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On the excess of a sequence of exponentials with perturbations at some subsequences of integers

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Abstract. It is known that the sequences of exponentials $\{1\} \cup \{e^{\pm i(n-\frac{1}{4})t}\}_{n=1}^{\infty}$ and $\{1\}\cup\{e^{\pm i(n+\frac{1}{4})t}\}_{n=1}^{\infty}$ have the excess 1 and 0 in $L^{2}[-\pi, \pi]$, respectively. In this article, we calculate the excess of a sequence of exponentials with perturbations at some subsequences of integers.

Key words: excess, exponential system, perturbation.

1. Introduction

Let $\lambda=\{\lambda_{n}\}$, $-\infty<n<\infty$, be a sequence of distinct complex numbers. A system $\{e^{i\lambda_{n}t}\}$ of complex exponentials is said to be *complete* in $L^{2}[-\pi, \pi]$ if the linear subspace spanned by $\{e^{i\lambda_{n}t}\}$ is dense in $L^{2}[-\pi, \pi]$. The system is said to be *minimal* in $L^{2}[-\pi, \pi]$ if each element of $\{e^{i\lambda_{n}t}\}$ lies outside the closed linear span of the others. We say the system $\{e^{i\lambda_{n}t}\}$ has *excess* N if it remains complete and becomes minimal when N terms $e^{i\lambda_{n}t}$ are removed and we define

$$E(\lambda)=N.$$

Conversely we define the excess

$$E(\lambda)=-N$$

if it becomes complete and minimal when N terms

$$e^{i\mu_{1}t}, \ldots, e^{i\mu_{N}t}$$

are adjoined. By convention we define $E(\lambda)=\infty$ if arbitrarily many terms can be removed without losing completeness and $E(\lambda)=-\infty$ if arbitrarily many terms can be adjoined without getting completeness. It is obvious that $\{e^{i\lambda_{n}t}\}$ is to be complete and minimal if and only if $E(\lambda)=0$. Also we denote by **PW** the Paley-Wiener space which is the set of all entire

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functions of exponential type at most π that are square integrable on the real axis. Now we define

$$\lambda_n = \begin{cases} n - \dfrac{1}{4}, & n > 0, \\ 0, & n = 0, \\ n + \dfrac{1}{4}, & n < 0, \end{cases}$$

then it is known that $E(\lambda) = 1$ (see [L, Ch.4, Theorem XIX] or [Y, Ch.3, §3, Theorem 5 and p.126, Problem 1]).

Next we define

$$\lambda_n = \begin{cases} n + \dfrac{1}{4}, & n > 0, \\ 0, & n = 0, \\ n - \dfrac{1}{4}, & n < 0, \end{cases}$$

then it is also known that $E(\lambda) = 0$ (see [RY, p.104, Lemma 1 and Remark]).

These results are unified as follows:

Theorem 1.1 *Let* $\lambda = \{\lambda_n\}$,

$$\lambda_n = \begin{cases} n + \alpha, & n > 0, \\ 0, & n = 0, \\ n - \alpha, & n < 0. \end{cases}$$

Then

$$E(\lambda) = \begin{cases} 2, & -1 \leq \alpha \leq -\dfrac{3}{4}, \\ 1, & -\dfrac{3}{4} < \alpha \leq -\dfrac{1}{4}, \\ 0, & -\dfrac{1}{4} < \alpha \leq \dfrac{1}{4}, \\ -1, & \dfrac{1}{4} < \alpha \leq \dfrac{3}{4}, \\ -2, & \dfrac{3}{4} < \alpha \leq 1. \end{cases}$$

Recently the result of $E(\lambda) \leq 0$ was obtained for some $\lambda = \{\lambda_n\}$ by [FNR].

Theorem 1.1 follows easily from the known results.

Remark 1.1 Of course we can also consider a non-symmetric case,

$$\lambda_n = \begin{cases} n+\alpha, & n>0, \\ 0, & n=0, \\ n+\beta, & n<0. \end{cases}$$

But this case can be reduced to the symmetric case in Theorem 1.1 by an isomorphism on $L^2[-\pi,\pi]$

$$\phi(t) \longmapsto e^{i\gamma t}\phi(t),$$

where γ is any real constant.

