqrt{A})\right)^k \frac{1}{2\sqrt{A}}\,[A, B] \\ + \left(\frac{1}{2\sqrt{A}}\mathrm{ad}(\sqrt{A})\right)^{N+2} [\sqrt{A}, B]\end{aligned}$$

Since the remainder term is of order $-N$, we have only to estimate the first term:

$$\begin{aligned}&\left\|\left(\frac{1}{2\sqrt{A}}\mathrm{ad}(\sqrt{A})\right)^k \frac{1}{2\sqrt{A}}[A, \Delta_{\mathcal{H}}]u\right\|_{-1} \\ &\quad\le C\|(\mathrm{ad}(\sqrt{A}))^k[A, \Delta_{\mathcal{H}}]u\|_{-2-k} \\ &\quad\le C'\sum_{l=0}^{k} \| [A, \Delta_{\mathcal{H}}]\, \sqrt{A}^{k-l} u\|_{-2-k+l} \\ &\quad\le C''\sum_{l=0}^{k} \|\Delta_{\mathcal{H}}\sqrt{A}^{k-l} u\|_{-1-k+l}.\end{aligned}$$

By using (5.4) for the case when $k - l$ is even and odd separately, the last side is estimated by

$$\le C\|\Delta_{\mathcal{H}} u\|_{-1} + C'\|[\sqrt{A}, \Delta_{\mathcal{H}}]u\|_{-2}.$$

It follows that

$$\|[\sqrt{A}, \Delta_{\mathcal{H}}]u\|_{-1} \le C\|\Delta_{\mathcal{H}} u\|_{-1} + C'\|[\sqrt{A}, \Delta_{\mathcal{H}}]u\|_{-2} + D\|u\|_{-N}.$$

Using the interpolation inequality to the second term of the right hand side, we have

$$\|[\sqrt{A}, \Delta_{\mathcal{H}}]u\|_{-1} \le C\|\Delta_{\mathcal{H}} u\|_{-1} + D\|u\|_{-N}. \tag{5.5}$$

By a similar estimate, we have easily

$$\left\|\left[\sqrt{A},[\sqrt{A},\Delta_{\mathcal{H}}]\right]u\right\|_{-2} \leq C\|\Delta_{\mathcal{H}}u\|_{-1} + D\|u\|_{-N}. \tag{5.6}$$

Applying the interpolation inequality to $\|\Delta_{\mathcal{H}}u\|_{-1}$ in (5.5) and (5.6), we have the desired inequalities (3.1). □

If (M,N,π) is a G-principal bundle with the compact group G, and the connection is G-invariant (G-connection), then $\Delta_{\mathcal{H}}$ commute with every element Y of the Lie algebra $\mathfrak{g}$ of G viewed as a vector field on M which tangents to the fiber. Hence for a linear basis $Y_1,\ldots,Y_r$ of $\mathfrak{g}$, the operator $\Delta_{\mathcal{H}} + \sum_i {Y_i}^* Y_i$ is elliptic and satisfies $[\Delta_{\mathcal{H}}, \Delta_{\mathcal{H}} + \sum_i {Y_i}^* Y_i] = 0$. Thus, we have no need to show the above inequalities in this case.

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