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L^p-L^q asymptotic behaviors of the solutions to the perturbed Schrödinger equations

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Abstract. We consider the asymptotic behaviors of a solution to the Schrödinger equation as t goes to $\pm\infty$. We present the sharp asymptotics in $L^\infty(\mathbf{R}^m)$. In particular, the low energy part of the perturbed dynamics is dominant to the L^∞ scattering and an explicit asymptotic form is shown in the uniform convergence topology.

Our approach to prove the results is based on the application of two facts, i.e., the local energy decay of $e^{-itH}P_{ac}(H)$ due to Jensen-Kato [4] and the L^p-boundedness properties of wave operators due to Yajima [10].

Key words: scattering theory, wave operators, L^∞, zero energy resonance, Schrödinger group.

1. Introduction

In this paper, we consider the asymptotic behaviors of the solution to the linear Schrödinger equation of the following type:

$$\begin{cases} i\partial_t u = Hu, \\ u|_{t=0} = \phi, \end{cases} \tag{1.1}$$

where u is a complex valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^m$ $(m \geq 3)$. H is the Hamiltonian of the form $H = H_0 + V$, where $H_0 = -\Delta$ and V is a short range scalar potential.

If the potential V is rapidly decreasing as $|x| \to \infty$, a particle governed by the above dynamics is expected to be asymptotically free for large time t. In view of the quantum mechanics, this means that the quantum state approaches to the free state. Mathematically, the solution $u(t,x)$ of (1.1) tends to the solution $v(t,x)$ of the free equation $i\partial_t v = H_0 v$. More precisely, by introducing the wave operators $W_\pm \equiv s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ in $L^2(\mathbf{R}^m)$, it is well-known that, if ϕ belongs to the absolutely continuous part of H, then $\|e^{-itH}\phi - e^{-itH_0}W_\pm^*\phi\|_{L^2} \to 0$ as $t \to \pm\infty$, where $W_\pm^*$ are the adjoint operators of $W_\pm$. This scattering result can be extended into the general

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situation in $L^q(\mathbf{R}^m)$. Let $P_{ac}(H)$ is the orthogonal projection onto the absolutely continuous subspace of H. For some appropriate potential V, we note that the L^p-L^q estimate $\|e^{-itH}P_{ac}(H)\phi\|_{L^q} \le C|t|^{-m(1/2-1/q)}\|\phi\|_{L^p}$ holds with $1 \le p \le 2$ and $1/p + 1/q = 1$. Also we have the L^p-boundedness of $W_\pm^*$. If θ, p' and q' satisfy $1/p' + 1/q' = 1$ and $1/q = \theta/2 + (1-\theta)/q'$ for $0 < \theta \le 1$, then, by the Hölder inequality, it follows that, for $\phi \in L^2(\mathbf{R}^m) \cap L^{p'}(\mathbf{R}^m)$,

$$\begin{aligned} &\|e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi\|_{L^q} \\ &\le \|e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi\|_{L^2}^\theta \|e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi\|_{L^{q'}}^{1-\theta} \\ &\le C|t|^{-m(1/2-1/q)}\|e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi\|_{L^2}^\theta \|\phi\|_{L^{p'}}^{1-\theta}. \end{aligned} \tag{1.2}$$

Since $\|e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi\|_{L^2} = o(1)$ as $t \to \pm\infty$, this estimate shows that the left hand side of (1.2) decays at a rate faster than $|t|^{-m(1/2-1/q)}$.

Our main concern is now to study the case when $q = \infty$. We shall show that, in this case, the decay rate is just equal to $|t|^{-m/2}$ but not smaller order in general.

To state our result precisely, we specify the potential V as follows.

(V) **Assumptions on V:**

(V–1) V is a real valued function.

(V–2) There exists some $\delta > 3m/2 + 1$ such that

$$\sup_{x \in \mathbf{R}^m} \Big(\langle x\rangle^\delta |\partial^\alpha V(x)|\Big) < \infty$$

holds for any α with $|\alpha| \le m - 3$.

(V–3) In addition, if $3 \le m \le 6$, we assume that

$$\widehat{V} \in L^1(\mathbf{R}^m),$$

where $\widehat{V}$ stands for the Fourier transform of V.

Note that, if $m \ge 7$, then the smoothness condition (V–2) automatically implies (V–3).

