On the theory of KM₂O-Langevin equations for stationary flows (3): extension theorem

Yasunori OKABE and Masaya MATSUURA

(Received April 12, 1999; Revised July 22, 1999)

Abstract. In this paper, we prove an extension theorem for a given stationary flow from the pointview of the fluctuation-dissipation theorem and apply it to an extension problem for a given positive definite matrix function with Toeplitz condition defined on a finite set.

Key words: pair of flows, stationarity, fluctuation-dissipation theorem, KM₂O-Langevin equations, KM₂O-Langevin matrix.

1. Introduction

From a practical and theoretical motivation to get a method that stands the test of analysis for time series in complex system, we have in the previous two papers ([2], [3]) studied the pair of flows in a metric vector space. We have in [2] characterized a notion of stationarity for the pair of flows in terms of the fluctuation-dissipation theorem ((DDT) and (FDT)) that hold among the KM₂O-Langevin matrix associated with the pair of flows. Moreover, we have obtained a formula ((PAC)) which builds a bridge between the KM₂O-Langevin matrix and the covariance matrix function of the stationary pair of flows.

In [3], for any positive definite matrix function R defined on a finite set $\{0, 1, \ldots, N\}$ with Toeplitz condition, we have constructed a KM₂O-Langevin matrix $\mathcal{LM}(R)$ associated with R, by using (DDT), (FDT) and (PAC) as an algorithm that holds among $\mathcal{LM}(R)$. Further, we have constructed a stationary pair of flows with the matrix function R its covariance matrix function, by solving the KM₂O-Langevin equation describing the time evolution for the pair of flows.

The purpose of this paper is to prove an extension theorem of a given stationary flow $\mathbf{X} = (X(n); 0 \le n \le N)$ from the pointview of the fluctuation-fluctuation theorem ((FFT)) and to apply it to an extension problem for a given positive definite matrix function defined on a finite set $\{0, 1, \ldots, N\}$ with Toeplitz condition.

2.   An extension problem for stationary flow

Let $\mathbf{X} = (X(n); N_1 \le n \le N_2)$ be any d-dimensional flow in a metric vector space W with an inner product $(\star, \ast)$ over real field $\mathbf{R}$. We shall say that the flow $\mathbf{X}$ has a linearly independent property if and only if $\{X_j(n); 1 \le j \le d, N_1 \le n \le N_2\}$ is linearly independent in the vector space W. On the other hand, we shall say that the flow $\mathbf{X}$ has a stationary property if and only if there exists a matrix function $R : \{N_1 - N_2, N_1 - N_2 + 1, \ldots, N_2 - N_1 - 1, N_2 - N_1\} \to M(d; \mathbf{R})$ such that

$$(X(m), {}^tX(n)) = R(m - n) \qquad (N_1 \le m, n \le N_2). \tag{2.1}$$

Then we call the function R the covariance matrix function of the stationary flow $\mathbf{X}$. When it needs to manifest the time domain of the flow $\mathbf{X}$ and the function R, we shall write $\mathbf{X}$ and R into $\mathbf{X}^{(N_1,N_2)}$ and $R^{(N_2-N_1)}$, respectively.

Throughout the remainder of this section, it will be assumed, unless stated otherwise, that the flow $\mathbf{X} = (X(n); 0 \le n \le N)$ is a d-dimensional stationary flow in the space W with the covariance matrix function $R = (R(n); |n| \le N)$. Moreover we assume that the flow $\mathbf{X}$ has a linearly independent property. We denote by $\mathcal{LM}(R)$ the KM$_2$O-Langevin matrix associated with the matrix function R:

$$\mathcal{LM}(R) = \{\gamma_\pm(n, k), \delta_\pm(n), V_\pm(\ell); 0 \le k < n \le N, 0 \le \ell \le N\}. \tag{2.2}$$

In order to treat an extension problem for the stationary flow $\mathbf{X}$, for any d-dimensional vector η_- in W^d, we define a d-dimensional vector $X(-1)$ in W^d and two $d \times d$-matrices $R(\pm(N+1))$ by

$$X(-1) \equiv -\sum_{k=0}^{N-1} \gamma_-(N, k)X(N - k - 1) + \eta_-, \tag{2.3}$$

$$R(N + 1) \equiv (X(N), {}^tX(-1)), \tag{2.4}$$

$$R(-N-1) \equiv {}^t R(N+1). \tag{2.5}$$

We note that any d-dimensional vector can be represented into the righthand side of (2.3) with some d-dimensional vector η_-. Then we shall consider the flow $\mathbf{X}^{(-1,N)} \equiv \{X(n); -1 \le n \le N\}$ and the matrix function $R^{(N+1)} \equiv \{R(n); |n| \le N+1\}$.

