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Sheaves on the category of periodic observation

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Abstract. A Grothendieck topology on the subgroup category of the additive group of integers is defined and the sheafification of the presheaves induced from discrete dynamical systems are determined.

Key words: category, dynamical systems, Grothendieck topology, sheafification.

Introduction

Suppose various observers record the activity of one object periodically with their own time units and each obtains his own dynamical model of the object. How should we obtain a comprehensive model of the object starting from these personal models?

This question may be regarded as a special case of the universal problem of recovering the global information from coherent pieces of local information, which is often analyzed succinctly by the sheaf theory.

In this paper, we introduce a Grothendieck topology on the category of observers with different time units and show that the sheafification procedure gives us an effective method of synthesizing the personal dynamical models of observers whose time units generates the unit ideal of the integer.

1. The category of observers with different time units

1.1. The category $\mathbf{N}^{\circlearrowleft}$

Let $\mathbf{N}^{\circlearrowleft}$ be the category whose objects are natural integers and whose arrows are generated by $\{ \beta_{n,m} : m \to n \mid n|m \}$ and $\{ \alpha_n : n \to n \mid n \in \mathbf{N} \}$ with the following relations:

$$\beta_{nn} = 1_n$$

$$\beta_{\ell,m}\beta_{m,n} = \beta_{\ell,n}$$

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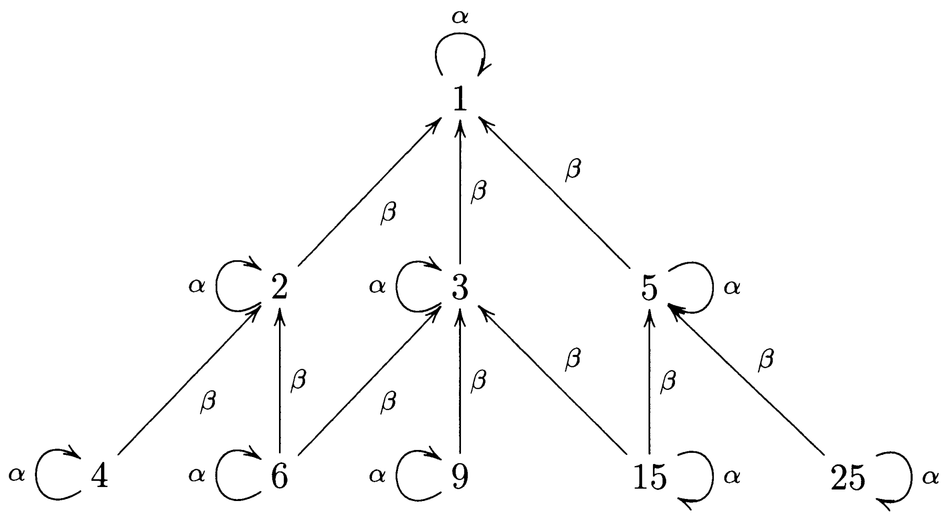
and

$$\alpha_m^n \beta_{m,mn} = \beta_{m,mn} \alpha_{mn}$$

for $m \in \mathbf{N}$ and $n > 1$. The latter relation can be expressed as the commutativity of the diagram

$$\begin{array}{ccc} m & \xrightarrow{\alpha_m^n} & m \\ \uparrow {\scriptstyle \beta_{m,mn}} & & \uparrow {\scriptstyle \beta_{m,mn}} \\ mn & \xrightarrow[\alpha_{mn}]{} & mn \end{array}$$

The following picture illustrates this category.



The object n stands for the observer with time unit n. The arrow α_n is the time flow of the observer n and the arrow $\beta_{m,nm} : nm \to m$ means that the observer mn can obtain his own data via the observer m.

Alternatively we can define $\mathbf{N}^{\circlearrowleft}$ as follows: Its objects are natural numbers and for each m and its divisor d, there are countable arrows

$$\alpha_d^p \beta_{d,m} : m \to d, \quad p = 0, 1, 2, \ldots .$$

We often write them as $\alpha^p \beta$ when there is no danger of confusion. When $d = m$, we write $\alpha^p \beta$ simply as α^p. We also denote $\alpha^0 \beta$ by β.

For each n, the identity arrow is $\alpha^0 = \alpha_n^0 \beta_{n,n}$.

The composition is defined by

$$(\alpha^p \beta_{d,dk}) \circ (\alpha^q \beta) = \alpha^{p+qk} \beta.$$

Lemma 1.1 *The composition satisfies the axioms of category.*

Proof. The identity axiom is obvious. A simple calculation shows that both $(\alpha^{p_1}\beta_{d,dn_1} \circ \alpha^{p_2}\beta_{dn_1,dn_1n_2}) \circ \alpha^{p_3}\beta$ and $\alpha^{p_1}\beta_{d,dn_1} \circ (\alpha^{p_2}\beta_{dn_1,dn_1n_2} \circ \alpha^{p_3}\beta)$ coincide with

$$\alpha^{p_1+p_2n_1+p_3n_1n_2}\beta.$$

□

We note that $\alpha^p \circ \beta = \alpha^p\beta$, which shows our notation for arrows conforms with the composition and the abbreviation. Hereafter we omit the composition symbol $\circ$.

1.2. The categories $\mathbf{Q}^{\circlearrowleft}$ and $\mathbf{R}^{\circlearrowleft}$

When the multiplicative monoid $\mathbf{N}^{\times}$ acts on a set X by $x \mapsto n.x$ ($n \in \mathbf{N}$), we can define a category $X^{\circlearrowleft}$ as follows. Its objects are elements of X. For each element x of X, there is an arrow

$$\alpha_x : x \to x$$

which is aperiodic, namely, for every natural number k, $\alpha_x^k \neq 1_x$. For each element x of X and a natural number n, there is an arrow

$$\beta_{x,n} : n.x \to x$$

satisfying

$$\beta_{x,n} \circ \beta_{n.x,m} = \beta_{x,mn}$$

for each $x \in X$ and $n, m \in \mathbf{N}$ and

$$\alpha_x^n \circ \beta_{x,n} = \beta_{x,n} \circ \alpha_{n.x}$$

for each $x \in X$ and $n \in \mathbf{N}$.

Note that when $X = \mathbf{N}$ with the action $n.x = nx$, the two meanings of $\mathbf{N}^{\circlearrowleft}$ coincide.

When $\mathbf{N}^{\times}$ acts on the sets of rational numbers and real numbers by multiplication, we obtain categories $\mathbf{Q}^{\circlearrowleft}$ and $\mathbf{R}^{\circlearrowleft}$ respectively.