We call that zero is the resonance of H if the equation $-\Delta u + Vu = 0$ (in distribution sense) has a solution u such that $u \notin L^2(\mathbf{R}^m)$ but $(1 + |x|^2)^{-\gamma/2}u \in L^2(\mathbf{R}^m)$ for any $\gamma > 1/2$. We assume the second assumption according to [4].

(S) **Spectral assumption of H:** Zero is neither an eigenvalue nor a resonance of H.

Under the assumption (V), it is well-known that

(i) H is the self-adjoint operator on $L^2(\mathbf{R}^m)$ with the domain $\mathcal{D}(H) = H^2(\mathbf{R}^m)$,

(ii) the wave operators $W_\pm$ exist,

(iii) $W_\pm$ are complete (see for example [1], [6]).

We are now in the position to state our result.

Theorem 1.1 *Suppose that the assumptions* (V) *and* (S) *are satisfied. Then, the operators $e^{-itH}P_{ac}(H) - e^{-itH_0}W_\pm^*$ can be extended from $\mathcal{B}(L^2(\mathbf{R}^m))$ to $\mathcal{B}(L^1(\mathbf{R}^m); L^\infty(\mathbf{R}^m))$ for $t \neq 0$. Besides, for $\phi \in L^1(\mathbf{R}^m)$, we have*

$$\lim_{t\to\pm\infty}(4\pi it)^{m/2}\left(e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi\right)$$
$$= -\langle (I+G_0V)^{-1}1, \phi\rangle G_0V(I+G_0V)^{-1}1, \quad in \ \ L^\infty(\mathbf{R}^m), \quad (1.3)$$

where $G_0(=(-\Delta)^{-1})$ is the integral operator defined by $G_0\phi(x) = \frac{1}{(m-2)\omega_m}\int |x-y|^{-m+2}\phi(y)dy$ (ω_m is the surface measure of the unit sphere).

We note some technical remarks.

Remark 1.1 According to [2]–[4], the inverse of $I + G_0V$ exists in the weighted Sobolev spaces if the spectral assumption (S) is satisfied. Since we can show that both $(I+G_0V)^{-1}1$ and $G_0V(I+G_0V)^{-1}1$ belong to $L^\infty(\mathbf{R}^m)$ (see Lemma 2.3 and 2.4), the right hand side of (1.3) is well-defined for any $\phi \in L^1(\mathbf{R}^m)$.

Remark 1.2 If $\phi \in L^1(\mathbf{R}^m)$, then $(4\pi it)^{m/2}e^{-itH_0}W_\pm^*\phi \to \langle 1, W_\pm^*\phi\rangle 1$ as $t \to \pm\infty$ in $H^{0,-\gamma} = \{f; \langle x\rangle^{-\gamma}f \in L^2\}$ with γ large enough. Applying Theorem 1.1 and the local energy decay estimate of $e^{-itH}P_{ac}(H)\phi$ (see Lemma 2.1 below), we can show that

$$\begin{aligned} -\langle 1, W_\pm^*\phi\rangle 1 &= \lim_{t\to\pm\infty}(4\pi it)^{m/2}(e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi) \\ &\quad - \lim_{t\to\pm\infty}(4\pi it)^{m/2}e^{-itH}P_{ac}(H)\phi \\ &= -\langle (I+G_0V)^{-1}1, \phi\rangle 1 \quad \text{in } H^{0,-\gamma}. \end{aligned}$$

Hence, $\widehat{W_\pm^*\phi}(0) = \langle 1, W_\pm^*\phi\rangle = \langle (I+G_0V)^{-1}1, \phi\rangle$ for any $\phi \in L^1(\mathbf{R}^m)$.

This implies, from the physical point of view, that the zero momentum component of the state $W_\pm^*\phi$ contributes to the right hand side of (1.3). Note that $\langle (I+G_0V)^{-1}1, \phi\rangle$ is generally away from 0.

Remark 1.3 We note that the right hand side of (1.3) is not identically zero unless $V \equiv 0$. In fact, one can show that $G_0V(I+G_0V)^{-1}1 \equiv 0$ only if $V \equiv 0$. If $G_0V(I+G_0V)^{-1}1 \equiv 0$, then $1 = (I+G_0V)(I+G_0V)^{-1}1 = (I+G_0V)^{-1}1$ and it follows $G_0V1 = 0$, i.e., $V = (-\Delta)G_0V1 = 0$.