Now we consider the next problems:

Let $\lambda = \{\lambda_n\}$ be a symmetric sequence such that

$$\begin{cases} \lambda_0 = 0, & \\ \lambda_{2n} = 2n, & n>0, \\ \lambda_{2n-1} = (2n-1)+\alpha, & n>0, \\ \lambda_{-n} = -\lambda_n, & n>0, \end{cases}$$

where $1/4 \le |\alpha| < 1$, then what is each value of $E(\lambda)$? Similarly we can also consider to replace the odd numbers by the even numbers and replace the even numbers by the odd numbers for the above sequences. In this article we shall calculate $E(\lambda)$ in each case of that we replace kn by $kn+\alpha$ and replace m by $m+\beta$, $m \neq kn$, where k is an iteger, $k \ge 2$ and $-2 < \alpha$, $-1 < \beta$. We remark that the every cases of $|\alpha|$ and $|\beta| < 1/4$ are trivial by Kadec's 1/4 Theorem. In the proof of the main results, we shall use Lemma 3.1 ([RY, pp.104–105, Lemma 2]) which gives the equality to represent the infinite products by Γ functions and caluculate $E(\lambda)$ by using Lemma 3.2. In essence Lemma 3.2 has been given by [R, p.17, Theorem 22].

2. Main Results

In this section we state our main result. We shall suppose in what follows that $\lambda_n \neq \lambda_m$ for $n \neq m$.

Theorem 2.1 *Let $\lambda = \{\lambda_n\}$ be a sequence such that*

$$\begin{cases} \lambda_0 = 0, & \\ \lambda_{kn} = kn + \alpha, & n > 0, \\ \lambda_{kn-j} = (kn - j) + \beta \ (j = 1, 2, \ldots, k-1), & n > 0, \\ \lambda_{-n} = -\lambda_n, & n > 0, \end{cases}$$

for $-2 < \alpha$, $-1 < \beta$, where $k \geq 2$ is a fixed integer. If an integer r satisfies

$$\frac{\alpha}{1-k} - \frac{(1+2r)k}{4(k-1)} < \beta \leq \frac{\alpha}{1-k} + \frac{(1-2r)k}{4(k-1)},$$

then $E(\lambda) = r$.

Remark 2.1 Theorem 2.1 shows for $k \geq 2$ that $E(\lambda) = 0$ when $\alpha = -1/4$, $\beta = 0$ or $\alpha = 0$, $\beta = -1/4$. These results are different from the conclusion of $\alpha = -1/4$ in Theorem 1.1.

Remark 2.2 We can state the above results as the results including the complex numbers. Elsner or Peterson (see [R, p.12, Theorem 17]) obtained the following result:

"Let M be a positive constant and if

$$\operatorname{Re} \lambda_n = \operatorname{Re} \mu_n, \quad |\operatorname{Im} \lambda_n - \operatorname{Im} \mu_n| \leq M,$$

then $E(\lambda) = E(\mu)$."

Consequently each excess is unchanged even if $\{\lambda_n\}$ in Theorem 1.1 and 2.1 is replaced by $\{\lambda_n + i\mu_n\}$, where $\{\mu_n\}$ is any real bounded sequence.

3. Proof of Theorems

We shall give the proofs of our theorems. Theorem 1.1 follows easily from the known results. In the proof of Theorem 2.1, we shall represent the infinite products by Γ functions using Lemma 3.1 ([RY, pp.104–105, Lemma 2]) and investigate the convergence and the divergence of the integral in Lemma 3.2 to calculate $E(\lambda)$.

Proof of Theorem 1.1. The case of $-1/4 < \alpha < 1/4$ is trivial by Kadec's 1/4 Theorem, and [RY, Lemma 1] shows $E(\lambda) = 0$ for $\alpha = 1/4$. Let

$\alpha = -1 + \epsilon$, $0 \le \epsilon \le 1$, then we can write

$$\lambda_n = \begin{cases} n - 1 + \epsilon, & n > 0, \\ 0, & n = 0, \\ n + 1 - \epsilon, & n < 0. \end{cases}$$

By Kadec's 1/4 Theorem and [RY, Lemma 1], it is trivial that $E(\lambda) = 2$ for $0 \le \epsilon \le 1/4$. Next we consider the case $1/4 < \epsilon \le 3/4$. Let $\epsilon = 1/4 + \delta$ $(0 < \delta \le 1/2)$ and

$$\lambda' = \{\lambda_n\}_{n \neq \pm 1}.$$

Then Theorem IV in [L] shows $E(\lambda') \le -1$ and Theorem V in [L] shows $E(\lambda') \ge -1$. Hence we get $E(\lambda') = -1$, i.e., $E(\lambda) = 1$.