The first aim is to obtain a necessary and sufficient condition for the flow $\mathbf{X}^{(-1,N)}$ to satisfy the following conditions:

$\mathbf{X}^{(-1,N)}$ is a stationary flow with its covariance matrix $R^{(N+1)}$,

$$\tag{2.6}$$

$\mathbf{X}^{(-1,N)}$ has a linearly independent property.   (2.7)

For convenience, we shall define a d-dimensional flow $\mathbf{Y} = (Y(\ell); -N \le \ell \le 0)$ by

$$Y(\ell) \equiv X(N+\ell-1) \qquad (-N \le \ell \le 0). \tag{2.8}$$

We note that equation (2.3) can be rewritten into

$$Y(-N) \equiv -\sum_{k=0}^{N-1} \gamma_-(N,k) Y(-k) + \eta_-. \tag{2.9}$$

Then it can be seen that

Lemma 2.1   *Condition* (2.6) *holds if and only if the pair* $[\mathbf{X}, \mathbf{Y}]$ *of flows has a stationary property.*

Moreover, we shall show

Lemma 2.2   *The pair* $[\mathbf{X}, \mathbf{Y}]$ *of flows has a stationary property if and only if the vector* η_- *satisfies the following properties*:

$$\begin{cases} (X(n), {}^t\eta_-) = 0 & (0 \le n \le N-1), \\ (\eta_-, {}^t\eta_-) = V_-(N). \end{cases}$$

Proof.   We assume that the pair $[\mathbf{X}, \mathbf{Y}]$ of flows has a stationary property. Then we can see from Lemma 2.1 and (2.9) that $\eta_- = \nu_-(\mathbf{Y})(-N)$, where $\nu_-(\mathbf{Y})$ is the backward KM₂O-Langevin fluctuation flow associated with the flow $\mathbf{Y}$. Therefore, the necessary part follows from Theorem 2.2 in [2]. The sufficient part follows from the characterization theorem for the sationarity

(Theorem 4.2 in [2]).   □

Concerning (2.7), we can show

Lemma 2.3   *Condition* (2.7) *holds if and only if the subset* $\{X_j(n), \eta_k^-; 1 \leq j, k \leq d, 0 \leq n \leq N\}$ *is linearly independent in* W, *where* η_k^- *is the* k*th component of the vector* η_- $(1 \leq k \leq d)$.

By combining Lemmas 2.1, 2.2 and 2.3, we have

Theorem 2.1   *The necessary and sufficient condition for the flow* $\mathbf{X}^{(-1,N)}$ *to have both the stationarity and the linearly independence is that the vector* η_- *satisfies the following conditions*:

$$\begin{cases} (X(n), {}^t\eta_-) = 0 \quad (0 \leq n \leq N - 1), \\ (\eta_-, {}^t\eta_-) = V_-(N), \\ \{X_j(n), \eta_k^-; 1 \leq j, k \leq d, 0 \leq n \leq N\} \text{ is independent in } W. \end{cases}$$

The results we have obtained so far for the stationary flow $\mathbf{X} = (X(n); 0 \leq n \leq N)$ can be generalized for any d-dimensional stationary flow $\mathbf{X}^{(N_1,N_2)} = (X(n); N_1 \leq n \leq N_2)$ satisfying the linearly independent property. We denote by $R^{(N_2-N_1)} = (R(n); |n| \leq N_2 - N_1)$ and $\mathcal{LM}(R) = \{\gamma_\pm(n,k), \delta_\pm(n), V_\pm(\ell); 0 \leq k < n \leq N_2 - N_1, 0 \leq \ell \leq N_2 - N_1\}$ the covariance matrix function and the KM$_2$O-Langevin matrix associated with the stationary flow $\mathbf{X}^{(N_1,N_2)}$. Similarly as in (2.3) and (2.4), for any d-dimensional vector η_- in W^d, we define a d-dimensional vector $X(N_1 - 1)$ in W^d and two $d \times d$-matrices $R(\pm(N_2 - N_1 + 1))$ by

$$X(N_1 - 1) \equiv -\sum_{k=0}^{N_2-N_1} \gamma_-(N_2 - N_1, k)X(N_2 - k - 1) + \eta_-, \quad (2.10)$$

$$R(N_2 - N_1 + 1) \equiv (X(N_2), {}^tX(N_1 - 1)), \quad (2.11)$$

$$R(-N_2 - N_1 - 1) \equiv {}^tR(N + 1). \quad (2.12)$$

Then we consider the flow $\mathbf{X}^{(N_1-1,N_2)} \equiv \{X(n); N_1 - 1 \leq n \leq N_2\}$ and the matrix function $R^{(N_2-N_1+1)} \equiv \{R(n); |n| \leq N_2 - N_1 + 1\}$. By applying Theorem 2.1 to the new flow $(X(N_1 + n); 0 \leq n \leq N_2 - N_1)$, we can obtain