The result of Theorem 1.1 shows a large contrast with the scattering result in $L^q(\mathbf{R}^m)$ when $q < \infty$. In fact, the right hand side of (1.3) is basically the low energy part of the asymptotic expansion $i\int_t^\infty e^{-i(t-s)H_0}Ve^{-isH}P_{ac}(H)\phi\, ds$. This term is visible only when we observe the scattering in $L^\infty(\mathbf{R}^m)$ space.

Before closing this section, we introduce some notations. Let $\langle\cdot,\cdot\rangle$ be the inner product defined by

$$\langle f, g\rangle = \int_{\mathbf{R}^m} \bar{f}(x)g(x)dx.$$

$\mathcal{S}'$ denotes the space of tempered distributions and $H^{\sigma,\gamma}(\mathbf{R}^m)$ is the weighted Sobolev space given by

$$H^{\sigma,\gamma}(\mathbf{R}^m) = \{f \in \mathcal{S}'; \|f\|_{\sigma,\gamma} = \|(1+|x|^2)^{\gamma/2}(1-\Delta)^{\sigma/2}f\|_{L^2} < \infty\}.$$

Note that the norm $\|(1-\Delta)^{\sigma/2}(1+|x|^2)^{\sigma/2}f\|_{L^2}$ is equivalent to $\|f\|_{\sigma,\gamma}$. We often use the brief notations L^p, $H^{\sigma,\gamma}$ in place of $L^p(\mathbf{R}^m)$, $H^{\sigma,\gamma}(\mathbf{R}^m)$. The abbreviation $\mathcal{B}(X)$ stands for $\mathcal{B}(X;X)$.

2. Preliminaries

We start to state the asymptotic behaviors of the perturbed Schrödinger evolution due to Jensen [2]–[3] and Jensen-Kato [4].

Lemma 2.1 ([2]–[4]) *For $m \geq 3$, Let $\gamma, \gamma' > m/2+1$ (if m is odd) and $\gamma, \gamma' > m/2+2$ (if m is even). Under the assumptions* (V) *and* (S)*, we have*

$$\begin{aligned} &e^{-itH}P_{ac}(H) \\ &\quad = (4\pi it)^{-m/2}\langle (I+G_0V)^{-1}1, \cdot\rangle (I+G_0V)^{-1}1 + o(t^{-m/2}) \qquad (2.1) \end{aligned}$$

as $t \to \pm\infty$ in $\mathcal{B}(H^{0,\gamma}(\mathbf{R}^m); H^{0,-\gamma'}(\mathbf{R}^m))$.

For the proof of Lemma 2.1, see [4] (cf. [2], [3]). □

To extend the operators $e^{-itH}P_{ac}(H) - e^{-itH_0}W_\pm^*$ from $\mathcal{B}(L^2)$ to $\mathcal{B}(L^1; L^\infty)$, we need the L^p-boundedness of $W_\pm^*$. This properties of $W_\pm^*$ was proved by Yajima [10].

Lemma 2.2 ([10]) *Let $m \geq 3$ and $1 \leq p \leq \infty$. Then, under the assumptions* (V) *and* (S), *the wave operators $W_\pm$ are extended from $\mathcal{B}(L^p(\mathbf{R}^m))$ to $\mathcal{B}(L^p(\mathbf{R}^m))$ and it follows that there exists a positive constant C such that*

$$\|W_\pm\|_{\mathcal{B}(L^p)} \leq C. \tag{2.2}$$

For the proof of lemma 2.2, see [8] and [10] (cf. [9]). □

Remark 2.1 In [8], Yajima proved the L^p-L^q estimate of $e^{-itH}P_{ac}(H)$ by applying the L^p-boundedness of $W_\pm$. Since $e^{-itH}P_{ac}(H) = W_\pm e^{-itH_0}W_\pm^*$, it follows that, for $1/p + 1/q = 1$ and $1 \leq p \leq 2$,

$$\|e^{-itH}P_{ac}(H)\|_{\mathcal{B}(L^p;L^q)} \leq C|t|^{-m(1/2-1/q)}. \tag{2.3}$$

Remark 2.2 In [10], Lemma 2.2 is proved under more general situation. For example, instead of the assumption (V–2), Lemma 2.2 holds if V satisfies

$$\sup_{x\in\mathbf{R}^m} \langle x\rangle^\delta \left(\int_{|x-y|\leq 1} |\partial^\alpha V(y)|^{p_0} dy \right)^{1/p_0} < \infty$$

for $|\alpha| \leq m-3$, where $p_0 > m/2$ if $m \geq 4$, and $p_0 = 2$ if $m = 3$. In this paper, however, we assume the condition (V) for simplicity of the proof.

