Hokkaido Mathematical Journal Vol. 28 (1999) p. 507–514

Regularity up to the boundary for the $\bar{\partial}$ complex

Giuseppe ZAMPIERI

(Received May 7, 1998; Revised October 12, 1998)

Abstract. We introduce a condition of *q-pseudoconvexity* for a domain W of $\mathbb{C}^N$, and prove that it is sufficient for solvability of the $\bar{\partial}$-complex over (antiholomorphic) forms of degree $\geq q+1$ with smooth coefficients up to the boundary. Our method applies to *wedges* of $\mathbb{C}^N$ and therefore it provides a useful tool to solve the tangential $\bar{\partial}$ system on real submanifolds of $\mathbb{C}^N$. The proof is very elementary. It consists in a variant of the L^2-estimates by Hörmander [4], [5] (in the non coordinate-free version) which permits a straight application of the method by Dufresnoy [2]. The plan of the paper is as follows.

In §1 we introduce generalized pseudoconvexity ((1.1) and (1.2)), prove that it can be formulated equivalently for defining functions of ∂W or exhaustion functions of W, and state our main result on solvability of $\bar{\partial}$ for forms with coefficients in $\mathbb{C}^\infty(\bar{W})$. In §2 we give the variant of the L^2 estimates by [4], [5] which fits our condition. It consists in a partial use of the commutation relations of [5, formula (4.2.6)], so that the terms involved in our condition (1.1), instead of the full Levi form, are obtained. The rest is just routine. The above estimates first entail existence in L^2 spaces with *universal* weight (i.e. independent of W) for $\bar{\partial}$, and then C^∞ regularity up to the boundary for $(\bar{\partial}, \bar{\partial}^*)$.

We aim to develop and refine our statements in our forthcoming paper [10].

Key words: q-convexity q-concavity, $\bar{\partial}$ and $\bar{\partial}_b$ Neumann problems.

1. Statement of the result

Let W_h, $h = 1, \ldots, m$ be C^2 half-spaces in a neighborhood of a point z_o in $\mathbb{C}^N$, with transversal boundaries $M_h = \partial W_h$, and let $W = \bigcap_{h=1,\ldots,m} W_h$. We assume $\bigcap_{h=1,\ldots,m} M_h$ *generic*, and set $\hat{M}_h := M_h \cap \partial W$, $N := \bigcup_{h \neq k} \hat{M}_h \cap \hat{M}_k$. For multiindices $J = (j_1, \ldots, j_k)$, we shall deal with vectors $w = (w_J)$ with complex alternate coefficients. We shall consider defining functions r_h for W_h (i.e. $W = \{r_h < 0\}$ with $\partial r_h \neq 0$). We assume there are positive integers a and q and local coordinates $z = x + iy$ on $\mathbb{C}^N$ at z_o such that

$$\sum'_{|K|=a+q} \sum_{i \text{ or } j \geq q+1} \bar{\partial}_j \partial_i r_h(z) \bar{w}_{jK} w_{iK} \geq 0$$
$$\forall z \in \hat{M}_h \quad \forall z \text{ close to } z_o \quad \forall (w_{iK})_i \in \partial r_h(z)^\perp. \qquad (1.1)$$

1991 Mathematics Subject Classification : 32F10, 32F20.

(Here $\sum'$ denotes summation over ordered indices and $\perp$ indicates the (complex) orthogonal.) Particular emphasis shall be put in the case $a=0$. If $\partial' := (\partial_1,\dots,\partial_q)$ we are also assuming that Span $\partial' \subset \partial r_h(z)^\perp$ $\forall z \in \hat{M}_h$. Let $\partial'' = \sum_{q+1}^N a''_j(z)\partial_j$ $\left(a'' = (a''^h_j)_{\substack{j=q+1\cdots N\\ h=q+1\cdots N-1}}\right)$ be the orthogonal completion of ∂' in $T^{(1,0)}M_h$; then a sufficient condition for (1.1) with $a=0$ is clearly:

$$\bar\partial'\partial'' r_h(z) = 0 \quad \bar\partial''\partial'' r_h \geq 0 \quad \forall z \in \hat{M}_h \text{ close to } z_o. \tag{1.2}$$

Note that both (1.1) and (1.2) are independent of the choice of the defining functions r_h. The other extremal case is when $q=0$. To treat it, let $\mu_1^h \leq \mu_2^h \leq \dots$ denote the eigenvalues of $\bar\partial\partial r_h|_{\partial r_h^\perp}$.