Theorem 2.2   *The necessary and sufficient condition for the flow* $\mathbf{X}^{(N_1-1,N_2)}$ *to have both the stationarity and the linearly independence with*

$R^{(N_2-N_1+1)}$ *its covariance matrix function is that the vector η_- satisfies the following conditions*:

$$\begin{cases} (X(n), {}^t\eta_-) = 0 & (N_1 \le n \le N_2 - 1), \\ (\eta_-, {}^t\eta_-) = V_-(N_2 - N_1), \\ \{X_j(n), \eta_k^-; 1 \le j, k \le d, N_1 \le n \le N_2\} \text{ is independent in } W. \end{cases}$$

By making a repeated use of Theorem 2.2, we have one of the main theorems in this paper.

Theorem 2.3 *If the dimension of the vector space W is not less than $2Nd$, then any stationary flow $\mathbf{X} = (X(n); 0 \le n \le N)$ in the space W satisfying the linearly independence can be extended to a stationary flow in W with the linearly independence whose time domain is the set $\{-N, -N + 1, \ldots, N-1, N\}$. That is, there exist d-dimensional vectors $X(-j)$ $(1 \le j \le N)$ in W^d such that $\mathbf{X}^{(-N,N)} = (X(n); |n| \le N)$ has both the stationarity and the linearly independence.*

The following algorithm gives the proof of Theorem 2.3.

Algorithm 2.1

(**step 0**)   Since $\dim W \ge 2Nd$, we can choose a system $\Xi = \{\xi_{jn}^-; 1 \le j \le d, -N \le n \le -1\}$ of vectors in the vector space W such that

$$\begin{cases} \left(\xi_{jm}^-, \xi_{kn}^-\right) = \delta_{jk}\delta_{mn} & (1 \le j, k \le d, -N \le m, n \le -1), \\ \left(X_j(m), \xi_{kn}^-\right) = 0 & (1 \le j, k \le d, 0 \le m \le N - 1, -N \le n \le -1), \\ \{X_j(m), \xi_{kn}^-; 1 \le j, k \le d, 0 \le m \le N, -N \le n \le -1\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{is linearly independent in } W, \end{cases}$$

and construct d-dimensional vectors $\xi_-(n) \equiv {}^t(\xi_{1n}^-, \xi_{2n}^-, \ldots, \xi_{dn}^-)$ $(-N \le n \le -1)$.

(**step 1-1**)   Calculate $\mathcal{LM}([\mathbf{X}, \tilde{\mathbf{X}}]) = \mathcal{LM}(R) = \{\gamma_\pm(n, k), \delta_\pm(n),$ $V_\pm(m); 0 \le k < n \le N, 0 \le m \le N\}$, where $\tilde{\mathbf{X}} = (\tilde{X}(\ell); -N \le \ell \le 0)$ is defined by $\tilde{X}(\ell) \equiv X(N + \ell)$ $(-N \le \ell \le 0)$.

(**step 1-2**)   Choose $W_-(N) \in M(d; \mathbf{R})$ such that $V_-(N) = W_-(N)^t W_-(N)$.

(**step 1-3**)   Define a d-dimensional vector $\eta_-(-1) \equiv W_-(N)\xi_-(-1)$.

(**step 1-4**)   Define a d-dimensional vector $X(-1) \equiv -\sum_{k=0}^{N-1} \gamma_-(N, k) X(N - k - 1) + \eta_-(-1)$.

It is easy to see that $\eta_-(-1)$ satisfies

$$
\begin{cases}
(X(n), {}^t\eta_-(-1)) = 0 \quad (0 \le n \le N-1), \\
(\eta_-(-1), {}^t\eta_-(-1)) = V_-(N), \\
\{X_j(n), \eta_k^-(-1); 1 \le j, k \le d, 0 \le n \le N\} \\
\hspace{6cm} \text{is linearly independent in } W,
\end{cases}
$$

where $\eta_k^-(-1)$ denotes the kth component of the vector $\eta_-(-1)$. Hence the extended flow $\mathbf{X}^{(-1,N)} = (X(n); -1 \le n \le N)$ is a stationary flow satisfying the linearly independence with $N_1 = -1$ and $N_2 = N$.