1.3. Localization of $\mathbf{N}^{\circlearrowleft}$

We also consider the category $\mathbf{N}^{\circledcirc}$ which can be obtained from $\mathbf{N}^{\circlearrowleft}$ by inverting the α's. Its objects are natural numbers and arrows $nd \to n$ are described as $\alpha^p\beta$ with $p \in \mathbf{Z}$ now. The composition is defined by the same formulas as before. Hence, $\mathbf{N}^{\circlearrowleft}$ is a full and faithful subcategory of $\mathbf{N}^{\circledcirc}$.

1.4. $\mathbf{N}^{\circledcirc}$ as the subgroup category of the additive group $\mathbf{Z}$

Every group G induces a category $\mathrm{Sub}(G)$ called *the subgroup category.* Its objects are nontrivial subgroups of G and if $H_1 \subseteq H_2$ then an element h of H_2 gives an arrow $(H_1, h, H_2) : H_1 \to H_2$. If $(H_1, h, H_2) : H_1 \to H_2$ and $(H_2, k, H_3) : H_2 \to H_3$, then its composition is $(H_1, kh, H_3) : H_1 \to H_3$ where the product kh is possible since $h \in H_2 \subseteq H_3$.

If $G = \mathbf{Z}$, then the subgroups are $\{ n\mathbf{Z} \mid n \in \mathbf{N} \}$ and $k\mathbf{Z} \subseteq n\mathbf{Z}$ iff $n|k$ and arrows $k\mathbf{Z} \to n\mathbf{Z}$ are $\{ n\ell \mid \ell \in \mathbf{Z} \}$. Hence there is an isomorphic functor $J : \mathrm{Sub}(\mathbf{Z}) \to \mathbf{N}^{\circledcirc}$ defined by

$$\begin{aligned} J(n\mathbf{Z}) &= n, \\ J((nk\mathbf{Z}, np, n\mathbf{Z})) &= \alpha^p \beta_{n,nk}. \end{aligned}$$

2. Some properties of $\mathbf{N}^{\circlearrowleft}$

2.1. Comma category $\mathbf{N}^{\circlearrowleft} \downarrow p$

A remarkable property of $\mathbf{N}^{\circlearrowleft}$ is that the comma categories are all isomorphic. In fact, for each $p \in \mathbf{N}$, there is an isomorphic functor

$$I_p : \mathbf{N}^{\circlearrowleft} \to \mathbf{N}^{\circlearrowleft} \downarrow p$$

defined by

$$\begin{aligned} I_p(n) &= np, \\ I_p(\alpha_n) &= \alpha_{np}, \\ I_p(\beta_{n,nk}) &= \beta_{np,nkp}. \end{aligned}$$

2.2. Relation with $\mathbf{N}^{\div}$

Let $\mathbf{N}^{\div}$ denotes the thin category whose objects are natural numbers and $n \to m$ if and only if n is divided by m. There is a functor

$$\iota : \mathbf{N}^{\div} \to \mathbf{N}^{\circlearrowleft}$$

which is identity on objects and maps $n \to m$ to $\beta_{m,n}$.

There is a right inverse to ι

$$\pi : \mathbf{N}^{\circlearrowleft} \to \mathbf{N}^{\div}$$

which is the identity map on objects and maps $\beta_{m,n}$ to $n \to m$ and α_n to the identity arrow of n.

2.3. Properties of arrows

Proposition 2.1 *Every arrow is both monic and epic.*

Proof. Let $f : nd \to n$. Suppose $f \circ h_1 = f \circ h_2$ for some $h_i : nde \to nd$ $(i = 1,\ 2)$. If $f = \alpha^s\beta$, $h_i = \alpha^{t_i}$ $(i = 1,\ 2)$, then $f \circ h_i = \alpha^{dt_i+s}\beta$ $(i = 1,\ 2)$. Hence $dt_1 + s = dt_2 + s$ which implies $h_1 = h_2$.

Similar arguments show that every arrow is an epic. □

Proposition 2.2 $\left\{ nd_1 \overset{\alpha^{p_1}\beta}{\to} n \overset{\alpha^{p_2}\beta}{\leftarrow} nd_2 \right\}$ *has a pull-back in* $\mathbf{N}^{\circledcirc}$ *if and only if* $p_1 - p_2$ *is divisible by the greatest common divisor of* d_1, d_2.

Proof. Let $\left\{ nd_1 \overset{\alpha^{p_1}\beta}{\to} n \overset{\alpha^{p_2}\beta}{\leftarrow} nd_2 \right\}$. If there is its pull-back, it must be of the form $\left\{ nd_1 \overset{\alpha^{q_1}\beta}{\leftarrow} nd \overset{\alpha^{q_2}\beta}{\to} nd_2 \right\}$ with d the common least multiplier of d_1 and d_2. The necessary condition for this is the commutativity of the square, which means

$$d_1q_1 + p_1 = d_2q_2 + p_2. \qquad (*)$$

Hence $p_1 - p_2$ must be divisible by $GCD(d_1, d_2)$. Suppose now that this condition is satisfied. Then there are q_1, q_2 which satisfy $(*)$.

Let $\left\{ nd_1 \overset{\alpha^{r_1}\beta}{\leftarrow} ndu \overset{\alpha^{r_2}\beta}{\to} nd_2 \right\}$ be any arrow which makes the square commutative, namely

$$d_1r_1 + p_1 = d_2r_2 + p_2.$$

Then we have

$$d_1(q_1 - r_1) = d_2(q_2 - r_2).$$

We have to show the existence of r such that

$$\alpha^{q_i}\beta \circ \alpha^r\beta = \alpha^{r_i}\beta \quad (i = 1,\ 2),$$

which means

$$e_ir = r_i - q_i \quad (i = 1,\ 2), \qquad (**)$$

where $e_i = d/d_i$ $(i = 1,\ 2)$. Since e_1, e_2 are mutually prime, we can solve $(**)$. The uniqueness of r is obvious. □

We note that pull-backs may not exist in $\mathbf{N}^{\circlearrowleft}$ since the equation $(**)$ has no positive solutions when the right hand side is negative.

Proposition 2.3 *The category $\mathbf{N}^{\circlearrowleft}$ does not have the followings:*

1. *initial objects,*
2. *terminal objects,*
3. *products,*
4. *pull-backs,*
5. *coproduct,*
6. *equalizers,*
7. *coequalizers.*

Proof. We have no arrows from m to $2m$ for any m, whence there are no initial objects.

The only candidate for the terminal object is 1, but 1 has non trivial endoarrows α_1^n.

The products do not exist in general. For example, the product cone of 2 and 3 if existed must be of the form

$$2 \xleftarrow{\alpha^k\beta} 6 \xrightarrow{\alpha^\ell\beta} 3.$$

Let $\alpha^p\beta : 12 \to 6$. Then

$$\alpha^k\beta_{2,6}\alpha^p\beta_{6,12} = \alpha^{k+3p}\beta_{2,12}, \quad \alpha^\ell\beta_{3,6}\alpha^p\beta_{6,12} = \alpha^{\ell+2p}\beta_{3,12}.$$

Hence, if we take $f := \alpha^{k+1}\beta_{2,12} : 12 \to 2$ and any $g : 12 \to 3$, there are no $h : 12 \to 6$ with $\alpha^k\beta \circ h = f$.