For $1/4 < \alpha \le 3/4$, we get $E(\lambda) = -1$ by the similar arguement to the case of $-3/4 < \alpha \le -1/4$. Finally for $3/4 < \alpha \le 1$, let

$$\alpha = \frac{3}{4} + \epsilon, \quad 0 < \epsilon \le \frac{1}{4},$$

then we can write

$$n + \alpha = (n+1) + \epsilon - \frac{1}{4}, \quad 0 < \epsilon \le \frac{1}{4}$$

for $n > 0$. Hence we see that $E(\lambda) = -2$ by Kadec's 1/4 Theorem. □

We need the following lemmas to prove Theorem 2.1. Lemma 3.1 was given in the proof of [RY, pp.104–105, Lemma 2]. In essence Lemma 3.2 was given by [R, p.17, Theorem 22], but we shall give the proof of Lemma 3.2.

Lemma 3.1 *If $\lambda_n = n + \epsilon$ $(n = 1, 2, \ldots)$ and $\epsilon > -1$, then*

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) = \frac{\Gamma^2(\mu)}{\Gamma(\mu + z)\Gamma(\mu - z)},$$

where $\mu = 1 + \epsilon$.

Lemma 3.2 *Let $\{\lambda_n\}$, $-\infty < n < \infty$, be a symmetric sequence of real numbers satisfying*

$$|\lambda_n - n| \le L,$$

where L is a positive constant and let r be an integer. We define the infinite *product*

$$P(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right).$$

Then we have $E(\lambda) = r$ *if and only if* $P(z)$ *satisfies the next conditions,*

$$\int_0^{\infty} \frac{|P(x)|^2}{(1+x^2)^r}dx = \infty \tag{1}$$

and

$$\int_0^{\infty} \frac{|P(x)|^2}{(1+x^2)^{r+1}}dx < \infty. \tag{2}$$

Proof of Lemma 3.2. We first remark that $E(\lambda)$ is finite under the above hypothesis $|\lambda_n - n| \leq L$ for $-\infty < n < \infty$ ([R, p.34, Theorem 47]).

We suppose that the case of $r = 0$ holds. Let be $r = 1$. We can write

$$P(z) = (z - \lambda_k)g(z),$$

where $g(z)$ is an entire function of exponential type π, $g(\lambda_n) = 0$ for $n \neq k$. Then the conditions (1) and (2) for $P(z)$ are equivalent to the next conditions for $g(z)$,

$$\int_0^{\infty} |g(x)|^2 dx = \int_0^{\infty} \frac{|P(x)|^2}{(x-\lambda_k)^2}dx = \infty$$

and

$$\int_0^{\infty} \frac{|g(x)|^2}{1+x^2}dx = \int_0^{\infty} \frac{|P(x)|^2}{(1+x^2)(x-\lambda_k)^2}dx < \infty.$$

Hence, by our hypothesis of the case of $r = 0$, if we set

$$\lambda' = \{\lambda_n\}_{n \neq k},$$

then we have $E(\lambda') = 0$ and the converse. Consequently we see that the lemma holds for $r = 1$. By the similar arguements we can also obtain the conclusion for the other integer r supposing that the case of $r = 0$ holds.

So we have only to prove the case of $r = 0$ in Lemma 3.2. First we suppose $P(z)$ satisfies (1) and (2) for $r = 0$. Let

$$f(z) \equiv \frac{P(z)}{z},$$

then $f(z) \in \mathbf{PW}$ by (2) and $f(\lambda_n) = 0$ for $n \neq 0$. Hence as

$$\{e^{i\lambda_n t}\}_{n \neq 0}$$

is not complete, we have

$$E(\lambda) \leq 0. \tag{3}$$

If $E(\lambda) = -1$, there exists a number μ such that $\mu \neq \lambda_n$ for all n and

$$\{e^{i\lambda_n t}\} \cup \{e^{i\mu t}\}$$

becomes complete and minimal. Consequently there exists an entire function $h(z) \in \mathbf{PW}$ such that $h(\lambda_n) = 0$ for all n and $h(\mu) \neq 0$. Now, as shown by [Y, p.149], we can write

$$h(z) = e^{Az} P(z),$$

where A is a constant. Let $A = A_1 + iA_2$, where A_1, A_2 are real. If $A_1 \geq 0$, we have

$$\begin{aligned}
\int_0^\infty |P(x)|^2 dx &\leq \int_0^\infty e^{2A_1 x} |P(x)|^2 dx \\
&\leq \int_{-\infty}^\infty e^{2A_1 x} |P(x)|^2 dx \\
&= \int_{-\infty}^\infty |h(x)|^2 dx \\
&< \infty.
\end{aligned}$$