Let p be any integer such that $1 \le p \le N-1$. We assume that the flow $\mathbf{X}^{(-p,N)} = (X(n); -p \le n \le N)$ obtained through (step 0) $\sim$ (step p - 4) is a stationary flow satisfying the linearly independence with $N_1 = -p$ and $N_2 = N$. We now construct a d-dimensional vector $X(-p-1)$ as follows:

(**step** $(p+1)$-**1**) Calculate $R(-N-p)$, $\delta_\pm(N+p)$, $\gamma_\pm(N+p,k)$, $V_\pm(N+p)$ by using (DDT), (FDT) and (PAC):

$$
R(-N-p) = (X(-p), \, {}^t X(N)), \tag{2.13}
$$

$$
\delta_-(N+p) = -\Big\{ R(-N-p) \tag{2.14}
$$

$$
+ \sum_{k=0}^{N+p-2} \gamma_-(N+p-1,k)R(-k-1) \Big\} V_+(N+p-1)^{-1},
$$

$$
\delta_+(N+p) = V_+(N+p-1){}^t\delta_-(N+p)V_-(N+p-1)^{-1}, \tag{2.15}
$$

$$
\gamma_\pm(N+p,k) = \gamma_\pm(N+p-1,k-1) \tag{2.16}
$$
$$
+ \delta_\pm(N+p)\gamma_\mp(N+p-1,N+p-k-1),
$$

$$
V_\pm(N+p) = (I - \delta_\pm(N+p)\delta_\mp(N+p))V_\pm(N+p-1). \tag{2.17}
$$

(**step** $(p+1)$-**2**) Choose $W_-(N+p) \in M(d; \mathbf{R})$ such that $V_-(N+p) = W_-(N+p){}^t W_-(N+p)$.

(**step** $(p+1)$-**3**) Define a d-dimensional vector $\eta_-(-p-1) \equiv W_-(N+p)\xi_-(-p-1)$.

(**step** $(p+1)$-**4**) Define a d-dimensional vector $X(-p-1)$ by

$$
X(-p-1) \equiv - \sum_{k=0}^{N+p-1} \gamma_-(N+p,k)X(N-k-1) + \eta_-(-p-1).
$$

$$
\tag{2.18}
$$

It is easily verified that $\eta_-(-p-1)$ defined in (step p-3) satisfies

$$\begin{cases} \left(X(n), {}^t\eta_-(-p-1)\right) = 0 & (-p \le n \le N-1), \\ \left(\eta_-(-p-1), {}^t\eta_-(-p-1)\right) = V_-(N+p), \\ \{X_j(n), \eta_k^-(-p-1); 1 \le j,k \le d, -p \le n \le N\} \\ \hspace{4cm} \text{is linearly independent in } W, \end{cases}$$

where $\eta_k^-(-p-1)$ denotes the kth component of the vector $\eta_-(-p-1)$. Hence from Theorem 2.2, the extended flow $\mathbf{X}^{(-p-1,N)} = (X(n); -p-1 \le n \le N)$ has both the stationarity and the linealry independence with $N_1 = -p-1$ and $N_2 = N$.

Thus we can extend the stationary flow $\mathbf{X} = (X(n); 0 \le n \le N)$ to a stationary flow $\mathbf{X}^{(-N,N)} = (X(n); |n| \le N)$.

Theorem 2.2 gives the solution of the backward extension problem for the stationary flows. Similar results can be obtained for the forward extension problem. Let $\mathbf{X}^{(N_1,N_2)} = (X(n); N_1 \le n \le N_2)$ be any d-dimensional stationary flow satisfying the linearly independence in the space W. Let $R^{(N_2-N_1)} = (R(n); |n| \le N_2 - N_1)$ be the covariance matrix function of the stationary flow $\mathbf{X}$ and $\mathcal{LM}(R) = \{\gamma_\pm(n,k), \delta_\pm(n), V_\pm(l); 0 \le k < n \le N_2 - N_1, 0 \le l \le N_2 - N_1\}$ be the KM₂O-Langevin matrix associated with the matrix function $R^{(N_2-N_1)}$. Then we define a d-dimensional vector $X(N_2 + 1)$ by

$$X(N_2 + 1) \equiv - \sum_{k=0}^{N_2-N_1-1} \gamma_+(N_2 - N_1, k)X(N_1 + k + 1) + \eta_+,$$

$$\tag{2.19}$$

where η_+ is some vector in W^d.