Let $f, g : m \to n$ be parallel arrows. If $f \circ h = g \circ h$ for some $h : p \to m$, then $f = g$. Hence there are no equalizers except for the trivial case $f = g$.

Similarly parallel arrows f, g with $f \neq g$ have no coequalizers.

We can show similarly that coproducts and coequalizers do not exist in general. □

3. Presheaves on $\mathbf{N}^{\circlearrowleft}$

3.1. Presheaves

A presheaf on $\mathbf{N}^{\circlearrowleft}$ is a family of discrete dynamical systems with different time units with comparison morphisms from one with time unit k to another with time unit nk. More precisely, a presheaf over $\mathbf{N}^{\circlearrowleft}$ is given by the following data:

- A family of sets $\{ X_n \mid n \in \mathbf{N} \}$ indexed by natural numbers,
- a family of endomaps $\tau_n : X_n \to X_n$ for $n \in \mathbf{N}$,

- a family of maps $\sigma_{n,mn} : X_n \to X_{mn}$, for $m, n \in \mathbf{N}$,

satisfying

(P_A) $\sigma_{mn,\ell mn} \circ \sigma_{n,mn} = \sigma_{n,\ell mn}$,

(P_B) $\sigma_{n,kn} \circ \tau_n^k = \tau_{kn} \circ \sigma_{n,kn}$.

Hence, for each $n \in N$, we have a discrete dynamical system[1] (X_n, τ_n), which we regard as the model conceived by the observer n.

Note that P_B means that $\sigma_{n,kn} : X_n \to X_{kn}$ induces a morphism of dynamical systems[2]

$$(X_n, \tau_n^k) \to (X_{kn}, \tau_{kn}).$$

This morphism compares the model of the observer n with that of the observer kn, which is possible because we can extract, from the model of the observer n, the information at the time intervals $nk, 2nk, 3nk, \ldots$ and compare them with the information extractable from the model of the observer nk.

For example, the periodic points of (X_n, τ_n) with periods dividing k are mapped to fixed points of (X_{nk}, τ_{nk}) by $\sigma_{n,kn}$.

3.2. Presheaf induced by a discrete dynamical systems

Suppose we know a dynamical system model of an object. Then we obtain a presheaf as follows: Let $D = (X, \tau)$ be the discrete dynamical system. For each natural number n, put $P_D(n) = X$ and $P_D(\alpha_n) = \tau^n$. Furthermore define $P_D(\beta_{x,n}) = \text{id}$ for every x, n. Then P_D is a presheaf on $\mathbf{N}^{\circlearrowleft}$, called *the presheaf induced by the dynamical system P_D*.

3.3. Fixed point functor

Each presheaf $X = (X_n, \tau_n, \sigma_{n,kn})$ over $\mathbf{N}^{\circlearrowleft}$ induces the presheaf $Fix(X) = (Fix(X_n, \tau_n), \sigma_{n,kn})$ over $\mathbf{N}^{\div}$, where $Fix(X_n, \tau_n) := \{\, x \in X_n \mid \tau_n x = x \,\}$.

[1]A pair (X, τ) is called a discrete dynamical system, if X is a set and $\tau : X \to X$ is an endomap. X is called **the state space** and τ **the transition map**.

[2]When (X_i, τ_i) $(i = 1,\ 2)$ are discrete dynamical systems, a map $f : X_1 \to X_2$ is called a morphism of dynamical systems when $f \circ \tau_1 = \tau_2 \circ f$.

4. The category of presheaves on $\mathbf{N}^{\circlearrowleft}$

4.1. Topos structure

The presheaves on $\mathbf{N}^{\circlearrowleft}$ form a category $\mathcal{S}\mathbf{et}^{\mathbf{N}^{\circlearrowleft op}}$, which is the functor category from $\mathbf{N}^{\circlearrowleft op}$ to $\mathcal{S}\mathbf{et}$. An arrow $F : X \to Y$ is a family of morphisms $F_n : X_n \to Y_n$ of dynamical systems which commute with the comparison operators, i.e.,

$$F_{nm} \circ X(\beta_{nm,n}) = Y(\beta_{nm,n}) \circ F_n.$$

The category $\mathcal{S}\mathbf{et}^{\mathbf{N}^{\circlearrowleft op}}$ has the following properties.

1. It is complete and cocomplete, with pointwise limit and colimit operations. For example, a product of X and Y is defined as $(X_n \times Y_n, \tau_n^X \times \tau_n^Y)$.
2. It has an exponentiation.
3. It has a subobject classifier.

Hence it is a topos. See [2] for generalities on topos.

4.2. Yoneda embedding

We first write explicitly the Yoneda embedding $\mathbf{y} : \mathbf{N}^{\circlearrowleft} \to \mathcal{S}\mathbf{et}^{\mathbf{N}^{\circlearrowleft op}}$, which we need to describe the subobject classifier. The presheaf $\mathbf{y}(n)$ is defined by

$$\mathbf{y}(n)_m := \mathbf{N}^{\circlearrowleft}(m,n) = \begin{cases} \emptyset & \text{if } n \nmid m \\ \{ \alpha_n^p \beta_{n,m} \mid p = 0,1,2,\cdots \} & \text{if } n|m \end{cases}$$

Since $\mathbf{N}^{\circlearrowleft}$ is a small category, we identify

$$\mathbf{y}(n) = \mathbf{N}^{\circlearrowleft}(-,n).$$

The arrows with codomain n are written uniquely as $\alpha_n^p \beta_{n,nk}$ with $(p,k) \in \mathbf{Z}_+ \times \mathbf{N}$. Denote the bijection $\mathbf{Z}_+ \times \mathbf{N} \to \mathbf{N}^{\circlearrowleft}(-,n)$ by Γ_n:

$$\Gamma_n(p,k) := \alpha^p \beta_{n,nk}.$$

We will identify $\mathbf{y}(n)$ with $\mathbf{Z}_+ \times \mathbf{N}$ by the bijection Γ_n.

Lemma 4.1 *For $(p,k) \in \mathbf{y}(n)$, we have*

$$(p,k) \circ \alpha_{nk} = (p+k,k),$$

$$(p,k) \circ \beta_{nk,nk\ell} = (p,k\ell).$$

Proof. These are just the following identities:

$$\alpha_n^p \beta_{n,nk} \alpha_{nk} = \alpha_n^{p+k} \beta_{n,nk},$$
$$\alpha_n^p \beta_{n,nk} \beta_{nk,nk\ell} = \alpha_n^p \beta_{n,nk\ell}.$$

□

Define transformations on $\mathbf{Z}_+ \times \mathbf{N}$ as follows:

$$A : \mathbf{Z}_+ \times \mathbf{N} \ni (p,k) \mapsto (p+k,k),$$
$$B_\ell : \mathbf{Z}_+ \times \mathbf{N} \ni (p,k) \mapsto (p,k\ell) \quad (\ell \in \mathbf{N})$$

Then the composition of α from the right is described by A and that of $\beta_{nk,nk\ell}$ from the right is by B_ℓ.

