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# Boundedness of the multiple singular integral operators on product spaces*

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**Abstract.** In this paper, we consider the $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ boundedness for the multiple singular integral operators of Fefferman type, defined by

$$Tf(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}^m \times \mathbb{R}^n} h(|y_1|, |y_2|) \frac{\Omega(y_1', y_2')}{|y_1|^m |y_2|^n} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2,$$

where $y_1 \in \mathbb{R}^m$, $y_2 \in \mathbb{R}^n$ and $y_i' = y_i/|y_i|$, $h(r,s)$ is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$, $\Omega$ satisfies the cancellation condition

$$\int_{S^{m-1}} \Omega(y_1', y_2') dy_1' = \int_{S^{n-1}} \Omega(y_1', y_2') dy_2' = 0.$$

We show that if $\Omega \in L(\log^+ L)^2(S^{m-1} \times S^{n-1})$, then $T$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for all $1 < p < \infty$.

*Key words*: multiple singular integral operator, Fourier transform estimate, Littlewood-Paley theory.

## 1. Introduction and Statement of the Result

Let $h(r, s)$ be a bounded function on $\mathbb{R}_+ \times \mathbb{R}_+$, and $\Omega(y_1, y_2)$ a function defined on $S^{m-1} \times S^{n-1}$ $(m, n \geq 2)$ satisfying

$$\int_{S^{m-1}} \Omega(y_1', y_2') dy_1' = \int_{S^{n-1}} \Omega(y_1', y_2') dy_2' = 0, \tag{1}$$

where $S^{m-1}$ (resp. $S^{n-1}$) is the unit sphere of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). For $y \in \mathbb{R}^m$, let $y' = y/|y|$. Define the multiple singular integral operator

$$Tf(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}^m \times \mathbb{R}^n} h(|y_1|, |y_2|) \frac{\Omega(y_1', y_2')}{|y_1|^m |y_2|^n} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2. \tag{2}$$

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Operators of this type have been considered by many authors. For the special case of $h(r,s) = 1$, R. Fefferman [2] proved that if $\Omega$ satisfies appropriate regularity condition, then $T$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for all $1 < p < \infty$. By Fourier transform estimate and the Littlewood-Paley estimate, Duoandikoetxea [1] showed that for $\Omega \in L^q(\mathbb{R}^m \times \mathbb{R}^n)$, $T$ is a bounded operator on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < p < \infty$. In this paper, we give a weeker condition than others under which the bounded result on $L^p$ would hold. Our statement is as following.

**Theorem 1** *Let $\Omega$ and $h$ be the same as above. Suppose that $\Omega$ belongs to the space $L(\log^+ L)^2(S^{m-1} \times S^{n-1})$, then the operator $T$ defined by (2) is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for all $1 < p < \infty$.*

## 2. Proof of Theorem 1

We begin with some preliminary lemmas.

**Lemma 1** *Let $\Omega(y_1', y_2')$ be integrable on $S^{m-1} \times S^{n-1}$. Then the maximal operator*

$$M_\Omega f(x_1,x_2) = \sup_{r,s>0} r^{-m}s^{-n}\left|\int_{|y_1|<r,\,|y_2|<s} \frac{\Omega(y_1',y_2')}{|y_1|^m|y_2|^n} f(x_1-y_1,x_2-y_2)dy_1dy_2\right|$$

*is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for all $1 < p < \infty$ with bound $C_{n,m,p}\|\Omega\|_1$.*

This Lemma can be proved by the standard method of rotation of Calderón and Zygmund, see also [2, p. 885].

**Lemma 2** *Let $\{\sigma_{u,v}\}_{u,v\in\mathbb{Z}}$ be a sequence of Borel measures on $\mathbb{R}^m \times \mathbb{R}^n$ such that $\|\sigma_{u,v}\| \le 1$. Suppose that the maximal operator*

$$\sigma^* f(x_1,x_2) = \sup_{u,v} \left||\sigma_{u,v}| * f(x_1,x_2)\right|$$

*is bounded on $L^{p_0}(\mathbb{R}^m \times \mathbb{R}^n)$ for some $p_0$ with $1 < p_0 < \infty$. Then the inequality*

$$\left\|\Big(\sum_{u,v} |\sigma_{u,v} * g_{u,v}|^2\Big)^{1/2}\right\|_p \le C\left\|\Big(\sum_{u,v} |g_{u,v}|^2\Big)^{1/2}\right\|_p$$

*holds for all $p$ with $\left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{1}{2p_0}$.*

For the proof of Lemma 2, the readers see the analogous result in [1, p.198].