Then the same techniques as those developed in the course of proving Theorem 2.2 are applicable to the forward extension problem and thereby we have the following theorem.

Theorem 2.4 (i) *The flow* $\mathbf{X}^{(N_1,N_2+1)} = (X(n); N_1 \le n \le N_2 + 1)$ *has a stationary property if and only if* η_+ *satisfies the following properties*:

$$\begin{cases} (X(n), {}^t\eta_+) = 0 & (N_1 + 1 \le n \le N_2), \\ (\eta_+, {}^t\eta_+) = V_+(N_2 - N_1). \end{cases}$$

(ii)   *The flow* $\mathbf{X}^{(N_1,N_2+1)}$ *satisfies the linearly independent property if and only if* η_+ *satisfies the following property*

$$\{X_j(m), \eta_k^+ \, ; 1 \leq j, k \leq d, N_1 \leq m \leq N_2\}$$

is linearly independent in W,

where η_k^+ *denotes the* kth *component of the vector* η_+.

One conclusion from Theorems 2.2 and 2.4 is the following.

Theorem 2.5 *Let* M_1 *and* M_2 *be two integers such that* $M_1 \leq N_1$ *and* $N_2 \leq M_2$. *If the dimension of the vector space* W *is not less than* $(M_2 - M_1)d$, *then any stationary flow* $\mathbf{X}^{(N_1,N_2)} = (X(n); N_1 \leq n \leq N_2)$ *in the space* W *satisfying the linearly independent property can be extended to a stationary flow the linearly independent property whose time domain is the set* $\{M_1, M_1 + 1, \ldots, M_2 - 1, M_2\}$.

3.   The fluctuation-fluctuation theorem

Let $\mathbf{X} = (X(n); |n| \leq N)$ be any d-dimensional stationary flow in the metric vector space W and $R = (R(n); |n| \leq 2N)$ the covariance matrix function of the flow $\mathbf{X}$. We define two flows $\mathbf{X}_+ = (X_+(n); 0 \leq n \leq N)$ and $\tilde{\mathbf{X}} = (\tilde{X}(\ell); -N \leq \ell \leq 0)$ by

$$\begin{cases} X_+(n) \equiv X(n) & (0 \leq n \leq N), \\ \tilde{X}(\ell) \equiv X(\ell) & (-N \leq l \leq 0). \end{cases} \tag{3.1}$$

Then the pair $[\mathbf{X}_+, \tilde{\mathbf{X}}]$ of flows has a stationary property. Moreover the pair $[\mathbf{X}_+, \tilde{\mathbf{X}}]$ of flows satisfies

$$\begin{cases} X_+(0) = \tilde{X}(0), \\ (X_+(n), {}^t\tilde{X}(-m)) = R(n+m) & (0 \leq n, m \leq N). \end{cases} \tag{3.2}$$

In [1], we have used the above relation (3.2) to derive the so-called fluctuation-fluctuation theorem ((FFT)) which gives a relation between the forward KM_2O-Langevin fluctuation flow associated with $\mathbf{X}_+$ and the backward KM_2O-Langevin fluctuation flow associated with $\tilde{\mathbf{X}}$, that is, a relation among the $d \times d$ matrces $I(m,n)$ defined by

$$I(m,n) \equiv (\nu_+(m), {}^t\nu_-(-n)) \qquad (1 \leq m, n \leq N). \tag{3.3}$$

In §2, we have used (3.2) to get (2.14) and (2.15) in Algorithm 2.1 which gives the structure of extension of the stationary flow $\mathbf{X}_+$ (Theorem 2.3). We shall show that the latter derives (FFT). Let $\mathcal{LM}([\mathbf{X}_+, \tilde{\mathbf{X}}]) = \{\gamma_\pm(n, k), \delta_\pm(n), V_\pm(l); 0 \le k < n \le N, 0 \le l \le N\}$ be the KM₂O-Langevin matrix associated with the pair $[\mathbf{X}_+, \tilde{\mathbf{X}}]$ of flows. By replacing p by $p - 1$ in (2.18) and noting the stationary property of the flow $\mathbf{X}^{(-p, N)} = (X(n); -p \le n \le N)$, we have

$$
\begin{aligned}
R(-N - P) &= \left(X(-p), {}^tX(N)\right) \\
&= -\sum_{k=0}^{N+p-2} \gamma_-(N + p - 1, k)R(-k - 1) \\
&\quad + (\eta_-(-p), {}^tX(N)).
\end{aligned}
$$