On account of the following lemmas, the right hand side of (1.3) makes a sense for any $\phi \in L^1(\mathbf{R}^m)$.

Lemma 2.3 *Let V satisfy the assumption* (V). *Then,*

$$(I + G_0V)^{-1} \in \mathcal{B}(L^\infty(\mathbf{R}^m)). \tag{2.4}$$

Moreover, if $0 \leq \sigma \leq m-3$ and $m/2 < \gamma$, then

$$(I + G_0V)^{-1} \in \mathcal{B}(H^{\sigma,-\gamma}(\mathbf{R}^m)). \tag{2.5}$$

Proof of Lemma 2.3. Since $L^\infty \subset H^{0,-\gamma}$ for $\gamma > m/2$, $g \equiv (I + G_0V)^{-1}f$

is well-defined in $H^{0,-\gamma}$. To show $g \in L^\infty$, we rewrite g as

$$\begin{aligned} g &= \sum_{j=0}^{N-1} (-G_0V)^j f + (-G_0V)^N (I + G_0V)^{-1} f \\ &\equiv \sum_{j=0}^{N-1} A_j(f) + R_N(f), \end{aligned} \tag{2.6}$$

where N is the largest integer not to exceed $(m+1)/2$. We apply the Hölder inequality to obtain

$$\begin{aligned} &|A_j(f)(x)| \\ &\le C\left(\int_{|x-y|\le 1} + \int_{|x-y|>1}\right)|x-y|^{-(m-2)}V(y)|A_{j-1}(f)(y)|dy \\ &\le C\|VA_{j-1}(f)\|_{L^{p_1}} + C\|VA_{j-1}(f)\|_{L^1}, \end{aligned}$$

where $p_1 > m/2$ if $m \ge 4$ and $p_1 = 2$ if $m = 3$. From the Hardy-Littlwood-Sobolev inequality, we see that $G_0 \in \mathcal{B}(L^p; L^q)$ with $1/q = 1/p - 2/m$. On the other hand, $G_0 \in \mathcal{B}(H^{0,\gamma}; H^{0,-\gamma'})$ for some $\gamma, \gamma' > 1/2$ and $\gamma + \gamma' > 2$ (see, e.g., [3]; Lemma 2.3). Then, it follows that

$$\begin{aligned} \|VA_{j-1}(f)\|_{L^{p_1}} &\le C\|\langle x\rangle^\delta V\|_{L^\infty}^{j-1}\|Vf\|_{L^{p_j}} \\ &\le C\|\langle x\rangle^\delta V\|_{L^\infty}^{j}\|f\|_{L^\infty}, \\ \|VA_{j-1}(f)\|_{L^1} &\le C\|\langle x\rangle^\delta V\|_{L^\infty}^{j-1}\|Vf\|_{0,\gamma} \\ &\le C\|\langle x\rangle^\delta V\|_{L^\infty}^{j}\|f\|_{L^\infty}, \end{aligned}$$

where $p_j = m/2j + \epsilon$ and δ can be seen in the assumptions on V. Hence, there exists $C_V > 0$ depending on V such that

$$\|A_j(f)\|_{L^\infty} \le C_V\|f\|_{L^\infty}. \tag{2.7}$$

Similarly, we can show that

$$\begin{aligned} &\|R_N(f)\|_{L^\infty} \\ &\le C\|\langle x\rangle^\delta V\|_{L^\infty}^{N-1}\left(\|V(I+G_0V)^{-1}f\|_{L^{p_N}} + \|V(I+G_0V)^{-1}f\|_{0,\gamma}\right) \\ &\le C\|\langle x\rangle^\delta V\|_{L^\infty}^{N}\|f\|_{L^\infty}. \end{aligned} \tag{2.8}$$