The functoriarity of the Yoneda embedding $\mathbf{y}$ is described by

Lemma 4.2 1. *$\beta^*_{n,ns}((p,k)) = (ps,ks)$, where the map $\beta^*_{n,ns} : \mathbf{y}(\beta_{n,ns}) : \mathbf{y}(ns) \to \mathbf{y}(n)$ is the induced map.*

2. *$\alpha^*(p,k) = (p+1,k)$, where the map $\alpha^* : \mathbf{y}(\alpha_n) : \mathbf{y}(n) \to \mathbf{y}(n)$ is the induced map.*

4.3. Sieves

We describe the subobject classifier Ω of the presheaf topos $\mathcal{S}\mathbf{et}^{\mathbf{N}^{\circlearrowleft op}}$ using the Yoneda lemma:

$$\Omega_n \simeq \mathcal{S}\mathbf{et}^{\mathbf{N}^{\circlearrowleft op}}(\mathbf{y}_n, \Omega)$$
$$\simeq \mathrm{Sub}(\mathbf{y}(n)).$$

A subobject S of $\mathbf{y}_n$ is a subset of $\mathbf{N}^{\circlearrowleft}(-,n) = \mathbf{Z}_+ \times \mathbf{N}$ closed by compositions from the right, which is called *a sieve on n* in $\mathbf{N}^{\circlearrowleft}$.

Proposition 4.3 *Sieves on n are the subsets of $\mathbf{Z}_+ \times \mathbf{N}$ which are closed under the transformations A, B_ℓ.*

Proof. Obvious from Lemma 4.1 □

By Lemma 4.2, the action of arrows on sieves can be described as follows:

Lemma 4.4 1. *$\Omega_\beta : \Omega_n \to \Omega_{ns}$ induced by $\beta : ns \to n$ is given by*

$$\Omega_\beta(S) = \{\, (n,k) \mid (ns,ks) \in S \,\}).$$

2. *$\Omega_\alpha : \Omega_n \to \Omega_n$ induced by $\alpha_n : n \to n$ is given by*

$$\Omega_\alpha(S) = \{ (n,k) \mid (n+1,k) \in S \}.$$

Define maps $M_s, \sigma : \mathbf{Z}_+ \times \mathbf{N} \to \mathbf{Z}_+ \times \mathbf{N}$ by

$$M_s(n,k) := (sn, sk) \qquad \sigma(n,k) = (n+1,k).$$

Then the above lemma can be written as

Lemma 4.5 1. *$\Omega_\beta : \Omega_n \to \Omega_{ns}$ induced by $\beta : ns \to n$ is given by*

$$\Omega_\beta(S) = M_s^{-1}(S).$$

2. *$\Omega_\alpha : \Omega_n \to \Omega_n$ induced by $\alpha_n : n \to n$ is given by*

$$\Omega_\alpha(S) = \sigma^{-1}(S).$$

Let $T \in \Omega_n \subseteq \mathcal{P}(\mathbf{Z}_+ \times \mathbf{N})$. We have a smallest subsieve $\widehat{T}$ containing T. In fact we add to T those elements obtained by A and B_ℓ ($\ell \in \mathbf{N}$) actions. This operation is a closure operator $T \mapsto \widehat{T}$ on $\mathcal{P}(\mathbf{Z}_+ \times \mathbf{N})$ and its closed sets are precisely the sieves. Hence the set of sieves forms a complete meet sublattice of $\mathcal{P}(\mathbf{Z}_+ \times \mathbf{N})$.

Proposition 4.6 *The lattice structure of Ω_n is given by*

1. $S_1 \le S_2 \iff S_1 \subseteq S_2$,
2. $S_1 \bigwedge S_2 = S_1 \bigcap S_2$,
3. $S_1 \bigvee S_2 = \widehat{S_1 \bigcup S_2}$.

4.4. Canonical sieves

For a finite subset $K \subseteq \mathbf{N}$, we denote by $S(n;K) \in J(n)$ the sieve generated by the arrows $\{ \alpha^s \beta_{n,nkt} \mid s,t \in \mathbf{N},\, k \in K \}$. This can be written also as

$$S(n;K) = \{ (p,\ell) \mid \ell \in \mathbf{N},\, \ell \in K^* \}.$$

Here K^* denotes the multipliers of elements of K. A sieve is called *canonical* if it can be expressed as $S(n;K)$ with a finite $K \subseteq \mathbf{N}$.

Lemma 4.7 *$S(n,K_1) \bigcap S(n,K_2) = S(n, K_1 \bigwedge K_2)$, where*

$$K_1 \bigwedge K_2 := \left\{ k_1 \bigwedge k_2 \mid k_i \in K_i \quad (i = 1,\ 2) \right\}$$

with $k_1 \bigwedge k_2$ *denoting the least common multiplier.*

Proof. The right hand side obviously is contained in the left hand side. Suppose (m, k) is in the left side hand. Then there are k_i with $k_i | k$ and $k_i \in K_i$ for $i = 1, 2$. Hence $k_1 \bigwedge k_2 | k$ and (m, k) is in the right hand side. □

We describe the actions of α_n and $\beta_{n,nk}$ on canonical sieves.

From Lemma 4.2, we have obviously the following

Proposition 4.8 *The arrow* α_n *leaves the canonical sieves invariant, namely,* $\alpha_n^* S(n; K) = S(n; K)$.

Similarly, we have

Proposition 4.9 *The arrow* $\beta_{n,ns}$ *maps* $S(n, K)$ *to* $S(ns, K/s)$*, where*

$$K/s := \left\{ \frac{k}{k \vee s} \mid k \in K \right\},$$

with $k \vee s$ *denoting the greatest common divisor of* k *and* s.

Proof. Since

$$\beta_{n,ns}^* S(n, K) = \{ (p, \ell) \mid (ps, \ell s) \in S(n, K) \}$$

and $(ps, \ell s) \in S(n, K)$ is equivalent to $k | \ell s$ for some $k \in K$, the assertion follows from

$$k \mid \ell s \iff (k/k \wedge s) \mid \ell.$$

□

5. A Grothendieck topology on $\mathbf{N}^{\circlearrowleft}$

5.1. Definition

Let S be a sieve on n identified with a subset of $\mathbf{Z}_+ \times \mathbf{N}$. Define

$$\mu(S) := \{ k \in \mathbf{N} \mid (p, k) \in S \text{ for all } \ p \in \mathbf{Z}_+ \}.$$

A sieve S is called *dense* if $\bigvee \mu(S) = 1$, i.e., the greatest common divisor of $\mu(S)$ is 1. Let $J(n)$ be the set of dense sieves on n.