Hence (2.14) and (2.15) in (step $(p + 1)$ - 1) are reduced to

$$
\delta_-(N + p) = -(\eta_-(-p), {}^tX(N))V_+(N + p - 1)^{-1}, \tag{3.4}
$$

$$
\delta_+(N + p) = -(X(N), {}^t\eta_-(-p))V_-(N + p - 1)^{-1}. \tag{3.5}
$$

From (3.4) and (3.5), we have

Fluctuation-Fluctuation Theorem-1

(i) $I(0, 0) = V_+(0)$,

(ii) $I(n, 0) = I(0, n) = 0 \quad (1 \le n \le N)$,

(iii) $I(n, 1) = I(1, n) = -\delta_+(n + 1)V_-(n) \quad (1 \le n \le N - 1)$.

More generally, by taking the same procedure as in [1], we have

Fluctuation-Fluctuation Theorem-2

(i)   *For any* m, n $(1 \le m \le N - 1, 2 \le n \le N)$,

$$
\begin{aligned}
I(m, n) &= I(m + 1, n - 1) + \left\{\sum_{k=1}^{n-2} I(m + 1, k) {}^t\delta_+(k + 1)\right\} {}^t\delta_-(n) \\
&\quad - \delta_+(m + 1)\left\{\sum_{k=1}^{m-1} \delta_-(k + 1)I(k, n)\right\}.
\end{aligned}
$$

(ii) *For any m, n $(2 \leq m \leq N, 1 \leq n \leq N - 1)$,*

$$I(m, n) = I(m - 1, n + 1) + \delta_+(m) \left\{ \sum_{k=1}^{m-2} \delta_-(k + 1) I(k, n + 1) \right\}$$

$$- \left\{ \sum_{k=1}^{n-1} I(m, k) \, {}^t\delta_-(k + 1) \right\} \, {}^t\delta_-(n + 1).$$

4. An extension problem for positive definite matrix function

For an $M(d; \mathbf{R})$-valued function $R = (R(n); |n| \leq N)$ defined on the set $\{-N, -N + 1, \ldots, N - 1, N\}$, we say that R is a positive definite matrix function if and only if the following Toeplitz conditions hold:

$${}^t R(n) = R(-n) \quad (0 \leq n \leq N), \tag{4.1}$$

$$(T_\pm(n)\xi, \xi) \geq 0 \quad \text{for any } \xi \in \mathbf{R}^{nd} \quad (1 \leq n \leq N + 1), \tag{4.2$_\pm$}$$

$$T_\pm(n) \in GL(nd; \mathbf{R}) \quad (1 \leq n \leq N), \tag{4.3$_\pm$}$$

where for each integer n $(1 \leq n \leq N + 1)$ two $nd \times nd$ block matrices $T_\pm(n)$ are defined by

$$T_\pm(n) = \begin{pmatrix} R(0) & R(\pm 1) & \cdots & R(\pm(n-1)) \\ R(\mp 1) & R(0) & \cdots & R(\pm(n-2)) \\ \vdots & \vdots & \ddots & \vdots \\ R(\mp(n-1)) & R(\mp(n-2)) & \cdots & R(0) \end{pmatrix}. \tag{4.4$_\pm$}$$

Let $R^{(N)} = (R(n); |n| \leq N)$ be any positive definite matrix function. By a construction theorem in [3], we can construct the KM$_2$O-Langevin matrix associated with the matrix function $R^{(N)}$ by

$$\mathcal{LM}(R) \tag{4.5}$$
$$= \{\gamma_\pm(R)(n, k), \delta_\pm(R)(n), V_\pm(R)(l); 0 \leq k < n \leq N, 0 \leq l \leq N\}.$$

In order to extend the matrix function $R^{(N)}$ on the set $\{-N-1, -N, \ldots, N, N+1\}$, for any element Q of $M(d; \mathbf{R})$, we define $R(\pm(N+1))(\in M(d; \mathbf{R}))$

by

$$R(N+1) \equiv -\sum_{k=0}^{N-1} \gamma_+(R)(N,k)R(k+1) + Q, \tag{4.6}$$

$$R(-N-1) \equiv {}^t R(N+1). \tag{4.7}$$

It is to be noted from Burg's relation that (4.6) and (4.7) can be rewritten into

$$R(-N-1) \equiv -\sum_{k=0}^{N-1} \gamma_-(R)(N,k)R(-k-1) + {}^t Q, \tag{4.6'}$$

$$R(N+1) \equiv {}^t R(-N-1). \tag{4.7'}$$

We now prove the following theorem.

