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Modular group algebras of coproducts of countable abelian groups

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Abstract. Suppose G is a coproduct (= a direct sum) of countable abelian groups and F is a perfect field of char $F = p \neq 0$. Then $S(FG)/G_p$ is a coproduct of countables and so G_p is a direct factor of $S(FG)$ with a complement which is a coproduct of countables. Moreover, $FH \cong FG$ as F-algebras for any group H implies $H_p \cong G_p$. In particular, if G_t is p-torsion, then G is a direct factor of $V(FG)$ with the same complementary factor. Besides, $H_t \cong G_t$ and there is a totally projective p-group T of length $\leq \Omega$ such that $H \times T \cong G \times T$. Thus H is a coproduct of countables.

The present results generalize statements obtained by Hill-Ullery in Comm. Algebra (1997).

Key words: group algebras, isomorphisms, direct factors, coproducts, countable groups.

1. Introduction and known facts

Our global aim in the present article is to give a detailed analysis of the commutative modular group algebra FG when the group G is a coproduct (i.e. a restricted, in other words, bounded direct product) of countable abelian groups and the field F is perfect of characteristic $p > 0$. For such a F-algebra FG, $V(FG)$ designates the group of all normed units (i.e. normalized invertible elements) in FG, and its p-component is denoted by $S(FG)$. For a subgroup A of G, we define $I(FG; A)$ as a relative augmentation ideal of FG with respect to A, and $I_p(FG; A)$ is its nilideal. Recall that G_t and G_p are the torsion part (= maximal torsion subgroup) and its p-primary component, respectively. All other notations and terminologies follow essentially the excellent books [4].

Further, in the sequel, we shall examine more specially *The Direct Factor Problem* for $S(FG)$ and $V(FG)$ and *The Isomorphism Problem* for FG. Our strategy is based precisely on the connection between the direct decompositions of G and $S(FG)$. Besides, our technique generalizes and extends May [7] and is absolute different to Hill-Ullery [5].

Now, for the sake of completeness and for the convenience of the reader,

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we summarize below some well-known and documented classical facts concerning the commutative group rings of coproducts of abelian groups. And so, we can formulate a fundamental [7, 8]

Theorem (May, 1979–1988) *Let G be a coproduct of countable abelian p-groups and R be a commutative ring with identity of prime characteristic p which is perfect. Then G is a direct factor of $V(RG)$ and $V(RG)/G$ is a coproduct of countables. Moreover, $RH \cong RG$ as R-algebras for any group H if and only if $H \cong G$.*

Remark Actually, the original May's result [8] is very stronger than the present, but we consider only this formulation of our interest.

More recently, Hill and Ullery generalized the May's result for p-mixed groups with totally projective torsion parts, but and with a length restriction. More precisely, they have proved the following [5].

Theorem (Hill and Ullery, 1997) *Suppose G is an abelian group of countable torsion - free rank whose G_t is totally projective p-primary of length less than $\Omega + \omega$ and F is a perfect field of characteristic $p > 0$. Then G is a direct factor of $V(FG)$ with totally projective complement. Moreover, the F-isomorphism $FH \cong FG$ for some group H implies that $H_t \cong G_t$ and that there is a totally projective p-group T such that $H \times T \cong G \times T$.*

Left-open is still the question of whether the group claim can be extended for lengths $\geq \Omega + \omega$. Besides, a crucial feature in the cited above work (cf. [5]) is that G can be decomposed as $A \times B$ in the chief case for length Ω, where A is countable and B is a coproduct of countable p-groups.

Here, we proceed by proving a strong generalization and extension to the above fact, as we study an arbitrary coproduct of countable abelian groups. Our major assertions are selected in the next paragraph.

2. Main results

Well, we are in position to state the following our goals.

Theorem (Direct Factor) *Suppose that G is an abelian coproduct of countables, R and F are a perfect unitary commutative ring with zero nil-radical and a perfect field of characteristics $p \neq 0$, respectively. Then G_p is a direct factor of $S(RG)$ and $S(RG)/G_p$ is a coproduct of countables. As* *a corollary, G is a direct factor of $V(FG)$ and $V(FG)/G$ is a coproduct of countables, provided G is p-mixed.*

We continue with the other paramount

Theorem (Isomorphism) *Suppose G is an abelian coproduct of countables and F is a field of characteristic $p > 0$. Then $FH \cong FG$ as F-algebras for an arbitrary group H yields $H_p \cong G_p$. As a consequence, $H_t \cong G_t$ and there exists a totally projective p-group T with length not exceeding Ω such that $H \times T \cong G \times T$, provided G_t is a p-group. In particular, H is a coproduct of countables.*

Remark This theorem partially settles a question posed by W. May in [8] (see also [2, 3]).