**Lemma 3** *Let $\Omega$ be a bounded function on $S^{m-1}\times S^{n-1}$. For $u,v\in\mathbb{Z}$, denote*

$$\begin{aligned}&K_{u,v}(x_1,x_2)\\&=|x_1|^{-m}|x_2|^{-n}h(|x_1|,|x_2|)\Omega(x_1',x_2')\chi_{\{2^u\le|x_1|\le 2^{u+1},\,2^v<|x_2|\le 2^{v+1}\}}(x_1,x_2).\end{aligned}$$

*Then there exist positive constants $C$ and $\varepsilon$ which are independent of $\Omega$, $u$ and $v$, such that for $\xi_1\in\mathbb{R}^m$, $\xi_2\in\mathbb{R}^n$, $|\xi_1|$, $|\xi_2|\ne 0$,*

$$|\widehat{K_{u,v}}(\xi_1,\xi_2)|\le C\|\Omega\|_\infty(|2^u\xi_1||2^v\xi_2|)^{-\varepsilon}. \tag{3}$$

*On the other hand, if $\Omega$ is integrable on $S^{m-1}\times S^{n-1}$ and satisfies the cancellation condition* (1)*, then*

$$|\widehat{K_{u,v}}(\xi_1,\xi_2)|\le C\|\Omega\|_1|2^u\xi_1||2^v\xi_2|, \tag{4}$$

*where $\widehat{K_{u,v}}$ is the Fourier transform of $K_{u,v}$.*

The proof of Lemma 3 is implied in [1, p.193–194].

**Remark** Let $\sigma_{u,v}=K_{u,v}(x_1,x_2)$ in Lamma 2. In the situation, by the estimates in (3) and (4), following the same proofs as in [3], we can prove the maximal operator $\sigma^*$ is bounded on $L^2$. By boot-strap method, we can obtain that the maximal operator $\sigma^*$ is bounded on $L^p$ for $1<p<\infty$ and Lemma 2 holds for all $1<p<\infty$.

*Proof of Theorem* 1. Let $\phi^1$ and $\phi^2$ be two Schwarz functions on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, such that □

(a) $0\le\phi^1$, $\phi^2\le 1$, $\operatorname{supp}\phi^1\subset\{x\in\mathbb{R}^m,\ 1/2\le|x|\le 2\}$, $\operatorname{supp}\phi^2\subset\{y\in\mathbb{R}^n,\ 1/2\le|y|\le 2\}$;

(b) $\sum_{k=-\infty}^{\infty}\phi^1(2^kx)^2=\sum_{l=-\infty}^{\infty}\phi^2(2^ly)^2=1$ for $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$ such that $|x|$, $|y|\ne 0$.

Set $\phi_k^1(x)=\phi^1(2^kx)$ and $\phi_l^2(y)=\phi^2(2^ly)$. Define the operators $S_k^1$ in $\mathbb{R}^m$ and $S_l^2$ in $\mathbb{R}^n$ by

$$\widehat{S_k^1f}(\xi_1)=\phi_k^1(\xi_1)\hat f(\xi_1),\quad \widehat{S_l^2h}(\xi_2)=\phi_l^2(\xi_2)\hat h(\xi_2)$$

and $S_k^1 \otimes S_l^2$ in $\mathbb{R}^m \times \mathbb{R}^n$ by

$$(\widehat{S_k^1 \otimes S_l^2 f})(\xi_1, \xi_2) = \phi_k^1(\xi_1)\phi_l^2(\xi_2)\widehat{f}(\xi_1, \xi_2).$$

For fixed $k, l \in \mathbb{Z}$ and $\sigma_{u,v}$ as in the remark as above, denote by $U_{k,l}$ the operator defined by

$$U_{k,l}f(x_1, x_2) = \sum_{u,v} S_{u-k}^1 \otimes S_{v-l}^2 \sigma_{u,v} * ((S_{u-k}^1 \otimes S_{v-l}^2)f)(x_1, x_2).$$

Lemma 1 and Remark via the Littlewood-Paley theory (see [5]) state that

$$\|U_{k,l}f\|_p \le C\|\Omega\|_1\|f\|_p, \quad 1 < p < \infty. \tag{5}$$

Decompose the operator $T$ as

$$\begin{aligned}
&Tf(x_1, x_2) \\
&\quad = \sum_{k,l}\sum_{u,v} S_{u-k}^1 \otimes S_{v-l}^2 \sigma_{i,j} * ((S_{u-k}^1 \otimes S_{v-l}^2)f)(x_1, x_2) \\
&\quad = \sum_{k,l\le 0} U_{k,l}f(x_1, x_2) + \sum_{k\le 0,\, l>0} U_{k,l}f(x_1, x_2) \\
&\qquad + \sum_{k>0,\, l\le 0} U_{k,l}f(x_1, x_2) + \sum_{k>0,\, l>0} U_{k,l}f(x_1, x_2) \\
&\quad = T_{\mathrm{I}}f(x_1, x_2) + T_{\mathrm{II}}f(x_1, x_2) + T_{\mathrm{III}}f(x_1, x_2) + T_{\mathrm{IV}}f(x_1, x_2).
\end{aligned}$$