Theorem 4.1 *The function* $R^{(N+1)} = (R(n); |n| \le N+1)$ *is a positive definite matrix functon if and only if there exist two d-dimensional vectors* ζ *and* η *in some metric vector space* W^d *such that*

$$\begin{cases} (\zeta, {}^t\eta) = Q, \\ (\zeta, {}^t\zeta) = V_+(R)(N), \\ (\eta, {}^t\eta) = V_-(R)(N), \\ \{\zeta_j, \eta_k; 1 \le j, k \le d\} \text{ is linearly independent in } W, \end{cases} \tag{4.8}$$

where ζ_j *denotes the jth component of the vector* ζ *and* η_k *denotes the kth component of the vector* η.

Proof. We assume that $R^{(N+1)} = (R(n); |n| \le N+1)$ is a positive definite matrix function. By the construction theorem obtained in [3], there exist a d-dimensional stationary flow $\mathbf{X} = (X(n); 0 \le n \le N+1)$ satisfying the linearly independent property in some metric vector space W such that $(X(n), {}^t X(m)) = R(n-m)$ $(0 \le n, m \le N+1)$. We define a flow $\mathbf{X}'^{(-1,N)} = (X'(n); -1 \le n \le N)$ by $X'(n) \equiv X(n+1)$ $(-1 \le n \le N)$. Then we see that $(X'(n), {}^t X'(m)) = R(n-m)$ $(-1 \le n, m \le N)$. Hence it follows from Theorem 2.1 that $X'(-1)$ can be written into

$$X'(-1) = -\sum_{k=0}^{N-1} \gamma_-(R)(N,k)X'(N-k-1) + \eta_-, \tag{4.9}$$

where η_- is a d-dimensinal vector in W^d such that

$$(X'(n),{}^t\eta_-) = 0 \quad (0 \le n \le N - 1), \tag{4.10}$$

$$(\eta_-,{}^t\eta_-) = V_-(R)(N), \tag{4.11}$$

$\{X'_j(n), \eta_k^- ; 1 \le j, k \le d, 0 \le n \le N\}$ is linearly independent in W.

$$\tag{4.12}$$

Thus, we find from (4.9) that

$$\begin{aligned}
R(-N - 1) &= (X'(-1),{}^t X'(N)) \\
&= -\sum_{k=0}^{N-1} \gamma_-(R)(N,k)R(-k-1) + (\eta_-,{}^t X'(N)).
\end{aligned}$$

$$\tag{4.13}$$

On the other hand, $X'(N) = -\sum_{k=0}^{N-1}\gamma_+(R)(N,k)X'(k) + \nu_+(\mathbf{X}')(N)$, where $\nu_+(\mathbf{X}')$ is the forward KM$_2$O-Langevin fluctuation flow associated with the flow $\mathbf{X}'$. It follows from (4.10) that $(\eta_-,{}^t X'(N)) = (\eta_-,{}^t\nu_+ (\mathbf{X}')(N))$, which derives with (4.13) that

$$\begin{aligned}
&R(-N - 1) \\
&= -\sum_{k=0}^{N-1} \gamma_-(R)(N,k)R(-k-1) + (\eta_-,{}^t\nu_+(\mathbf{X}')(N)). \tag{4.14}
\end{aligned}$$

Here we note that $\nu_+(\mathbf{X}')(N)$ satisfies

$$(\nu_+(\mathbf{X}')(N),{}^t\nu_+(\mathbf{X}')(N)) = V_+(R)(N), \tag{4.15}$$

$$\{X'_j(n), \nu_{+k}(\mathbf{X}')(N); 1 \le j, k \le d, 0 \le n \le N - 1\}$$

is linearly independent in W. $\tag{4.16}$

Thus from (4.11), (4.12), (4.14), (4.15) and (4.16), we find that condition (4.8) holds with $\zeta = \nu_+(\mathbf{X}')(N)$ and $\eta = \eta_-$.