Proposition 1.1 (1.1) *for* $q=0$ *is equivalent to*

$$\sum_{j=1,\dots,a+1} \mu_j^h \geq 0 \;\forall h. \tag{1.3}$$

Proof. It is a general fact that

$$\sum_{|K|=a}{}' \sum_{ij=1,\dots,N} \bar\partial_j\partial_i r_h \bar w_{jK} w_{iK} \geq \Big(\sum_{j=1\dots a+1} \mu_j^h\Big)|w|^2, \tag{1.4}$$

(which proves that (1.3) implies (1.1)). Moreover when $\bar\partial_j\partial_i r_h|_{\partial r_h^\perp}$ is diagonal, and $w = (w_1 \cdots a+1)$, then (1.4) becomes equality (which proves that (1.1) implies (1.3)). $\square$

We represent now $\hat{M}_h$ as a graph $x_1 = g_h$, and ∂W as $x_1 = g$. We put $r := -x_1 + g$, $\delta := -r$, $\phi = -\log\delta + c|z|^2$. Let $S = \{z : g_h = g_k \text{ for } h \neq k\}$. This is a manifold (because the M_h's intersect transversally) with conormals $\pm n = \frac{\pm\partial(g_h - g_k)}{|\partial(g_h-g_k)|}$. Denote by $J(\cdot)$ the *jump* between the h's and k's side of S. We have

$$+n = \frac{J(\partial r)}{|J(\partial r)|} = \frac{J(\partial\phi)}{|J(\partial\phi)|}. \tag{1.5}$$

It is also clear that

$$\partial'|_S \subset T^{\mathbb{C}}S. \tag{1.6}$$

Proposition 1.2 *Assume* (1.1). *Then there is a defining function* r *of* W *such that if we set* $\phi = -\log\delta + c|z|^2$ $(\delta := -r)$, *for suitable* c, *we obtain* *an exhaustion function of W at z_o such that for some λ ($\lambda(z) > 0$ $z \in W$) and for any $k \geq q + a + 1$:*

$$\sum_{|K|=k-1}' \sum_{i \text{ or } j \geq q+1} \bar{\partial}_j \partial_i \phi(z) \bar{w}_{jK} w_{iK} \geq \lambda |w|^2$$

$$\forall z \in W \setminus S \ \text{close to } z_o. \qquad (1.7)$$

Proof. One of the problems here is that in (1.1) z ranges in ∂W whereas in (1.7) it ranges through W. Recall the functions $r = -x_1 + g$, $r_h = -x_1 + g_h$, and the surface $S = \{z : g_h = g_k \text{ for } h \neq k\}$. We then consider the local foliation $W = \bigcup_\epsilon M_\epsilon$ (where $M_\epsilon = \{r = -\epsilon\}$). Let z, z^* be two points in $M_\epsilon \setminus S$ and M_0 ($= \partial W$) respectively with the same (y_1, z')-components. We have

$$T_z M_\epsilon = T_{z^*} M_0 \quad \bar{\partial}\partial r(z) = \bar{\partial}\partial r(z^*).$$

Under our choice of r, (1.1) holds for any $z \in W \setminus S$ (not only $z \in \partial W$). We then put $\phi = -\log(-r) + c|z|^2$. We have $\forall K$:

$$\bar{\partial}\partial\phi(\bar{w}_{\cdot K}, w_{\cdot K}) = r^{-2} \partial r w_{\cdot K} \bar{\partial} r \bar{w}_{\cdot K} - r^{-1} \bar{\partial}\partial r(\bar{w}_{\cdot K}, w_{\cdot K}) + c|w_{\cdot K}|^2. \qquad (1.8)$$

When $w_{\cdot K} \perp \partial r$, then the first term on the right of (1.8) vanishes whereas for the second (1.1) applies ($\forall z \in W$). Observe here that any $|J| \geq q + 1$ can be written, up to order, as $J = iK$ for $i \geq q + 1$, $|K| = k - 1$. Then (1.7) follows.