By (5) together with Plancherel's theorem we see that

$$\begin{aligned}
&\|U_{k,l}f\|_2^2 \\
&\le C \sum_{u,v=-\infty}^{\infty} \left\|\sigma_{u,v} * ((S_{u-k}^1 \otimes S_{v-l}^2)f)\right\|_2^2 \\
&= \sum_{u,v=-\infty}^{\infty} \int_{\mathbb{R}^m \times \mathbb{R}^n} |\widehat{\sigma_{u,v}}(\xi_1, \xi_2)\hat{f}(\xi_1, \xi_2)\phi^1(2^{u-k}\xi_1)\phi^2(2^{v-l}\xi_2)|^2 d\xi_1 d\xi_2 \\
&\le C(2^k 2^l)^2 \sum_{u,v=-\infty}^{\infty} \|(S_{u-k}^1 \otimes S_{v-l}^2)f\|_2^2 \\
&\le C(2^k 2^l)^2 \|f\|_2^2.
\end{aligned} \tag{6}$$

Interpolation between the inequalities (5) and (6) shows that for $1 < p < \infty$, there exists a positive constant $\delta = \delta_p > 0$ such that

$$\|U_{k,l}f\|_p \le C2^{\delta l}2^{\delta k}\|f\|_p.$$

This in turn leads to the estimate

$$\|T_{\mathrm{I}}f\|_p \le C\|f\|_p \sum_{k<0} 2^{\delta k} \sum_{l<0} 2^{\delta l} \le C\|f\|_p, \quad 1 < p < \infty.$$

Now we turn our attention to $T_{\mathrm{IV}}$. Let $E_0 = \{(x_1', x_2') \in S^{m-1} \times S^{n-1}, |\Omega(x_1', x_2')| \le 1\}$ and $E_d = \{(x_1', x_2') \in S^{m-1} \times S^{n-1}, 2^{d-1} < |\Omega(x_1', x_2')| \le 2^d\}$ for positive integer $d$. Denote by $\Omega_d$ the restriction of $\Omega$ on $E_d$. Our assumption implies that $\sum_{d>0} d^2 2^d |E_d| < \infty$. Set

$$\begin{aligned}&\sigma_{u,v}^d(y_1, y_2)\\ &= h(|y_1|, |y_2|)|y_1|^{-m}|y_2|^{-n}\Omega_d(y_1, y_2)\chi_{\{2^u < |y_1| \le 2^{u+1}, 2^v < |y_2| \le 2^{v+1}\}}(y_1, y_2),\end{aligned}$$

and $U_{k,l}^d$ defined in the same way as that in the definition of $U_{k,l}$, but with $\sigma_{u,v}$ replacing by $\sigma_{u,v}^d$. Again by Lemma 1 and Lemma 2,

$$\|U_{k,l}^d f\|_p \le C\|\Omega_d\|_1 \|f\|_p, \quad 1 < p < \infty. \tag{7}$$

Let $N$ be a integer and $N > 2\varepsilon^{-1}$, where $\varepsilon$ is the positive constant in Lemma 3. Write

$$\begin{aligned}&T_{\mathrm{IV}}f(x_1, x_2)\\ &= \sum_{l>0}\sum_{k>0} U_{k,l}^0 f(x_1, x_2) + \sum_{d>0}\sum_{0<k\le Nd}\sum_{0<l\le Nd} U_{k,l}^d f(x_1, x_2)\\ &\quad + \sum_{d>0}\sum_{0<k\le Nd}\sum_{l>Nd} U_{k,l}^d f(x_1, x_2) + \sum_{d>0}\sum_{k>Nd}\sum_{l>0} U_{k,l}^d f(x_1, x_2)\\ &= T_{\mathrm{IV}}^0 f(x_1, x_2) + T_{\mathrm{IV}}^1 f(x_1, x_2) + T_{\mathrm{IV}}^2 f(x_1, x_2) + T_{\mathrm{IV}}^3 f(x_1, x_2)\end{aligned}$$