We now prove the converse. That is, under condition (4.8), we derive the positive definite property of the matrix function $R^{(N+1)} = (R(n); |n| \le N + 1)$. From the construction theorem and the arguments in the course of proving it, we know that there exists a stationary flow $\mathbf{X} = (X(n); 0 \le n \le$

N) in the metric vector space W appearing in (4.8) such that

$$
\begin{cases}
(X(n),{}^{t}X(m)) = R(n-m) & (0 \le n, m \le N), \\
(X(n),{}^{t}\eta) = 0 & (0 \le n \le N-1), \\
\nu_{+}(\mathbf{X})(N) = \zeta, \\
\{X_{j}(n), \eta_{k}; 1 \le j, k \le d, 0 \le n \le N\} \text{ is linearly independent in } W.
\end{cases}
\tag{4.17}
$$

From Theorem 2.1, we can extend the stationary flow by defining

$$
X(-1) \equiv -\sum_{k=0}^{N-1} \gamma_{-}(R)(N,k)X(N-k-1) + \eta.
$$

By noting that $X(N) = -\sum_{k=0}^{N-1} \gamma_{+}(R)(N,k)X(k) + \zeta$, we have

$$
\begin{aligned}
\big(X(-1),{}^{t}X(N)\big) &= -\sum_{k=0}^{N-1} \gamma_{-}(R)(N,k)R(-k-1) + \big(\eta,{}^{t}X(N)\big) \\
&= -\sum_{k=0}^{N-1} \gamma_{-}(R)(N,k)R(-k-1) + \big(\eta,{}^{t}\zeta\big) \\
&= -\sum_{k=0}^{N-1} \gamma_{-}(R)(N,k)R(-k-1) + {}^{t}Q \\
&= R(-N-1),
\end{aligned}
$$

which shows that the matrix function $R^{(N+1)} = (R(n); |n| \le N+1)$ is a covariance matrix function of the stationary flow $\mathbf{X}^{(-1,N)} = (X(n); -1 \le n \le N)$. Thus $R^{(N+1)}$ is a positive definite matrix function. $\square$

Given an $M(d;\mathbf{R})$-valued positive definite matrix function $R^{(N)} = (R(n); |n| \le N)$, we can always choose two d-dimensional vectors ζ and η in some metric vector space W^{d} such that

$$
\begin{cases}
(\zeta,{}^{t}\zeta) = V_{+}(R)(N), \\
(\eta,{}^{t}\eta) = V_{-}(R)(N), \\
\{\zeta_{j}, \eta_{k}; 1 \le j, k \le d\} \text{ is linearly independent in } W.
\end{cases}
$$

Then Theorem 4.1 tells us that by defining

$$
\begin{cases}
R(N+1) \equiv -\displaystyle\sum_{k=0}^{N-1} \gamma_+(R)(N,k)R(k+1) + \left(\zeta, {}^t\eta\right), \\[2mm]
R(-N-1) \equiv {}^t R(N+1),
\end{cases}
$$

we obtain an extended positive definite matrix function $R^{(N+1)} = (R(n);$ $|n| \leq N+1)$. Repeating this process, we can conclude that the following statement holds.

Theorem 4.2 *Let M be any integer such that $M > N$. Then the positive definite matrix function $R^{(N)} = (R(n); |n| \leq N)$ can be extended to a positive definite matrix function $R^{(M)} = (R(n); |n| \leq M)$ defined on the set $\{-M, -M+1, \ldots, M-1, M\}$.*

References

[1] Okabe Y., *A new algorithm derived from the view-point of the fluctuation-dissipation principle in the theory of KM_2O-Langevin equations.* Hokkaido Math. J. **22** (1993), 199–209.

[2] Okabe Y., *On the theory of KM_2O-Langevin equations for stationary flows* (I): *characterization theorem.* J. Math. Soc. Japan, Vol.**51**, No.4, 1999, 817–841.

[3] Okabe Y., *On the theory of KM_2O-Langevin equations for stationary flows* (II): *construction theorem.* to appear in the special volume in honor of the 70th birthday of Professor Takeyuki Hida, Word Scientific Publishing Co.

Yasunori Okabe
Department of Mathematical Engineering
and Information Physics
Graduate School and Faculty of Engineering
University of Tokyo
Hongo, Bunkyo-ku 113, Japan
E-mail: okabe@langevin2.t.u-tokyo.ac.jp

Masaya Matsuura
Department of Mathematical Engineering
and Information Physics
Graduate School and Faculty of Engineering
University of Tokyo
Hongo, Bunkyo-ku 113, Japan
E-mail: masaya@lorentz.expm.t.u-tokyo.ac.jp