Proposition 5.1 J *is a Grothendieck topology on* $\mathbf{N}^{\circlearrowleft}$.

Proof. Obviously $t_n = \mathbf{y}(n) = \mathbf{Z}_+ \times \mathbf{N}$ is dense since $\mu(t_n) = \mathbf{N}$.

Let $f : ns \to n$ and $S \in J(n)$. We show that $f^*S \in J(ns)$. Since f is the composition of $\alpha_n^k : n \to n$ and $\beta_{n,ns} : ns \to n$, it suffices to show that $\alpha_n^* S \in J(n)$ and $\beta_{n,ns}^* S \in J(ns)$.

By Lemma 4.4, we have obviously $\mu(\alpha_n^* S) = \mu(S)$, whence $\alpha_n^* S \in J(n)$. By the same lemma,

$$\mu(\beta_{n,ns}^* S) \supseteq \{\, k \mid ks \in \mu(S) \,\} \supseteq \mu(S),$$

since $s\mu(S) \subseteq \mu(S)$ obviously. Hence from $\bigvee \mu(S) = 1$, we have $\bigvee \mu(\beta_{n,ns}^* S) = 1$.

Finally, we have to show the transitivity of J. Let $S \in J(n)$ and R be a sieve on n. Suppose, for every $f \in S$, $f^*R \in J(dom(f))$. Let $s_1, \ldots, s_m \in \mu(S)$ with $\bigvee_i s_i = 1$. For each i and $\ell \in \{\, 0, 1, \ldots, s_i - 1 \,\}$ we have $\alpha_n^\ell \beta_{n,ns_i} \in S$, whence $\mu(\beta_{n,ns_i}^* \alpha_n^{\ell *} R)$ has the greatest common divisor 1. This means there are $t_{i\ell j} \in \mu(\beta_{n,ns_i}^* \alpha_n^{\ell *} R)$ ($j \in I_{i\ell}$) such that $\bigvee_{j \in I_{i\ell}} t_{i\ell j} = 1$. Since $(p, nt_{i\ell j}) \in \beta_{n,ns_i}^* \alpha_n^{\ell *} R$ for all $p \in \mathbf{Z}_+$, we have

$$(*) \qquad (ps_i + \ell, t_{i\ell j} s_i) \in R \quad \text{for all} \quad p.$$

Let $I_i := \prod_{\ell=0}^{s_i - 1} I_{i\ell}$ and, for $J = (j_0, j_1, \ldots, j_{s_{i-1}}) \in I_i$, define $t_{iJ} := \bigwedge_{\ell=0}^{s_i-1} t_{i\ell j_\ell}$. Then by the distributivity of the poset $\mathbf{N}^{\div}$, we have

$$\bigvee_{J \in I_i} t_{iJ} = 1.$$

From $(*)$, we have

$$(ps_i + \ell, t_{iJ} s_i) \in R \ \text{ for all } \ p \ \text{ and } \ J \in I_i \ \text{ and } \ \ell,$$

since for all ℓ there is a j with $t_{iJ} | t_{i\ell j}$. Hence we have

$$(p, t_{iJ} s_i) \in R \quad \text{for all} \quad p,$$

which implies $t_{iJ} s_i \in \mu(R)$ for all i and $J \in I_i$. Since

$$\bigvee_i \bigvee_{J \in I_i} t_{ij} s_i = \bigvee_i s_i = 1,$$

we conclude that $R \in J(n)$. □

5.2. Canonical dense sieves

Since $\mu(S(n, K))$ is generated multiplicatively by K, the canonical sieve $S(n; K)$ is dense if and only if $\bigvee K = 1$.

Lemma 5.2 *Every dense sieve contains a canonical dense sieve.*

Proof. Let S be a dense sieve. Since $\bigvee \mu(S) = 1$, there are finite $K \subseteq \mu(S)$ with $\bigvee K = 1$. Hence S contains the canonical dense sieve $S(n, K)$. □

6. Sheaves

6.1. Matching family

Let P be a presheaf over $\mathbf{N}^{\circlearrowleft}$. Let $S \in \Omega_n$ be a sieve. A matching family x is described as follows. It is a family $(x_{i,k})_{(i,k)\in S}$ satisfying, for $i \in \mathbf{N}$ and $k, \ell \in K$,

(M1) $x_{i,k} \in P(nk)$,

(M2) $x_{i,k} \cdot \alpha_{nk} = x_{i+k,k}$,

(M3) $x_{i,k} \cdot \beta_{nk,nkp} = x_{i,kp}$.

Each $x \in P(n)$ defines a matching family $\kappa x := (x_{i,k})_{(i,k)\in S}$, where $x_{i,k} := x \cdot \alpha_n^i \beta_{n,nk}$, whence we have

$$(*) \qquad \kappa_S : P(n) \to \mathrm{Match}(S, P).$$

A presheaf P is called *separated* if and only if κ_S is injective for every n and for every dense sieve S on n. A presheaf P is called *a sheaf for the Grothendieck topology J* if and only if κ is bijective for every n and for every dense sieve S on n.

Lemma 6.1 *A presheaf P is a sheaf if κ_S is bijective for canonical sieves S.*

Proof. In fact, if S contains a dense $S(n, K)$, then we have

$$P_D(n) \xrightarrow{f} \mathrm{Match}(S, P_D) \xrightarrow{g} \mathrm{Match}(S(n, K), P_D).$$

Since f and g are obviously injective, if $g \circ f$ is bijective then f is surjective and hence bijective. □

6.2. Presheaves P_D

Let $D = (X, \tau)$ be a discrete dynamical system and let P_D be the induced presheaf defined in §3.2.

Then $P_D(n) = X$ for every n and β's act as the identity and $\alpha_n : n \to n$ acts as τ^n by definition.

We have the following description of matching families.

Proposition 6.2 *If $K = \{ n_1, n_2, \dots, n_k \}$, then matching families*

$$x \in \mathrm{Match}(S(n,k), P_D)$$

correspond bijectively to the sequences

$$(x_i)_{i \in \mathbf{N}} \in X^{\mathbf{N}}$$

satisfying

$$\tau^{nn_j} x_i = x_{i+n_j} \quad \text{for all} \quad i \in \mathbf{N} \quad \text{and} \quad j \in \{ 1, 2, \dots, k \},$$

by the correspondence $x_i = x_{i, \bigwedge K}$ for $i \in \mathbf{N}$, where $\bigwedge K$ is the least common multiplier. Moreover the $\kappa_{S(n,K)} : P_D(n) \to \mathrm{Match}(S(n,K), P_D)$ is given by

$$x \mapsto (x, \tau^n x, \tau^{2n} x, \dots, \tau^{kn} x, \dots).$$

Hence it is obviously injective and we have the following proposition.