In the general case, let $w^\tau_{\cdot K}$ (resp. $w^\nu_{\cdot K}$) be the component of $w_{\cdot K}$ orthogonal (resp. parallel) to ∂r. We have

$$\begin{aligned} &\sum_{|K|=k-1}' \sum_{i \text{ or } j \geq q+1} \bar{\partial}_j \partial_i \phi \bar{w}_{jK} w_{iK} \\ &\geq \sum_{|K|=k-1}' \Big(- \sum_{i \text{ or } j \geq q+1} r^{-1} \bar{\partial}_j \partial_i r \bar{w}^\tau_{jK} w^\tau_{iK} \Big) \\ &\quad + \sum_{|K|=k-1}' \Big(\frac{r^{-2}}{2} |w^\nu_{\cdot K}|^2 + c \sum_{i \geq q+1} |w_{iK}|^2 - b r^{-1} |w^\tau_{\cdot K}| |w^\nu_{\cdot K}| \Big). \end{aligned} \qquad (1.9)$$

The first term on the right of (1.9) is positive by assumption, while the second is positive for suitable $c = c_b$. (We are using again here the fact that any $|J| \geq q+1$ can be written as $J = iK$ for $i \geq q+1$.) Then (1.7) easily follows. $\square$

We are ready to state the main theorem of the paper

Theorem 1.3 *Assume* (1.1). *Then there is a fundamental system of neighborhoods* $\{U\}$ *of* z_o *such that for any* $\bar\partial$*-closed form* $f = \sum'_{|J|=k} f_J d\bar z_J$ *of degree* $k \geq \max(a,q)+1$ *and with coefficients in* $C^\infty(\overline{W \cap U})$, *there is a form* $u = \sum'_{|K|=k-1} u_K d\bar z_K$ *with coefficients in* $C^\infty(\overline{W \cap U})$ *which solves* $\bar\partial u = f$.

2. L^2 estimates and proof of Theorem 1.3

We provide here the variant of the L^2 estimates by Hörmander [4], [5] which fits our condition (1.1). We shall then recall the sequence of arguments which yields the proof of Th. 1.3 in the line of [2]. Let W be a domain of $\mathbb{C}^N$ with C^2 boundary, and ϕ a real positive C^2 function on W. We denote by $L^2_\phi(W)$ the space of functions f such that $||f|| := \int_W e^{-\phi}|f|^2 dV$ is finite (where dV denotes the Euclidean element of volume). We denote by $L^2_\phi(W)^k$ the space of antiholomorphic forms $f = \sum'_{|J|=k} f_J d\bar z_J$ with $L^2_\phi(W)$ coefficients. We consider the sequence of closed densely defined operators

$$L^2_\phi(W)^{k-1} \overset{\bar\partial}{\to} L^2_\phi(W)^k \overset{\bar\partial}{\to} L^2_\phi(W)^{k+1}, \tag{2.1}$$

and denote by $\bar\partial^*$ the adjoint operators. Let δ_i be the operator (on functions) defined by $\delta_i(f_J) = e^\phi \partial_i(e^{-\phi} f_J)$. The following equality holds for any positive ϕ:

$$\begin{aligned}\sum'_{|K|=k-1} \sum_{ij=1,\dots,N} \int_W e^{-\phi}(\delta_i(f_{iK})\overline{\delta_j(f_{jK})} - \bar\partial_j(f_{iK})\overline{\bar\partial_i(f_{jK})}dV \\ + \sum'_{|J|=k} \sum_{j=1,\dots,N} \int_W e^{-\phi}|\bar\partial_j(f_J)|^2 dV = ||\bar\partial^* f||^2_\phi + ||\bar\partial f||^2_\phi \\ \forall f \in C^\infty_c(W)^k.\end{aligned} \tag{2.2}$$

