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Solvability of convolution equations in $\mathcal{D}'_{L^p}$

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Abstract. In this paper we give a necessary condition on the Fourier transform of a convolution operator S of the space $\mathcal{D}'_{L^p}; 2 \leq p < \infty$, for the equation $S * u = v$ to have a solution u in $\mathcal{D}'_{L^p}$ for every v in $\mathcal{D}'_{L^p}$. In the case $p = 2$, this condition with the additional assumption $\widehat{S}(\xi) \neq 0$ for all $\xi \in \Re^n$, are sufficient for solvability of the convolution equation.

Key words: distributions of L^p-growth, convolution equations.

1. Introduction

Convolution equations in spaces of distributions and ultradistributions of L^p-growth were studied by several authors. In this work we study the problem of characterizing the convolution operators S for which the convolution equation $S * u = v$ have a solution u in $\mathcal{D}'_{L^p}$ for every v in $\mathcal{D}'_{L^p}$. Pahk [3] characterized hypoelliptic convolution operators in the space $\mathcal{D}'_{L^\infty}$, and left the problem of solvability of convolution equations in $\mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$ open. Pilipovič [4] has established necessary condition and sufficient condition on the convolution operator S to be invertible in $\mathcal{D}'^{(M_p)}_{L^2}$. Moreover, Pilipovič characterized hypoelliptic convolution operators in $\mathcal{D}'^{(M_p)}_{L^2}$. Here we give a necessary condition on $\widehat{S}$, the Fourier transform of the convolution operator S, for the convolution equation $S * u = v$ to have a solution u in $\mathcal{D}'_{L^p}$ for a given v in $\mathcal{D}'_{L^p}$. Moreover, in the case $p = 2$ we give sufficient conditions for solvability of the equation $S * u = v$. Characterizing invertible and hypoelliptic convolution operators in $\mathcal{D}'_{L^p}$ is difficult in general. This is due to lack of differentiability of $\widehat{S}$. It is known (see [1] part (c) of Theorem 2 and the remark which follows it on page 202) that the Fourier transform of any convolution operator in $\mathcal{D}'_{L^p}, 1 \leq p \leq \infty$, is a continuous function which is slowly increasing at infinity. We remark that in this work

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we did not assume any differentiability condition on the Fourier transform of the convolution operator. We will use the standard notations as in [2] and [5]. For more information on the space $O'_c(\mathcal{D}'_{L^p};\mathcal{D}'_{L^p})$ of convolution operators on $\mathcal{D}'_{L^p}$ and its topology, we refer the reader to [1].

We recall the definitions of the space $\mathcal{D}_{L^q}$; $1 < q \leq 2$, of test functions and the space $\mathcal{D}'_{L^p}$ of distributions of L^p-growth, $2 \leq p < \infty$. The space $\mathcal{D}_{L^q}$; $1 < q \leq 2$, consists of all infinitely differentiable functions φ such that $D^\alpha\varphi$ is in L^q for all α in $\aleph^n$, equipped with the topology generated by the norms

$$\|\varphi\|_{m,q} = \left\{ \sum_{|\alpha|\leq m} \|D^\alpha\varphi\|_q^q \right\}^{\frac{1}{q}}, \quad m = 0,1,2,3,\dots .$$

With this topology, the space $\mathcal{D}_{L^q}$ is a Frechet space.