Note that, we have used the fact that $(I+G_0V)^{-1} \in \mathcal{B}(H^{0,-\gamma})$ for $\gamma > m/2$ in the last inequality of (2.8). Hence (2.6), (2.7) and (2.8) give us (2.4) in Lemma 2.3. The proof of (2.5) is easily shown by the expansion (2.6) and the fact that $G_0 \in \mathcal{B}(H^{\sigma-2,\gamma}; H^{\sigma,-\gamma'})$ for $\gamma, \gamma' > 1/2$, $\gamma + \gamma' > 2$ and $\sigma \in \mathbf{R}$ (see, e.g., [3]; Lemma 2.4). □

Lemma 2.4 *Let V satisfy the assumption* (V).
If $f \in H^{[m/2]-1,-[m/2]-1}(\mathbf{R}^m)$, then $\int_0^{\pm\infty} e^{isH_0} V f ds \in L^\infty(\mathbf{R}^m)$ and we have

$$\int_0^{\pm\infty} e^{isH_0} V f ds = -iG_0 V f. \tag{2.9}$$

Proof of Lemma 2.4. We start the proof by showing that the integral in the left hand side of (2.9) makes a sense in L^∞. For sufficiently large N_1, N_2 with $N_1 > N_2$, we have, by using L^1-L^∞ estimate of e^{isH_0} that

$$\begin{aligned}
&\left\| \int_{N_2}^{N_1} e^{isH_0} V f ds \right\|_{L^\infty} \\
&\quad \le C \int_{N_2}^{N_1} s^{-m/2} \|Vf\|_{L^1} ds \\
&\quad \le C(N_1^{1-m/2} - N_2^{1-m/2}) \|\langle x\rangle^\delta V\|_{L^\infty} \|f\|_{0,-[m/2]-1}.
\end{aligned} \tag{2.10}$$

Hence, the left hand side of (2.10) tends to 0 as $N_1, N_2 \to \infty$.

On the other hand, for small ϵ_1 and ϵ_2, it follows that

$$\begin{aligned}
\left\| \int_{\epsilon_2}^{\epsilon_1} e^{isH_0} V f ds \right\|_{L^\infty} &= \left\| \int_{\epsilon_2}^{\epsilon_1} \int e^{is\xi^2 + ix\cdot\xi} \widehat{Vf}(\xi) d\xi ds \right\|_{L^\infty} \\
&= \left\| \int e^{ix\cdot\xi} (e^{is\epsilon_1\xi^2} - e^{is\epsilon_2\xi^2}) \xi^{-2} \widehat{Vf}(\xi) d\xi \right\|_{L^\infty} \\
&\le \int |e^{is(\epsilon_1-\epsilon_2)\xi^2} - 1| \xi^{-2} |\widehat{Vf}(\xi)| d\xi.
\end{aligned} \tag{2.11}$$

By Lebesgue's dominated convergence theorem, the right hand side of (2.11) tends to 0 as $\epsilon_1, \epsilon_2 \to 0$ if we show that $\xi^{-2}\widehat{Vf} \in L^1$. By applying the L^1-L^∞ estimate of the Fourier transform and the Schwarz inequality, we obtain

$$\begin{aligned}
\|\xi^{-2}\widehat{Vf}\|_{L^1} &\le \left(\int_{|\xi|\le 1} + \int_{|\xi|>1} \right) \xi^{-2} |\widehat{Vf}(\xi)| d\xi \\
&\le C(\|Vf\|_{L^1} + \|\langle\xi\rangle^{[m/2]-1} \widehat{Vf}\|_{L^2}) \\
&\le C \sup_{|\alpha|\le[m/2]-1} \|\langle x\rangle^\delta \partial^\alpha V\|_{L^\infty} \|f\|_{[m/2]-1,-[m/2]-1} < \infty.
\end{aligned}$$

Thus, $\lim_{N\to\infty,\epsilon\to 0}\int_\epsilon^N e^{isH_0}Vfds \in L^\infty$. To show the identity (2.9), we put $\psi \in C_0^\infty(\mathbf{R}^m)$ and we observe, by Fubini's theorem,

$$\begin{aligned}\left\langle \psi, \int_0^\infty e^{isH_0}Vfds \right\rangle &= \lim_{N\to\infty,\epsilon\to 0}\int_\epsilon^N \left\langle \hat{\psi}, e^{is\xi^2}\widehat{Vf}\right\rangle ds \\ &= \lim_{N\to\infty,\epsilon\to 0}\left\langle \hat{\psi}, -i(e^{iN\xi^2}-e^{i\epsilon\xi^2})\xi^{-2}\widehat{Vf}\right\rangle \\ &= \left\langle \hat{\psi}, i\xi^{-2}\widehat{Vf}\right\rangle = \langle \psi, iG_0Vf\rangle .\end{aligned}$$

The case $N\to -\infty$ can be shown similarly. This completes the proof. □

The following lemma is well known. (see e.g. Journé-Soffer-Sogge [5]).