Note that by the trivial choice $\phi = 0$, (2.2) gives

$$\sum'_{|J|=k} \sum_{j=1,\dots,N} ||\bar\partial_j f_J||^2 = ||\bar\partial^* f||^2_\phi + ||\bar\partial f||^2_\phi \quad \forall f \in C^\infty_c(W)^k, \tag{2.3}$$

where $||\cdot||$ is the norm in $L^2(W)$. ((2.3) will be used in the sequel as the main ingredient in proving the *ellipticity* of the system $(\bar\partial, \bar\partial^*)$.) Let us introduce now a new $\psi \geq 0$. Then (2.1) modifies to

$$L^2_{\phi-2\psi}(W)^{k-1} \overset{\bar\partial}{\to} L^2_{\phi-\psi}(W)^k \overset{\bar\partial}{\to} L^2_\phi(W)^{k+1}. \tag{2.4}$$

Let (I) be the term on the left side of (2.2). By introducing the new function ψ, (2.2) modifies to

$$(I) \leq 2||\bar\partial^* f||^2_{\phi-2\psi} + ||\bar\partial f||^2_\phi + 2||\,|\partial\psi| f||^2_\phi \quad \forall f \in C^\infty_c(W)^k. \tag{2.5}$$

Let $D_{\bar\partial}$ and $D_{\bar\partial^*}$ denote the domains in (2.4) of $\bar\partial$ and $\bar\partial^*$ respectively.

Proposition 2.1 *Assume* (1.7) $(\forall z \in W\backslash S)$ *and let* $k \geq \max(a+1, q+1)$. *Then we may find* ϕ *and* ψ *such that*

$$||f||^2_{\phi-\psi} \leq ||\bar\partial^* f||^2_{\phi-2\psi} + ||\bar\partial f||^2_\phi \quad \forall f \in D_{\bar\partial} \cap D_{\bar\partial^*}. \tag{2.6}$$

Moreover, for any fixed compact subset $C \subset\subset W$, *we can choose* $\psi|_C \equiv 0$ *and* $\phi|_C \equiv 2|z|^2$.

Proof. We choose ψ according to the density result [5, Lemma 4.1.3] (in particular $\psi|_C \equiv 0$). By this choice it shall be enough to prove (2.6) only on forms with $C^\infty_c(W)$ coefficients. We fix the coordinates in which (1.7) holds. We observe that the sum of the terms in (I) of (2.5) with both i and $j \leq q$ equals $||\bar\partial'^* f||^2_\phi + ||\bar\partial' f||^2_\phi$. In particular it is positive. We want to rewrite now those terms where either of i or j is $\geq q+1$. We recall that $\delta_i = -\bar\partial^*_i$ (for the inner product underlying to the $L^2_\phi(W)$ norm) and observe that

$$\delta_i \bar\partial_j - \bar\partial_j \delta_i = \bar\partial_j \partial_i \phi. \tag{2.7}$$

We also recall the notation n for the conormal to S. By (2.7) and by Stokes formula, we get

$$\begin{aligned}
&\sum'_{|K|=k-1} \sum_{i \text{ or } j \geq q+1} \cdot + \sum'_{|J|=k} \sum_{j \geq q+1} \cdot \\
&\quad = \sum'_{|K|=k-1} \sum_{i \text{ or } j \geq q+1} \int_W e^{-\phi} \bar\partial_j \partial_i \phi \bar f_{jK} f_{iK} \mathrm{d}V \\
&\quad\quad + \sum'_{|K|=k-1} \sum_{i \text{ or } j \geq q+1} \int_S e^{-\phi} \bar n_j n_i |J(\partial\phi)| \bar f_{jK} f_{iK} \mathrm{d}V.
\end{aligned} \tag{2.8}$$

Now since $n' = 0$ (by (1.5)) then in the last term in (2.8) we can extend the sum to all indices ij and conclude that it is positive (because it contains a square). Collecting all the previous remarks, we get

$$(I) \geq \sum_{|K|=k-1}{}' \sum_{i \text{ or } j \geq q+1} \int_W e^{-\phi} \bar{\partial}_j \partial_i \phi \bar{f}_{jK} f_{iK} \mathrm{d}V. \tag{2.9}$$