Next, we come to the proofs of our central statements, given in the following paragraph.

3. Proofs of the theorems

Before proving the claims, we will establish a few preliminaries and multiplicities very needed for our good presentation. As usual, R is an abelian unitary (i.e. with unity) ring of prime characteristic p.

We start with a key technical

Lemma (Intersection) *Let $A, B \leq G$. Then*

$$RA \cap I(RG; B) = I(RA; A \cap B) \quad \textit{and}$$
$$S(RA) \cap (1 + I(RG; B)) = 1 + I_p(RA; A \cap B).$$

Proof. To prove the first relation, we take x in the left-hand side. Hence $x = \sum_{a \in A} r_a a$, where $r_a \in R$ and $\sum_{a \in \bar{a}B} r_a = 0$ for every $\bar{a} \in A$. But $\bar{a}B \cap A = \bar{a}(B \cap A)$ because $\bar{a} \in A$. As a final we deduce $\sum_{a \in \bar{a}(A \cap B)} r_a = 0$, i.e. it is trivial that $x \in I(RA; A \cap B)$, as desired.

For the second dependence, given y in the left-hand side. Therefore $y = \sum_{a \in A} r_a a$, where $r_a \in R$ and $\sum_{a \in \bar{a}B} r_a = \begin{cases} 0, & \bar{a} \notin B \\ 1, & \bar{a} \in B \end{cases}$ for each $\bar{a} \in A$. Since $\bar{a}B \cap A = \bar{a}(B \cap A)$ it follows that $\sum_{a \in \bar{a}(A \cap B)} r_a = \begin{cases} 0, & \bar{a} \notin B \cap A \\ 1, & \bar{a} \in B \cap A \end{cases}$, hence y lies in $1 + I(RA; A \cap B)$. □

Now we can attack the next valuable.

Proposition (Direct decomposition) *Suppose $G = A \times B$ is abelian. Then $S(RG) = S(RA) \times (1 + I_p(RG; B))$. Inductively, $G = \coprod_{\alpha<\tau} G_\alpha$ implies $S(RG) = \coprod_{\beta<\tau}(1 + I_p(RC_{\beta+1}; G_\beta))$, where $C_{\beta+1} = \coprod_{\alpha<\beta+1} G_\alpha$.*

Proof. Foremost, let $G = A \times B$. Since $RG = (RA)B$ may be regarded as a group algebra of the group B over a ring RA, for every $x \in S(RG)$ we may write $x = \sum_{b\in B} x_{ab}b$, $x_{ab} \in RA$. Besides, it is clear that $x = \sum_{b\in B} x_{ab} + \sum_{b\in B} x_{ab}(b-1)$. Because $x \in S(RG)$ we derive $1 = \sum_{b\in B} x_{ab}^{p^t} + \left(\sum_{b\in B} x_{ab}(b-1)\right)^{p^t}$ for some $t \in N$. Consequently by virtue of the lemma, $1 - \sum_{b\in B} x_{ab}^{p^t} \in RA \cap I(RG; B) = 0$. Thus $\left(\sum_{b\in B} x_{ab}\right)^{p^t} = 1$ and immediately $\sum_{b\in B} x_{ab} \in S(RA)$ since evidently it is a normed element. By the Intersection Lemma, $x = \sum_{b\in B} x_{ab}\left(1 + \sum_{b\in B}\left(\sum_{b\in B} x_{ab}\right)^{-1} x_{ab}(b-1)\right) \in S(RA) \times (1 + I_p(RG; B))$.

Further, the transfinite inductive procedure is organized thus: choose arbitrary β with $\beta < \tau$ and such that $C_\beta = \coprod_{\alpha<\beta} G_\alpha$. Clearly $C_{\beta+1} = C_\beta \times G_\beta$. Employing the first half of the statement, we yield $S(RC_{\beta+1}) = S(RC_\beta) \times (1 + I_p(RC_{\beta+1}; G_\beta))$. By induction hypothesis, $S(RC_\beta) = \coprod_{\gamma<\beta}(1+I_p(RC_{\gamma+1}; G_\gamma))$ where $C_{\gamma+1} = \coprod_{\alpha<\gamma+1} G_\alpha$ and hence $S(RC_{\beta+1}) = \coprod_{\gamma\leq\beta}(1 + I_p(RC_{\gamma+1}; G_\gamma))$. Finally, it is a routine exercise to verify that $S(RG) = \coprod_{\beta<\tau}(1 + I_p(RC_{\beta+1}; G_\beta))$ since $G = \coprod_{\beta<\tau} C_\beta$. $\square$

Of some interest and importance is also the next significant.