Lemma 2.5 *Let $\hat{V}\in L^1(\mathbf{R}^m)$. Then, for $1\le p\le\infty$, $e^{i\sigma H_0}Ve^{-i\sigma H_0}\in \mathcal{B}(L^p(\mathbf{R}^m))$. Moreover, it follows that*

$$\|e^{i\sigma H_0}Ve^{-i\sigma H_0}\|_{\mathcal{B}(L^p)} \le (2\pi)^{-m/2}\|\widehat{V}\|_{L^1}. \tag{2.12}$$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we use the representation due to Cook–Kuroda.

$$\begin{aligned}&e^{-itH}P_{ac}(H)\phi - e^{-itH_0}W_\pm^*\phi \\ &= i\int_t^{\pm\infty} e^{-i(t-s_1)H_0}Ve^{-is_1H}P_{ac}(H)\phi ds_1 \\ &= i\int_0^{\pm\infty} e^{is_1H_0}Ve^{-i(s_1+t)H}P_{ac}(H)\phi ds_1.\end{aligned}\tag{3.1}$$

We decompose the integral range into two parts.

$$\begin{aligned}&\text{(The right hand side of (3.1))} \\ &= i\int_{\pm\epsilon}^{\pm\infty} e^{is_1H_0}Ve^{-i(s_1+t)H}P_{ac}(H)\phi ds_1 \\ &\quad + i\int_0^{\pm\epsilon} e^{is_1H_0}Ve^{-i(s_1+t)H}P_{ac}(H)\phi ds_1 \\ &\equiv Q_\pm(t,\epsilon,\phi)+R_\pm(t,\epsilon,\phi).\end{aligned}\tag{3.2}$$

To estimate $R_\pm(t,\epsilon,\phi)$, we need the following proposition.

Proposition 3.1 *For small $\epsilon > 0$, it follows that*

$$\|R_{\pm}(t,\epsilon,\phi)\|_{L^\infty} \le C_V \epsilon |t|^{-m/2} \|\phi\|_{L^1}, \tag{3.3}$$

where C_V is a positive constant depending on V but independent of ϵ.

Proof of Proposition 3.1. We only show the estimate of $R_+(t,\epsilon,\phi)$ since the result for $R_-(t,\epsilon,\phi)$ is proved by similar arguments. Putting $f = e^{-itH}P_{ac}(H)\phi$ for $\phi \in L^2 \cap L^1$, we have $R_+(t,\epsilon,\phi) = \int_0^\epsilon e^{is_1H_0}Ve^{-s_1H}fds_1$. Note that Remark 2.1 gives us $f \in L^2 \cap L^\infty$ for $t > 0$.

By Duhamel's formula, it follows that

$$\begin{aligned} e^{i\tau H_0}Ve^{-i\tau H}f &= e^{i\tau H_0}Ve^{-i\tau H_0}\phi \\ &\quad - i\int_0^\tau e^{i\tau H_0}Ve^{-i\tau H_0}(e^{i\sigma H_0}Ve^{-i\sigma H}f)d\sigma. \end{aligned} \tag{3.4}$$

Applying Lemma 2.5 to (3.4), we obtain

$$\begin{aligned} \|e^{i\tau H_0}Ve^{-i\tau H}f\|_{L^\infty} &\le C\|\hat{V}\|_{L^1}\|f\|_{L^\infty} \\ &\quad + C\|\hat{V}\|_{L^1}\int_0^\tau \|e^{i\sigma H_0}Ve^{-i\sigma H}f\|_{L^\infty}d\sigma. \end{aligned}$$