Proposition 6.3 *The presheaf P_D is separated.*

We introduce an equivalence relation $\sim_n$ on X by

$$x \sim_n y \overset{def}{\Longleftrightarrow} \tau^{nm} x = \tau^{nm} y \quad \text{for some} \quad m \in \mathbf{N}.$$

It is obvious that $\sim_n$ is in fact an equivalence relation.

Lemma 6.4 *Let $S(n, \{ n_1, \dots, n_k \}) \in J(n)$. If a sequence $(x_i)_{i \in \mathbf{N}} \in X^{\mathbf{N}}$ satisfies*

$$\tau^{nn_j} x_i = x_{i+n_j}$$

for all $i \in \mathbf{Z}_+$ and $j = 1, \dots, k$, then

$$\tau^n x_i \sim_n x_{i+1} \quad \forall i \in \mathbf{N}.$$

Proof. Since $1 = \sum_{1 \leq i \leq k} \ell_i n_i$, with $\ell_i \in \mathbf{Z}$, we have

$$1 + \sum_{\ell_i < 0} |\ell_i| n_i = \sum_{\ell_i \geq 0} \ell_i n_i,$$

which we denote by m. Then, for all $p \in \mathbf{N}$,

$$\tau^{nm} x_p = \left(\prod_{\ell_i > 0} \tau^{\ell_i n n_i} \right) x_p = x_{p + \sum_{\ell_i > 0} \ell_i n_i} = x_{p+m}$$

and

$$\tau^{nm-n}x_{p+1} = \prod_{\ell_i<0} \tau^{|\ell_i|nn_i}x_{p+1} = x_{p+\sum_{\ell_i<0}\tau^{|\ell_i|n_i}} = x_{p+m}.$$

Hence

$$\tau^{nm-n}(\tau^n x_p) = \tau^{mn-n}x_{p+1},$$

which implies $\tau^n x_p \sim_n x_{p+1}$. □

When τ is injective, the equivalence relation $\sim_n$ is the identity relation. Hence we have the following theorem.

Theorem 6.5 *The presheaf P_D induced from a discrete dynamical system $D = (X, \tau)$ is a sheaf if τ is injective.*

Proof. In fact we show that

$$P_D(n) \longrightarrow \mathrm{Match}(S, P_D)$$

is a bijection for $n \in \mathbf{N}^{\circlearrowleft}$ and $S \in J(n)$. By Lemma 6.1, we may also assume that

$$S = S(n, K) \in J(n).$$

By Proposition 6.2, it suffices to show that if a sequence $(x_i)_{i\in\mathbf{N}} \in X^{\mathbf{N}}$ satisfies

$$\tau^{nn_j}x_i = x_{i+n_j}$$

for all $i \in \mathbf{Z}_+$ and $j = 1, \ldots, k$, then

$$x_i = \tau^{ni}x_0 \quad \forall i \in \mathbf{N},$$

which is valid by Lemma 6.4. Hence we conclude that the matching family x comes from $x_0 \in P_D(1)$. □

7. Sheafification of discrete dynamical systems

7.1. Sheafification operation

There is a general method of converting presheaves to sheaves.

For a presheaf P, we can define another presheaf P^+ by

$$P^+(n) := \mathrm{colim}_{S\in J(n)} \mathrm{Match}(S, P).$$

Note that if $S \subseteq T$, then there is a natural restriction map

$$\mathrm{Match}(T, P) \to \mathrm{Match}(S, P),$$

and the colimit is taken with respect to the poset of sieves on n ordered by the inclusion order.

The κ_S's induce

$$\kappa(n) : P(n) \to P^+(n).$$

If P is separated, then $\kappa(n)$ is injective for all n and if P is a sheaf then κ is bijection for all n. In fact the converse is true.

Proposition 7.1 [2] *A presheaf is separated if κ is injective and a sheaf for the Grothendieck topology J, if κ is bijective.*

Theorem 7.2 [2, Lemma 4, Lemma 5, p.131] *The presheaf P^+ is separated. If P is already separated, then P^+ is a sheaf.*

7.2. Discrete dynamical systems

Let D be a discrete dynamical system. Since P_D is separated, the presheaf P_D^+ is a sheaf.

In this section, we examine the sheafification of the presheaf P_D induced from some concrete discrete dynamical systems D.

The following lemma gives us a method of calculating the matching family. We note that the arrow α_n leaves the canonical sieves invariant and whence induces an endomap of $\mathrm{Match}(S(n, K), P)$.

Obviously we have the following.

Lemma 7.3 *Let $K = \{p, q\}$ with $p < q$, then a matching family in $\mathrm{Match}(S(n, K), P_D)$ is determined by the sequence $\langle x_0, x_1, \ldots, x_{p-1}\rangle$ which satisfies*

$$\tau^{nq} x_i = (\tau^p)^s x_t$$

where $i + q \equiv t \bmod p$ with $0 \leq t < p$ and $s = \frac{i+q}{p}$.

The arrow α_n acts on $\mathrm{Match}(S(n, K), P)$ by

$$(x_0, x_1, \ldots, x_{p-1}).\alpha_n = (x_1, x_2, \ldots, x_{p-1}, \tau^p x_0).$$

Example Suppose D is as in Figure 1.