This contradicts (1) for $r = 0$. Similarly we also obtain the contradiction in the case of $A_1 < 0$. Hence the case $E(\lambda) = -1$ is impossible. By the same arguement we see that the case $E(\lambda) \leq -2$ is impossible too. Consequently we have $E(\lambda) = 0$.

Conversely we suppose $E(\lambda) = 0$. By Theorem 1 in [Y, pp.148–149], we have

$$\frac{P(z)}{z} \in \mathbf{PW}.$$

Hence we obtain (2) for $r = 0$. Moreover if

$$\int_0^\infty |P(x)|^2 dx < \infty,$$

then $P(z) \in \mathbf{PW}$ and this contradicts the completeness of $\{e^{i\lambda_n t}\}$. Consequently we obtain (1). $\square$

Proof of Theorem 2.1. We define the infinite product $P(z)$ as follows

$$P(z) \equiv z \prod_{n=1}^{\infty} \left\{ \prod_{j=1}^{k-1} \left\{ 1 - \left(\frac{z}{kn - j + \beta} \right)^2 \right\} \right\} \left\{ 1 - \left(\frac{z}{kn + \alpha} \right)^2 \right\}.$$

Then we can write $P(z)$ as follows

$$P(z) = zS(z) \prod_{j=1}^{k-1} Q_j(z),$$

where

$$S(z) = \prod_{n=1}^{\infty} \left\{ 1 - \left(\frac{z}{kn + \alpha} \right)^2 \right\}$$

and

$$Q_j(z) = \prod_{n=1}^{\infty} \left\{ 1 - \left(\frac{z}{kn - j + \beta} \right)^2 \right\}.$$

Now we shall obtain $E(\lambda) = r$ by Lemma 3.2 whenever

$$\int_0^{\infty} \frac{|P(x)|^2}{(1 + x^2)^r} dx = \infty \tag{4}$$

and

$$\int_0^{\infty} \frac{|P(x)|^2}{(1 + x^2)^{r+1}} dx < \infty. \tag{5}$$

After the substitution $x = kt$, we have

$$\int_0^{\infty} \frac{|P(x)|^2}{(1 + x^2)^r} dx = k^3 \int_0^{\infty} \frac{t^2 V^2(t) \prod_{j=1}^{k-1} W_j^2(t)}{(1 + k^2 t^2)^r} dt,$$

where

$$V(t) = \prod_{n=1}^{\infty} \left\{ 1 - \left(\frac{t}{n + \frac{\alpha}{k}} \right)^2 \right\}$$

and

$$W_j(t) = \prod_{n=1}^{\infty} \left\{ 1 - \left(\frac{t}{n - \frac{j-\beta}{k}} \right)^2 \right\}.$$

Since $-(j-\beta)/k > -1$ for $k \geq 2$ and $\beta > -1$, we can represent $W_j(t)$ by Γ functions using Lemma 3.1 as $\epsilon_1 = -(j-\beta)/k$, $\mu_1 = 1 + \epsilon_1$. Moreover, using the well known equality of Γ function, we have

$$\begin{aligned} W_j(t) &= \frac{\Gamma^2(\mu_1)}{\Gamma(\mu_1 + t)\Gamma(\mu_1 - t)} \\ &= \Gamma^2(\mu_1) \frac{\Gamma(t + (1-\mu_1))}{\Gamma(t+\mu_1)\Gamma(t+(1-\mu_1))\Gamma(1-(t+(1-\mu_1)))} \\ &= \Gamma^2(\mu_1) \frac{\Gamma(t+(1-\mu_1))}{\Gamma(t+\mu_1)} \frac{\sin \pi (t + (1-\mu_1))}{\pi}. \end{aligned}$$

Similarly, since $\alpha/k > -1$ for $k \geq 2$ and $\alpha > -2$, if we take $\epsilon_2 = \alpha/k$, $\mu_2 = 1 + \epsilon_2$ we have

$$V(t) = \Gamma^2(\mu_2) \frac{\Gamma(t+(1-\mu_2))}{\Gamma(t+\mu_2)} \frac{\sin \pi (t+(1-\mu_2))}{\pi}.$$