By combining (1.7), (2.5) and (2.9) we conclude:

$$\lambda ||f||^2_\phi \leq 2||\bar{\partial}^* f||^2_{\phi-2\psi} + ||\bar{\partial} f||^2_\phi + 2|| \, |\partial\psi| f||^2_\phi \quad \forall f \in C^\infty_c(W)^k.$$

Here we can take the same constant $\lambda = \lambda_C \; \forall f$ with $\operatorname{supp} f \subset C$. Note also that the subsets $C_t := \{z \in W : \phi(z) \leq t\}$ $t \in \mathbb{R}^+$ are an exhaustive family of compact of W. Thus with $\psi \equiv 0$ on C_c (c large), we replace ϕ by $\phi_1 = \chi(\phi) + 2|z|^2$ where χ has the properties: $\chi(t) \geq 0 \; \forall t$, $\chi(t) \equiv 0 \; \forall t \leq c$, $\chi'' \geq 0$, and finally

$$\chi'(t) \geq \frac{\sup_{C_t} 2(|\partial\psi|^2 + e^\psi)}{\lambda_{C_t}}. \tag{2.10}$$

Then (2.6) immediately follows for such a ϕ_1. □

End of proof of Theorem 1.3 We first prove existence in L^2, and then regularity in C^∞ for solutions of $(\bar{\partial}, \bar{\partial}^*)$. Finally we shall apply the technique by Dufresnoy. Let W be bounded and assume that in suitable coordinates, (1.7) holds $\forall z \in W \setminus S$. Then for any $f \in L^2_{2|z|^2}(W)^k$ with $\bar{\partial} f = 0$ there is $u \in L^2_{2|z|^2}(W)^{k-1}$ such that

$$(\bar{\partial} u = f, \bar{\partial}^* u = 0) \quad ||u||^2_{2|z|^2} \leq ||f||^2_{2|z|^2}. \tag{2.11}$$

This statement follows from Prop. 2.1 in the lines of [5, Lemma 4.4.1]. Let now $|| \cdot ||_{(s)}$ be the norm of the Sobolev space $H^s(W)$. Let $W_\epsilon = \{z \in W : \operatorname{dist}(z, \partial W) > \epsilon\}$. Let W be still bounded and (1.7) be fulfilled $\forall z \in W \setminus S$. Using (2.11) together with (2.3), we can in fact prove the following result on regularity of the solutions of the system $(\bar{\partial}, \bar{\partial}^*)$ (cf. [5, Th. 4.2.5]): For any $f \in C^\infty(W)^k$ with $\bar{\partial} f = 0$, there is $u \in C^\infty(W_\epsilon)^{k-1}$ such that $\forall s \geq 0$ and for suitable M_s we have

$$(\bar{\partial} u = f, \bar{\partial}^* u = 0) \quad ||u||_{(s+1)} \leq \frac{M_s}{\epsilon^{s+1}} ||f||_{(s)}. \tag{2.12}$$

We are ready to conclude. We aim to apply (2.12) to a sequence of domains $W_\nu \supset\supset W_{\nu+1} \supset\supset \cdots W$. We suppose W is defined locally by $r(= -x_1 + g) < 0$, and then define W_ν by $r < \frac{\eta^{2^\nu}}{2}$, for $0 < \eta < \frac{1}{2}$. Clearly we have in a neighborhood of z_o:

$$\{z \in \mathbb{C}^N : \operatorname{dist}(z, W) < \eta^{2^{\nu+1}}\}$$
$$\subset W_\nu \subset \left\{z \in \mathbb{C}^N : \operatorname{dist}(z, W) < \frac{\eta^{2^\nu}}{2}\right\}. \qquad (2.13)$$

We then observe that since the hypothesis of Th. 1.3 is local, whereas the techiques developped in the whole §2 are global, we need to replace W by $W \cap U$ and W_ν by $W_\nu \cap U$ for a system of neighborhoods U of z_o. We shall still use the notation W and W_ν instead of $W \cap U$ and $W_\nu \cap U$.