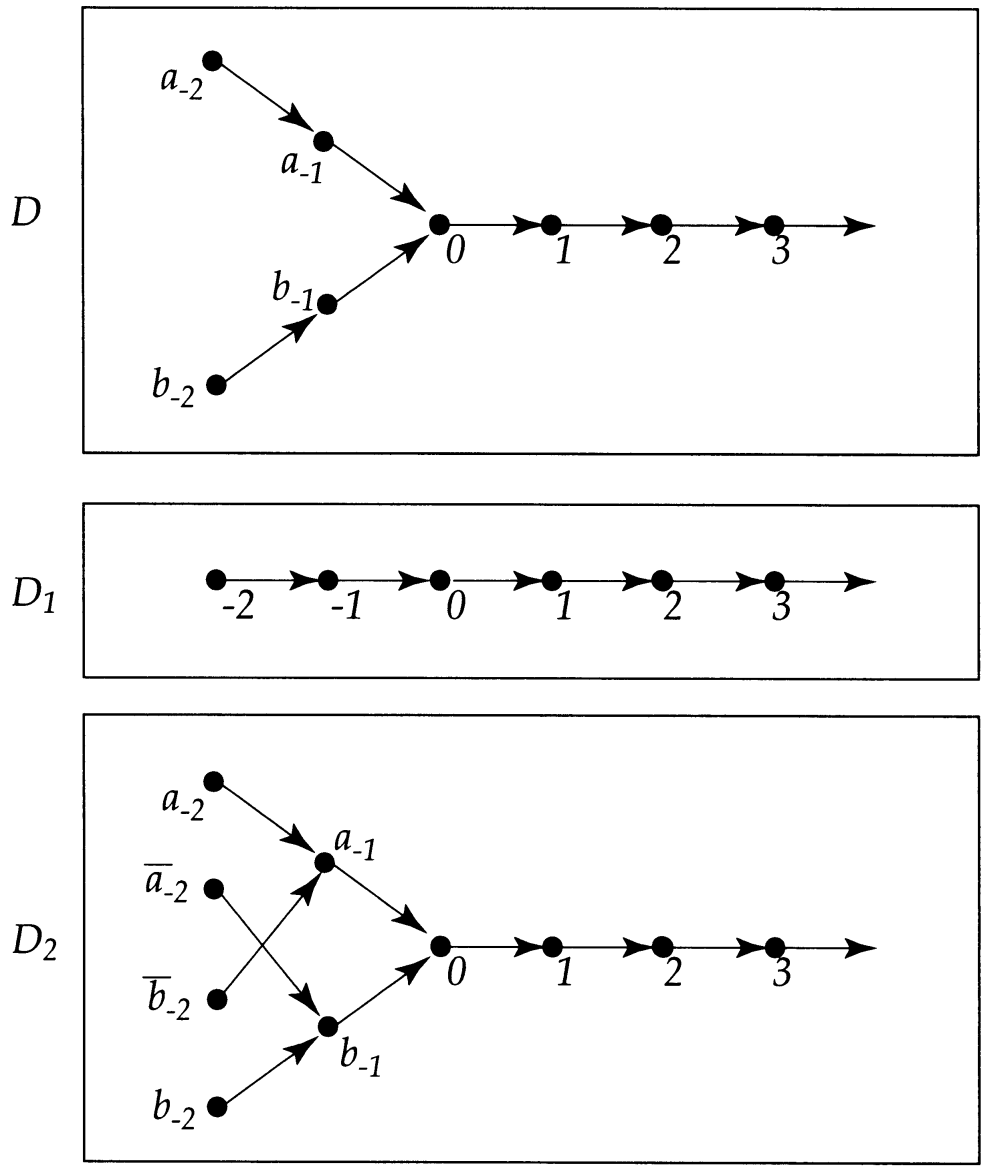


Fig. 1. Example of sheafification

When $K = \{2, 3\}$, then $x \in \mathrm{Match}(S(1, K), P_D)$ is determined by $(x, y) \in X^2$ with

$$\tau^3 x_0 = \tau^2 x_1,$$

whence $\tau^2(\tau x_0) = \tau^2(x_1)$. It is easy to show that the discrete dynamical system $(\mathrm{Match}(S(1, K), P_D), \alpha_1)$ is given by D_2 in Figure 1.

7.3. Sheafification of P_D

Let $D = (X, \tau)$ be a discrete dynamical system.

Define first *its reduced dynamical system* $\overline{D}$ as follows. Let $\pi : X \to \overline{X}$

be the quotient map of the equivalence relation $\sim_1$ introduced in §6.2. Then τ induces $\overline{\tau} : \overline{X} \to \overline{X}$ by $\overline{\tau}([x]) := [\tau x]$.

Define a discrete dynamical system $\widehat{D}$ as follows. Let $\widehat{X}$ be the set of sequences $(x_0, x_1, \cdots) \in X^{\mathbf{N}}$ which satisfy the following two conditions:

$$\overline{\tau}[x_i] = [x_{i+1}] \quad \text{for all} \quad i \in \mathbf{N},$$

and there is a natural number N such that

$$(*) \qquad \tau^i x_j = x_{i+j} \quad \text{for all} \quad i, j \quad \text{with} \quad i + j > N.$$

Define $\widehat{\tau}(x_0, x_1, x_2, \cdots) = (x_1, x_2, x_3, \cdots)$.

Example Let D be as in Figure 1. Then $D_1 = \overline{D}$ and $D_2 = \widehat{D}$.

Theorem 7.4 *Let* $D = (X, \tau)$ *be a discrete dynamical system, then* $P_D^+(n) = \widehat{(X, \tau^n)}$.

Proof. We show $P_D^+(1) = \widehat{(X, \tau)}$. The general case can be shown similarly.

Let $x = (x_i) \in P_D^+(1)$. Then $x \in \mathrm{Match}(S(1, \{n_1, \ldots, n_k\}, P_D))$ for some $n_1 < \cdots < n_k$. Then by Lemma 6.4, $\tau x_i \sim_1 x_{i+1}$ for all i. Since the second condition $(*)$ is obvious if we take $N = n_1$, we have $x \in \widehat{D}$.

Conversely suppose $x \in \widehat{D}$. Let N be an integer such that $(*)$ holds. Let $p, q > N$ be relatively prime integers so that $S(1, \{p, q\}) \in J(1)$. Then, by $(*)$, we have $\tau^p x_i = x_{i+p}$ and $\tau^q x_i = x_{i+q}$ for all i. This shows $x \in \mathrm{Match}(S(1, \{p, q\}), P_D)$. □

8. Concluding remarks

We considered the problem of reconstructing the dynamic behavior of an object from the data of observers who observe it periodically with mutually prime periods. We analyzed this problem by introducing the base category $\mathbf{N}^{\circlearrowleft}$ with a natural Grothendieck topology.

It turned out that when the original dynamics has no states which merge, then the original structure is recovered from the observations. If the observed system has merging states, then the presheaf P_D is not a sheaf, but the sheafification procedure recovers the structure of the quotient dynamical system obtained by identifying two states which eventually coincides.

We will consider in future the general case when the comparison maps β are not identities. Then the sheafification procedure gives rise to the new state spaces which are fibred products of the local observers.

Finally we note that the Grothendieck topology J is not the unique one. We show another natural Grothendieck topology in the appendix, whose sheafification operator however destroys the transition information among the transient states.

References

[1] Birkoff G., *Lattice Theory.* AMS Colloquium Publications, Vol. **25**, 1948.

[2] Maclane S. and Moerdijk L., *Sheaves in Geometry and Logic.* A first Introduction to Topos Theory. Second corrected printing. Springer, 1994.

A. Another Topology on $\mathcal{S}\mathbf{et}^{\mathbf{N}^{\circlearrowleft op}}$

There is another natural Grothendieck topology on $\mathbf{N}^{\circlearrowleft}$, which we define as a Lawvere-Tierney topology j on the presheaf topos $\mathcal{S}\mathbf{et}^{\mathbf{N}^{\circlearrowleft op}}$.