Hence, Gronwall's inequality gives us the estimate,

$$\begin{aligned} \|e^{i\tau H_0}Ve^{-i\tau H}f\|_{L^\infty} &\le C\|\hat{V}\|_{L^1}\exp(C\tau\|\hat{V}\|_{L^1})\|f\|_{L^\infty} \\ &\le C_V\|f\|_{L^\infty} \end{aligned} \tag{3.5}$$

for small $\tau > 0$. Using (3.5), we see that

$$\begin{aligned} \|R_+(t,\epsilon,\phi)\|_{L^\infty} &\le C_V\epsilon\|f\|_{L^\infty} \\ &\le C_V\epsilon|t|^{-m/2}\|\phi\|_{L^1}. \end{aligned}$$

This completes the proof of Proposition 3.1. □

In what follows, we use the notations; $(I + G_0V)^{-1}1 = u_0$ and $\langle (I + G_0V)^{-1}1, \phi\rangle = c_\phi$.

Proof of Theorem 1.1. Let ϕ be a rapidly decreasing function. By Lemma 2.1, we decompose $Q_+(t,\epsilon,\phi)$ in (3.2) as

$$\begin{aligned} &(4\pi it)^{m/2}Q_+(t,\epsilon,\phi) \\ &= ic_\phi\int_\epsilon^\infty e^{isH_0}Vt^{m/2}(s+t)^{-m/2}u_0ds \end{aligned}$$

$$+ i \int_\epsilon^\infty e^{isH_0} V t^{m/2} S(s+t)\phi ds$$
$$\equiv (4\pi i t)^{m/2} Q_+^{(1)}(t,\epsilon,\phi) + (4\pi i t)^{m/2} Q_+^{(2)}(t,\epsilon,\phi), \tag{3.6}$$

where $S(\sigma)$ is the operator satisfying $\|S(\sigma)\|_{\mathcal{B}(H^{0,\gamma};H^{0,-\gamma'})} = o(\sigma^{-m/2})$ as $\sigma \to \infty$. Since

$$\|(4\pi i t)^{m/2} Q_+^{(1)}(t,\epsilon,\phi) - ic_\phi \int_\epsilon^\infty e^{isH_0} V u_0 ds\|_{L^\infty}$$
$$\leq C \int_\epsilon^\infty s^{-m/2}\left(1 - t^{m/2}(s+t)^{-m/2}\right)\|Vu_0\|_{L^1} ds,$$

it is easy to see that the first term of (3.6) has a limit

$$\lim_{t\to\infty} (4\pi i t)^{m/2} Q_+^{(1)}(t,\epsilon,\phi)$$
$$= ic_\phi \int_\epsilon^\infty e^{isH_0} V u_0 ds \quad \text{in} \quad L^\infty. \tag{3.7}$$

Similarly, for the second term in (3.6), we can show that

$$\lim_{t\to\infty} (4\pi i t)^{m/2} Q_+^{(2)}(t,\epsilon,\phi) = 0 \quad \text{in} \quad L^\infty. \tag{3.8}$$

Hence, by (3.7), (3.8) and Proposition 3.1, we obtain

$$\limsup_{t\to\infty} \left\|(4\pi i t)^{m/2}(e^{-itH}P_{ac}(H)\phi - e^{-tH_0}W_+^*\phi) - ic_\phi \int_0^\infty e^{isH_0} V u_0 ds\right\|_{L^\infty}$$
$$\leq \left\|c_\phi \int_0^\epsilon e^{isH_0} V u_0 ds\right\|_{L^\infty} + C_V \epsilon \|\phi\|_{L^1}. \tag{3.9}$$

Since $u_0 \in H^{[m/2]-1,-[m/2]-1}$ by (2.5) in Lemma 2.3, we can apply Lemma 2.4 to obtain $\int_0^\infty e^{isH_0} V u_0 ds = -iG_0 V u_0$. According to (2.11) in the proof of Lemma 2.4, the first term in the right hand side of (3.9) vanishes as ϵ tends to 0. Hence, by letting $\epsilon \to 0$, we prove Theorem 1.1 for rapidly decreasing functions ϕ. Note that $\langle (I + G_0 V)^{-1}, \cdot\rangle$ can be extended to the operator in $\mathcal{B}(L^1; \mathbf{C})$ (see (2.4) in Lemma 2.3). Taking into account of the L^1-L^∞ estimates to $e^{-itH}P_{ac}(H)$ and $e^{-itH_0}W_\pm^*$ (see Remark 2.1) together with the simple density argument, we obtain Theorem 1.1 for any $\phi \in L^1$. □

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