Proposition (Structure) $(*)$ *If G is countable torsion and R is perfect, then $S(RG)/G_p$ is a coproduct of countables.*

$(**)$ *If G is p-mixed and F is perfect, then $V(FG)$ as a coproduct of countables yields that G is a coproduct of countables.*

Proof. $(*)$ Although that this claim is well-known and documented by us in a more general form in [1] (see [5], too), we give a new distinguish confirmation. Indeed, we can write $G = \bigcup_{k<\omega} A_k$, where $A_k \subseteq A_{k+1}$ and all A_k are finite. Furthermore $S(RG)/G_p = \bigcup_{k<\omega}[S(RA_k)G_p/G_p]$. Certainly $S(RA_k)$ are height-finite in $S(RG)$. Now, we show that all $S(RA_k)G_p/G_p$ have finite height spectrum in $S(RG)/G_p$. In fact, since G_p is balanced in $S(RG)$ (cf. [2]), for each element $aG_p \in S(RA_k)G_p/G_p$ such that $a \in S(RA_k)\backslash G_p$ it is true that height(aG_p) = height(ag_p) for some $g_p \in G_p$. Apparently height$(a) \leq$ height(aG_p) and immediately height$(a) \leq$ height(g_p); otherwise height$(a) \leq$ height$(g_p) <$ height(a) which is a false. If now height$(a) <$ height(g_p) there is nothing to prove. That is why, let height$(a) =$ height(g_p). Write $a = \sum_i r_k^{(i)} a_k^{(i)}$, where $r_k^{(i)} \in R$, $a_k^{(i)} \in A_k$, $i \in N$. Hence $ag_p = \sum_i r_k^{(i)} a_k^{(i)} g_p$ and besides it is no harm in presuming that $a_k^{(i)} = 1$ for some i, owing to the form of aG_p. Moreover, because height$(a) = \min_i$ height$(a_k^{(i)})$ and height$(ag_p) = \min_i$ height$(a_k^{(i)} g_p)$, it is a routine matter to verify that height$(ag_p) =$ height$(g_p) =$ height(a), as required. Finally, the criterion for total projectivity in [5] is applicable to complete the proof.

(**) Since $V(FG)$ is a coproduct of countables, it is evident that so is $S(FG)$. Next, we will show that the same holds for $S(FG)/G_p$. Really, if $\{N_\alpha\}_\alpha$ is a nice composition series for $S(FG)$, then $\{N_\alpha G_p/G_p\}_\alpha$ is (or can be refined to) a nice composition series for $S(FG)/G_p$. In order to prove this, let

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \qquad (\alpha < \mu)$$

be a smooth ascending chain of nice subgroups of $S(FG)$ with $S(FG) = \bigcup_{\alpha<\mu} N_\alpha$. Furthermore $S(FG)/G_p = \bigcup_{\alpha<\mu}(N_a G_p/G_p)$ and on the other hand it is easy to verify that

$$1 = N_0 G_p/G_p \subseteq N_1 G_p/G_p \subseteq \cdots \subseteq N_\alpha G_p/G_p \subseteq \cdots \qquad (\alpha < \mu)$$

is a smooth ascending chain. Now, in order to finish the claim in general, we must prove only that $N_\alpha G_p$ is nice (or can be expanded in a nice subgroup) in $S(FG)$. But the last follows automatically from the technique described in [5, Theorems 4.2 and 5.6]. So, finally $N_\alpha G_p/G_p$ or its extension will be nice in $S(FG)/G_p$ (see, for example, [4]). By the above arguments $S(FG)/G_p$ is a coproduct of countables. After this, owing to the fact G_p is balanced in $S(FG)$ (cf., for example, [9]), G_p must be a direct factor of $S(FG)$ [4]. But it follows from [8, 2] that $V(FG) = GS(FG)$ and thus we directly have that G is a direct factor of $V(FG)$. As a final, we need only apply [4, p.63, Proposition 9.10] to get the proposition. □

We begin now in the paper with all details of the proofs of two theorems. First, we are now prepared to obtain

Proof of the Direct Factor theorem. We shall show now that the quotient group $S(RG)/G_p$ is a coproduct of countables by using a standard transfinite induction on the power of G denoted by $|G|$. The countable case is done by [1, 5] (or the structure proposition for the torsion situation), therefore we may presume that G is uncountable. For this purpose, let τ be the first (i.e. the smallest) ordinal such that $|\tau| = |G|$. Thus we may assume that $G = \coprod_{\alpha<\tau} G_\alpha$, where each G_α is countable. For every $\beta < \tau$, put $C_\beta = \coprod_{\alpha<\beta} G_\alpha$; note that $G_0 = 1$. Next, we observe that the decomposition's Proposition is applicable to obtain that $S(RG) = \coprod_{\beta<\tau}(1+I_p(RC_{\beta+1}; G_\beta))$. Since $G = \coprod_{\beta<\tau} G_\beta$, the standard canonical isomorphism ensure

$$(\circ) \qquad S(RG)/G_p \cong \coprod_{\beta<\tau} \Big[(1 + I_p(RC_{\beta+1}; G_\beta))/(G_\beta)_p\Big].$$