Recall [2, p.219] that a Lawvere-Tierney topology j is an endo arrow of the subobject classifier Ω satisfying

LT1 $j \circ true = true$,

LT2 $j \circ j = j$,

LT3 $j \circ \bigwedge = \bigwedge \circ (j \times j)$.

Here $true : \mathbf{1} \to \Omega$ is the arrow classifying the identity arrow $1_{\mathbf{1}}$. The arrow

$$\bigwedge : \Omega \times \Omega \to \Omega$$

is the meet operation and $j \times j : \Omega \times \Omega \to \Omega \times \Omega$ is the product of j.

Define now $j_n : \Omega_n \to \Omega_n$ by

$$j_n(S) = \overline{S},$$

where $S \subseteq \mathbf{Z}_+ \times \mathbf{N}$ is a sieve and $\overline{S}$ is defined as follows: First

$$|S| := \{\, n \in \mathbf{Z}_+ \mid (n,p) \in S \quad \text{for some} \quad p \,\}.$$

For a subset $W \subseteq \mathbf{Z}_+$ we define

$$\overline{W} := \{\, (n,k) \mid n + k.\mathbf{N} \subseteq W \,\}.$$

Lemma A.1 $\overline{W}$ *is a sieve.*

Proof. Suppose $(n,k) \in \overline{W}$. Then obviously $(n+k,k) \in \overline{W}$. Moreover

$(n, k\ell) \in \overline{W}$ since $n + k\ell.\mathbf{N} \subseteq n + k.\mathbf{N} \subseteq W$. □

Finally we define, for $S \subseteq \mathbf{Z}_+ \times \mathbf{N}$,

$$\overline{S} := \overline{|S|}.$$

Lemma A.2 1. *$\overline{S}$ is a sieve containing S.*

2. $|\overline{S}| = |S|$.
3. *If $S_1 \subseteq S_2$, then $\overline{S_1} \subseteq \overline{S_2}$.*
4. $\overline{\overline{S}} = \overline{S}$.

Proof. By Lemma A.1, $\overline{S}$ is a sieve. Suppose $(n, k) \in S$. Then $n \in |S|$. Moreover, $(n + tk, k) \in S$ $(t \in \mathbf{N})$ implies, $n + k.\mathbf{N} \subseteq |S|$, which means $(n, k) \in \overline{S}$.

By definition,

$$n \in |\overline{S}| \iff (n, k) \in \overline{S} n + k.\mathbf{N} \subseteq |S| n \in |S|.$$

Hence $|\overline{S}| \subseteq |S|$. On the other hand, we have proved that $S \subseteq \overline{S}$, whence $|S| \subseteq |\overline{S}|$.

Hence

$$\overline{\overline{S}} = \overline{|\overline{S}|} = \overline{|S|} = \overline{S}.$$

□

Lemma A.3 *$j = (j_n) : \Omega \to \Omega$ is a presheaf map.*

Proof. By Lemma 4.5, we have to show, for $S \subseteq \mathbf{Z}_+ \times \mathbf{N}$,

$$\overline{M_s^{-1} S} = M_s^{-1} \overline{S} \tag{1}$$

$$\overline{\sigma^{-1} S} = \sigma^{-1} \overline{S} \tag{2}$$

First we note that

Lemma A.4

$$|M_s^{-1} S| = \frac{1}{s} |S| \bigcap \mathbf{Z}_+.$$

Proof. In fact

$$\begin{aligned} n \in |M_s^{-1} S| &\iff \exists k \, [(n, k) \in M_s^{-1} S] \\ &\iff \exists k \, [(sn, sk) \in S] \end{aligned}$$

$$\Longrightarrow ns \in |S| \Longrightarrow n \in \frac{1}{s}|S| \bigcap \mathbf{Z}_+.S$$

Conversely let $n \in \frac{1}{s}|S| \bigcap \mathbf{Z}_+$. Then $(ns, k) \in S$ for some k, whence $(ns, nk) \in S$, which implies $(n, k) \in M_s^{-1}S$. Hence $n \in |M_s^{-1}|$. □

Hence

$$\begin{aligned}(n,k) \in \overline{M_S^{-1}S} &\iff n + k.\mathbf{N} \subseteq |M_S^{-1}S| = \frac{1}{s}|S| \bigcap \mathbf{Z}_+ \\ &\iff n + k.\mathbf{N} \subseteq \frac{1}{s}|S| \\ &\iff sn + sk.\mathbf{N} \subseteq |S| \\ &\iff (sn, sk) \in \overline{S} \iff (n,k) \in M_S^{-1}S.\end{aligned}$$

Lemma A.5

$$|\sigma^{-1}S| = (|S| - 1) \bigcap \mathbf{Z}_+,$$

where $|S| - 1 := \{ s - 1 \mid s \in |S| \}$.

Proof.

$$\begin{aligned}n \in |\sigma^{-1}S| &\iff \exists k\ [(n,k) \in \sigma^{-1}S] \\ &\iff \exists k\ [(n+1,k) \in S] \iff n + 1 \in |S|\end{aligned}$$

□

Hence

$$\begin{aligned}(n,k) \in \overline{\sigma^{-1}S} &\iff n + k.\mathbf{N} \subseteq |\sigma^{-1}S| \\ &\iff n + 1 + k.\mathbf{N} \subseteq |S| \\ &\iff (n+1,k) \in \overline{S} \\ &\iff (n,k) \in \sigma^{-1}S\end{aligned}$$

□

Proposition A.6 *The endo arrow $j : \Omega \to \Omega$ is a Lawvere-Tierney topology.*

Proof. The conditions (LT1) and (LT2) follows from Lemma A.2. It remains to show that

$$\overline{S_1 \bigcap S_2} = \overline{S_1} \bigcap \overline{S_2},$$

for $S_1, S_2 \subseteq \mathbf{Z}_+ \times \mathbf{N}$. Since

$$\overline{S_1 \bigcap S_2} \subseteq \overline{S_1} \bigcap \overline{S_2}$$

is obvious, we have to show the other inclusion.

Let $(n,k) \in \overline{S_1} \bigcap \overline{S_2}$. Then $n \in |S_1| \bigcap |S_2|$ and

$$n + k.\mathbf{N} \subseteq |S_1| \qquad n + k.\mathbf{N} \subseteq |S_2|.$$

Then

$$n + k.\mathbf{N} \subseteq |S_1| \bigcap |S_2|$$

and we have $(n,k) \in \overline{S_1 \bigcap S_2}$, since we have

$$|S_1| \bigcap |S_2| = |S_1 \bigcap S_2|.$$

In fact, $n \in |S_1| \bigcap |S_2|$ means $(n, k_1) \in S_1$ and $(n, k_2) \in S_2$ for some $k_1, k_2 \in \mathbf{N}$. Then $(n, k_1 k_2) \in S_1 \bigcap S_2$ and hence we have $n \in |S_1 \bigcap S_2|$. □

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