Now if we notice the order of Γ function, i.e.

$$\frac{\Gamma(t+(1-\mu))}{\Gamma(t+\mu)} \sim t^{1-2\mu},$$

we see that (4) and (5) hold for k, α and β such that satisfy the next conditions for the sufficiently large $R > 0$:

$$\int_R^{\infty} t^{-\frac{4\{\beta(k-1)+\alpha\}}{k} - 2r} \sin^2 \pi \left(t - \frac{\alpha}{k}\right) \prod_{j=1}^{k-1} \sin^2 \pi \left(t + \frac{j-\beta}{k}\right) dt = \infty \tag{6}$$

and

$$\int_R^{\infty} t^{-\frac{4\{\beta(k-1)+\alpha\}}{k} - 2r - 2} dt < \infty. \tag{7}$$

We define

$$I_{R,k,\alpha,\beta} \equiv \int_R^{\infty} t^{-\frac{4\{\beta(k-1)+\alpha\}}{k} - 2r} \sin^2 \pi \left(t - \frac{\alpha}{k}\right) \prod_{j=1}^{k-1} \sin^2 \pi \left(t + \frac{j-\beta}{k}\right) dt$$

and

$$J_{n,k,\alpha,\beta} \equiv \int_n^{n+1} t^{-\frac{4\{\beta(k-1)+\alpha\}}{k}-2r} \sin^2 \pi \left(t - \frac{\alpha}{k}\right) \prod_{j=1}^{k-1} \sin^2 \pi \left(t + \frac{j-\beta}{k}\right) dt.$$

After the substitution $t = n + u$ for $R \leq n_0 \leq n$, we have

$$\begin{aligned} J_{n,k,\alpha,\beta} &= \int_0^1 (n+u)^{-\frac{4\{\beta(k-1)+\alpha\}}{k}-2r} \sin^2 \pi \left(n + u - \frac{\alpha}{k}\right) \\ &\qquad \prod_{j=1}^{k-1} \sin^2 \pi \left(n + u + \frac{j-\beta}{k}\right) du \\ &= \int_0^1 (n+u)^{-\frac{4\{\beta(k-1)+\alpha\}}{k}-2r} \sin^2 \pi \left(u - \frac{\alpha}{k}\right) \\ &\qquad \prod_{j=1}^{k-1} \sin^2 \pi \left(u + \frac{j-\beta}{k}\right) du. \end{aligned}$$

Hence if k, α, β satisfy the next condition

$$-\frac{4\{\beta(k-1)+\alpha\}}{k} - 2r \geq -1, \tag{8}$$

then we have

$$\begin{aligned} J_{n,k,\alpha,\beta} &\geq \int_0^1 \frac{1}{n+u} \sin^2 \pi \left(u - \frac{\alpha}{k}\right) \prod_{j=1}^{k-1} \sin^2 \pi \left(u + \frac{j-\beta}{k}\right) du \\ &\geq \frac{1}{n+1} \int_0^1 \sin^2 \pi \left(u - \frac{\alpha}{k}\right) \prod_{j=1}^{k-1} \sin^2 \pi \left(u + \frac{j-\beta}{k}\right) du \\ &= \frac{C_{k,\alpha,\beta}}{n+1}, \end{aligned}$$

where $C_{k,\alpha,\beta}$ is a positive constant depending only on k, α, β.

Consequently we have

$$\begin{aligned} I_{R,k,\alpha,\beta} &\geq \sum_{n \geq n_0} J_{n,k,\alpha,\beta} \\ &\geq \sum_{n \geq n_0} \frac{C_{k,\alpha,\beta}}{n+1} = \infty. \end{aligned}$$

Hence (6) holds. Next (7) holds for k, α, β, r such that

$$-\frac{4\{\beta(k-1)+\alpha\}}{k} - 2r - 2 < -1. \tag{9}$$

If we search for the conditions for k, α, β, r such that (8) and (9) simultaneously hold, we obtain

$$\frac{\alpha}{1-k} - \frac{(1+2r)k}{4(k-1)} < \beta \leq \frac{\alpha}{1-k} + \frac{(1-2r)k}{4(k-1)}.$$

Then we have $E(\lambda) = r$. $\square$

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