Moreover $C_{\beta+1} = C_\beta \times G_\beta$, and so applying again the cited above Proposition on the splitting, we deduce $S(RC_{\beta+1}) = S(RC_\beta) \times (1+I_p(RC_{\beta+1}; G_\beta))$ and consequently $S(RC_{\beta+1})/(C_{\beta+1})_p \cong S(RC_\beta)/(C_\beta)_p \times (1+I_p(RC_{\beta+1}; G_\beta))$ $/(G_\beta)_p$. Besides, without loss of generality τ may be choosen to be limit, whence $\beta + 1 < \tau$ and $|\beta + 1| < |G|$ implies $|C_{\beta+1}| < |G|$. Therefore $S(RC_{\beta+1})/(C_{\beta+1})_p$ is a coproduct of countables by an induction hypothesis. By [4], so is $(1 + I_p(RC_{\beta+1}; G_\beta))/(G_\beta)_p$. It now follows directly by the formula $(\circ)$ that $S(RG)/G_p$ is a coproduct of countables, as claimed.

After this, if $G_t = G_p$, then by [8, 2] $V(FG) = GS(FG)$ and hence $V(FG)/G \cong S(FG)/G_p$ is also a coproduct of countables. □

We begin with

Proof of the Isomorphism theorem. Without any restriction F may be choosen perfect. And so, by what we have just proved above, $S(FG) \cong S(FH)$ is a coproduct of countables, hence the same holds for H_p by application of a slight modification of [5, Theorem 5.6]. On the other hand, the Ulm-Kaplansky cardinal invariants of G_p and H_p are known by May [6] to be equal, therefore [4] guarantees $G_p \cong H_p$, as stated. Moreover, it is a simple matter to see that H_t is p-primary, too. Hence $G_t \cong H_t$.

Next, according to the first central theorem along with (**) we establish that $V(FG) = G \times T_1 \cong H \times T_2 = V(FH)$ for some totally projective p-groups T_1 and T_2 with lengths $\leq \Omega$. As in [5] we can select a totally projective p-group T such that T, $T \times T_1$ and $T \times T_2$ have equal Ulm-Kaplansky functions (such a group is easy to construct; as example the Ulm-Kaplansky invariants of T_1 and T_2 not exceed these of T). Thus $T \cong T \times T_1 \cong T \times T_2$ by [4] and now automatically $G \times T \cong H \times T$, which must be proved. □

Now, we conclude this paragraph with an interesting discussion. The first significant observation is that our major theorems proved above generate the stated above Hill-Ullery's theorem. Indeed, the case for lengths Ω follows immediately from the fact given by us, and the general case for lengths $\Omega+n$ ($n \in N$) is a trivial consequence of the first. Besides, using the isomorphism theorem and the technique of σ-summable groups developed by us in [2], we are ready to formulate and establish a new independent proof of the following our assertion more stronger than the corresponding result of Hill-Ullery [5, Corollary 5.9], namely:

Theorem ([3], 2000) *Suppose G is of countable torsion-free rank and G_p is totally projective of length $< \Omega^2$. Then the F-isomorphism $FH \cong FG$ for any group H implies $H_p \cong G_p$. Moreover, if G_t is a p-group, it is fulfilled that $H_t \cong G_t$ and even more that $H \cong G$ provided G is of torsion-free rank one.*

Further, combining the Direct Factor Theorem together with (**) we yield the statement listed below that is an expanson of the corresponding in [3].

Corollary *Let G be p-mixed and F be perfect. Then $V(FG)$ is a coproduct of countables if and only if G is a coproduct of countables.*

Well, we close the investigation with

4. Epilogy

The concluding discussion leads us to state the following general and very important group-theoretic question: What is the complete set of invariants for two p-mixed coproducts of countables to be isomorphic? The situation will be solved if it is done for the countable case.

In this light, numerous other actual problems remain unanswered; here are a few, namely:

- It is still unknown to the moment, whether in the main isomorphism theorem it is true that $G \cong H$? This probably is so for p-mixed groups.

- Does it follow that $S(FG)/G_p$ is totally projective provided F is perfect and G is of cardinality $\geq \aleph_1$ or of p-length $\geq \Omega$? If this is the case, then the central theorem will be itself valid for direct sums of such groups with the above power and length restrictions, following the present algorithm.

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