Let $f \in C^\infty(\bar{W})^k$ satisfy $\bar{\partial} f = 0$. Extend f to $\tilde{f}$ such that

$$||\bar{\partial}\tilde{f}||_{(s)} \leq M_{rs}\eta^{r2^\nu} \text{ on } W_\nu \text{ for any } r, s \text{ and for suitable } M_{rs}.$$

This is clearly possible because $\bar{\partial}\tilde{f} \equiv 0$ on W and $W_\nu \subset \{z : \operatorname{dist}(z, W) < \frac{\eta^{2^\nu}}{2}\}$. According to (2.12) there is a solution h_ν on $W_{\nu+1}$ of

$$\begin{cases} \bar{\partial} h_\nu = \bar{\partial}\tilde{f} \\ ||h_\nu||_{(s+1)} \leq M_s(\eta^{2^{\nu+1}})^{-s-1}||\bar{\partial}\tilde{f}||_{(s)}, \end{cases}$$

(due to $W_{\nu+1} \subset \{z : \operatorname{dist}(z, \partial W_\nu) > \frac{\eta^{2^{\nu+1}}}{2}\}$). Solve on W_2 the equation $\bar{\partial} g_1 = \tilde{f} - h_1$, and, inductively on $W_{\nu+2}$:

$$\bar{\partial} g_{\nu+1} = h_\nu - h_{\nu+1},$$

with the estimates

$$\begin{aligned} ||g_{\nu+1}||_{(s+2)} &\leq M_{s+1}(\eta^{2^{\nu+2}})^{-(s+2)}||h_\nu - h_{\nu+1}||_{(s+1)} \\ &\leq M'_s(\eta^{2^{\nu+2}})^{-2s-3}M_{rs}\eta^{r2^\nu} \\ &\leq M'_{rs}\frac{1}{2^\nu} \quad (r, \nu \text{ large}). \end{aligned}$$

Therefore $\sum_{\nu=1}^\infty g_\nu$ converges in $C^\infty(\bar{W})$ and solves on $\bar{W}$:

$$\bar{\partial}\Big(\sum_{\nu=1}^\infty g_\nu\Big) = \tilde{f} - \lim_\nu h_\nu = \tilde{f}.$$

□

References

[1] Andreotti A. and Grauert H., *Théorèmes de finitude pour la cohomologie des éspaces complexes.* Bull. Soc. Math. France **90** (1962), 193–259.

[2] Dufresnoy A., *Sur l'operateur $\bar{\partial}$ et les fonctions différentiables au sens de Whitney.* Ann. Inst. Fourier **29** (1) (1979), 229–238.

[3] Henkin G.M., *H. Lewy's equation and analysis on pseudoconvex manifolds* (Russian). I. Uspehi Mat. Nauk. **32** (3) (1977), 57–118.

[4] Hörmander L., *L^2 estimates and existence theorems for the $\bar{\partial}$ operator.* Acta Math. **113** (1965), 89–152.

[5] Hörmander L., *An introduction to complex analysis in several complex variables.* Van Nostrand, Princeton N.J. (1966).

[6] Kohn J.J., *Regularity at the boundary of the $\bar{\partial}$-Neumann problem.* Proc. Nat. Acad. Sci. U.S.A **49** (1963), 206–213.

[7] Michel V., *Sur la regularité C^∞ du $\bar{\partial}$ au bord d'un domaine de $\mathbb{C}^n$ dont la forme de Levi a exactement s valeurs propres strictement negatives.* Math. Ann. **195** (1993), 131–165.

[8] Tumanov A., *Extending CR functions on a manifold of finite type over a wedge.* Mat. Sb. **136** (1988), 129–140.

[9] Zampieri G., *L^2-estimates with Levi-singular weight, and existence for $\bar{\partial}$.* J. d'Analyse Math. **74** (1998), 331–344.

[10] Zampieri G., *q-psudoconvexity and regularity at the boundary for solutions of the $\bar{\partial}$-complex.* Preprint (1998).

Dip. Matematica, Università
v. Belzoni 7, Padova
Italy
E-mail: Zampieri@math.unipd.it