It follows from Lemma 3 that

$$\begin{aligned}&\|U_{k,l}^d f\|_2^2\\ &\le C \sum_{u,v=-\infty}^{\infty} \int_{\mathbb{R}^m \times \mathbb{R}^n} |\widehat{\sigma_{u,v}^d}(\xi_1, \xi_2)\hat{f}(\xi_1, \xi_2)\phi^1(2^{u-k}\xi_1)\phi^2(2^{v-l}\xi_2)|^2 d\xi_1 d\xi_2\\ &\le C(\|\Omega_d\|_\infty 2^{-\varepsilon k}2^{-\varepsilon l})^2 \sum_{u,v=-\infty}^{\infty} \|(S_{u-k}^1 \otimes S_{v-l}^2)f\|_2^2\\ &\le C(\|\Omega_d\|_\infty 2^{-\varepsilon k}2^{-\varepsilon l})^2 \|f\|_2^2.\end{aligned} \tag{8}$$

Combining the estimate (7) and (8) we thus have that for $1 < p < \infty$,

$$\|U^d_{k,l}f\|_p \le C\|\Omega^d\|^t_\infty 2^{-t\varepsilon k}2^{-t\varepsilon l}\|f\|_p,$$

where $0 < t < 1$ is a constant depending only on $p$. So,

$$\|T^0_{\mathrm{IV}}f\|_p \le C\sum_{k,l>0} 2^{-t\varepsilon k}2^{-t\varepsilon l}\|f\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

Similarly, we have

$$\begin{aligned}\|T^2_{\mathrm{IV}}f\|_p &\le C\sum_{d>0}\sum_{0<k\le Nd,\, l>Nd}\|U^d_{k,l}\|_p \\ &\le C\sum_{d>0}2^{td}\sum_{0<k<Nd,\, l>Nd}2^{-t\varepsilon k}2^{-t\varepsilon l}\|f\|_p \\ &\le C\sum_{d>0}2^{td}2^{-t\varepsilon Nd}\|f\|_p \le C\|f\|_p,\end{aligned}$$

and

$$\|T^3_{\mathrm{IV}}f\| \le C\|f\|_p.$$

On the other hand, it is easy to see that

$$\begin{aligned}\|T^1_{\mathrm{IV}}f\|_p &\le C\sum_{d>0}\|\Omega_d\|_1\sum_{0<k<Nd}\sum_{0<l<Nd}\|f\|_p \\ &\le C\sum_{d>0}d^2 2^d|E_d|\|f\|_p \le C\|f\|_p.\end{aligned}$$

It remains to estimate $T_{\mathrm{II}}$ and $T_{\mathrm{III}}$. We only consider $T_{\mathrm{II}}$, the other can be treated in the same way. Let $\widetilde{\Omega}(x_2') = \|\Omega(\cdot, x_2')\|_{L^1(S^{m-1})}$. Set $\widetilde{E_0} = \{x_2' \in S^{n-1},\ \widetilde{\Omega}(x_2') \le 1\}$, and $\widetilde{E_d} = \{x_2' \in S^{n-1},\ 2^{d-1} < \widetilde{\Omega}(x_2') \le 2^d\}$ for positive integer $d$. By Jensen's inequality, we see that $\widetilde{\Omega} \in L(\log^+ L)^2(S^{n-1})$ and so $\sum_{d>0} d^2 2^d|\widetilde{E_d}| < +\infty$. Denote by $\widetilde{\Omega_d}$ the restriction of $\Omega$ on $S^{m-1}\times\widetilde{E_d}$. Let

$$\widetilde{\sigma^d_{u,v}}(y_1, y_2) = h(|y_1|, |y_2|)|y_1|^{-m}|y_2|^{-n}\widetilde{\Omega_d}(y_1, y_2) \chi_{\{2^u<|y_1|\le 2^{u+1},\, 2^v<|y_2|\le 2^{v+1}\}}(y_1, y_2),$$

and $\widetilde{U^d_{k,l}}$ be defined in the same way as that in the definition of $U_{k,l}$, but with $\sigma_{u,v}$ replacing by $\widetilde{\sigma^d_{u,v}}$. We claim that

$$\left|\widehat{\widetilde{\sigma^d_{u,v}}}(\xi_1,\xi_2)\right| \le C\min\left\{2^d|2^u\xi_1||2^v\xi_2|^{-\varepsilon},\, 2^d|\widetilde{E_d}||2^u\xi_1|\right\}.$$

In fact, the estimate

$$\left|\widehat{\widetilde{\sigma^d_{u,v}}}(\xi_1,\xi_2)\right| \le C2^d|\widetilde{E_d}||2^u\xi_1|$$

follows from the cancellation property of $\Omega(x_1,x_2)$ on $x_1$. Write

$$\left|\widehat{\widetilde{\sigma^d_{u,v}}}(\xi_1,\xi_2)\right| \le C\int_{2^v}^{2^{v+1}}\int_{2^u}^{2^{u+1}} \left|\int_{S^{n-1}} e^{is\xi_2x_2'}\int_{S^{m-1}}(e^{ir\xi_1x_1'}-1)\widetilde{\Omega_d}(x_1',x_2')dx_1'dx_2'\right|\frac{dr}{r}\frac{ds}{s}.$$

For each fixed $r$, $\xi_1$, set

$$\Omega_{d;\,r,\xi_1}(x_2') = \int_{S^{m-1}}(e^{ir\xi_1x_1}-1)\widetilde{\Omega_d}(x_1',x_2')dx_1'.$$

A well-known result of Duoandikoetxea and Rubio de Francia [2] shows that

$$\begin{aligned}\int_{2^v}^{2^v+1} &\left|\int_{S^{n-1}} e^{s\xi_2x_2'}\int_{S^{m-1}}(e^{ir\xi_1x_1}-1)\widetilde{\Omega_d}(x_1',x_2')dx_1'dx_2'\right|\frac{ds}{s}\\ &\le C|2^v\xi_2|^{-\varepsilon}\|\Omega_{d;\,r,\xi_1}\|_{L^\infty(S^{n-1})}\\ &\le C|2^v\xi_2|^{-\varepsilon}|r\xi_1|\left\|\int_{S^{m-1}}|\widetilde{\Omega_d}(x_1',x_2')|dx_1'\right\|_{L^\infty(S^{n-1})}\\ &\le C2^d|2^v\xi_2|^{-\varepsilon}|r\xi_1|\end{aligned}$$

Straightforward computation then establishes our claim. Plancherel's theorem now tells us that

$$\|\widetilde{U^d_{k,l}}f\|_2 \le C\min\{2^d2^k2^{-\varepsilon l},\, 2^d|\widetilde{E_d}|2^k\}\|f\|_2.$$

On the other hand, We know that

$$\|\widetilde{U^d_{k,l}}f\|_p \le C\|\widetilde{\Omega_d}\|_1\|f\|_p \le C2^d|\widetilde{E_d}|\|f\|_p, \quad 1<p<\infty. \tag{9}$$

It follows from the last two inequalities that for each $1 < p < \infty$,

$$\begin{aligned}\|\widetilde{U^d_{k,l}}f\|_p &\le C\min\left\{2^d|\widetilde{E_d}|^{1-t}2^{tk}2^{-t\varepsilon l},\ 2^{tk}2^d|\widetilde{E_d}|\right\}\|f\|_p\\ &\le C\min\left\{2^{tk}2^{td}2^{-\varepsilon tl},\ 2^{tk}2^d|\widetilde{E_d}|\right\}\|f\|_p,\end{aligned}$$

with $t = t_p \in (0,1)$ (note that $2^d|\widetilde{E_d}| \le C$). Write

$$\begin{aligned}T_{\mathrm{II}}f(x_1,x_2) &= \sum_{k\le 0}\sum_{l>0}\widetilde{U^0_{k,l}}f(x_1,x_2) + \sum_{k\le 0}\sum_{d>0}\sum_{0<l\le Nd}\widetilde{U^d_{l,k}}f(x_1,x_2)\\ &\quad+\sum_{k\le 0}\sum_{d>0}\sum_{l>Nd}\widetilde{U^d_{l,k}}f(x_1,x_2).\end{aligned}$$

Therefore,

$$\sum_{k\le 0}\sum_{l>0}\|\widetilde{U^0_{k,l}}f\|_p \le C\sum_{k\le 0}\sum_{l>0}2^{tk}2^{-t\varepsilon l}\|f\|_p \le C\|f\|_p,$$

and

$$\sum_{k\le 0}\sum_{d>0}\sum_{l>Nd}\|\widetilde{U^d_{l,k}}f\|_p \le C\|f\|_p\sum_{k\le 0}2^{tk}\sum_{d>0}2^{td}\sum_{l>Nd}2^{-t\varepsilon l} \le C\|f\|_p.$$

Finally, we have,

$$\sum_{k\le 0}\sum_{d>0}\sum_{0<l\le Nd}\|\widetilde{U^d_{l,k}}f\|_p \le C\|f\|_p\sum_{k\le 0}2^{tk}\sum_{d>0}2^d d|\widetilde{E_d}| \le C\|f\|_p.$$

This completes the proof of Theorem 1.

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