The subspace of all functions in $\mathcal{D}_{L^\infty}$ which converge to 0 at infinity is denoted by $\dot{\mathcal{D}}_{L^\infty}$. The strong dual of $\mathcal{D}_{L^q}$ is $\mathcal{D}'_{L^p}$ the space of distributions with restricted L^p-growth, where $2 \leq p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. For any φ in $\mathcal{D}_{L^q}$; $1 < q \leq 2$, its Fourier transform $\widehat{\varphi}$ and its multiple with any polynomial are in L^p. The space $\mathcal{F}(\mathcal{D}_{L^q}) = \{\widehat{\varphi} : \varphi \in \mathcal{D}_{L^q}\}$ is a subspace of L^p, and will be provided with the induced L^p norm topology. It follows that the Fourier transformation from $\mathcal{D}_{L^q}$ into L^p is continuous. Given T in $\mathcal{D}'_{L^p}$, we define its Fourier transform $\widehat{T}$ in $\mathcal{F}(\mathcal{D}_{L^q})$ by $\langle \widehat{T}, \widehat{\varphi}\rangle = \langle T, \varphi\rangle$. It follows that $\widehat{T}$ is well defined and continuous onto $\mathcal{F}(\mathcal{D}_{L^q})$.

2. The Results

Our first result gives neccessary condition for solvability of convolution equations in $\mathcal{D}'_{L^p}$.

Theorem 1 *Let S be a convolution operator on $\mathcal{D}'_{L^p}$, $2 \leq p < \infty$. If the convolution equation*

$$S * u = v \tag{1}$$

has a solution u in $\mathcal{D}'_{L^p}$ for every v in $\mathcal{D}'_{L^p}$, then there exist positive constants c, d, and k such that

$$|\widehat{S}(\xi)| \geq c(1 + |\xi|)^{-d} \tag{2}$$

for all ξ in $\Re^n$ with $|\xi| \geq k$.

Proof. The proof is by contradiction. Suppose that condition (2) is not satisfied. Then there exists a sequence of points (ξ_j) such that $|\xi_{j+1}| > |\xi_j| + 1$, $|\xi_1| > 2$, $j^2 \le |\xi_j|$, for $j \ge 4$, and

$$|\widehat{S}(\xi_j)| < 2^{-j^3}(1+|\xi_j|)^{-5j}, \quad j \ge 1. \tag{3}$$

From the continuity of $\widehat{S}$ it follows that there exist open balls U_j centered at ξ_j with positive small radius ε_j such that

$$|\widehat{S}(\xi)| \le 2^{-j^3}(1+|\xi_j|)^{-5j} \tag{4}$$

for all $\xi \in U_j$, $j \in \aleph$.

For each $j \in \aleph$ we define

$$T_j(\xi) = \varepsilon_j^{-n}(1+|\xi|)^{-j}, \quad \xi \in U_j$$
$$0 \text{ if } \xi \text{ is in } U_j^c \tag{5}$$

and $T(\xi) = \sum_{j=1}^{\infty} T_j(\xi)$, where n is the dimension of $\Re^n$. We claim that T is in the set $\mathcal{F}(\mathcal{D}'_{L^p})$ of all Fourier transforms of the distributions in $\mathcal{D}'_{L^p}$. Indeed, for any $\Psi \in \mathcal{D}_{L^q}$ one has

$$\begin{aligned} |\langle T, \widehat{\Psi}\rangle| &= \left|\sum_{j=1}^{\infty} \int_{U_j} T_j(\xi)\widehat{\Psi}(\xi)d\xi\right| \\ &\le \sum_{j=1}^{\infty} \int_{U_j} \varepsilon_j^{-n}(1+|\xi|)^{-j}|\widehat{\Psi}(\xi)|\, d\xi && (6) \\ &\le \sum_{j=1}^{\infty} j^{-2j}\|\widehat{\Psi}\|_\infty && (7) \\ &\le \sum_{j=1}^{\infty} C_1 j^{-2j}\|\widehat{\Psi}\|_p = C\|\widehat{\Psi}\|_p; && (8) \end{aligned}$$

where C is a constant which is independent of Ψ. Thus T is a well defined continuous linear functional on $\mathcal{F}(\mathcal{D}_{L^q})$ considered as a subspace of L^p. We remark that the above argument shows that T is in $\mathcal{D}'_{L^p}$.

Next we construct a function which is in $\mathcal{F}(\mathcal{D}_{L^q})$. Let U_j be as above. For each j, let B_j be a ball with center ξ_j and radius $\frac{1}{2}\varepsilon_j$. Let φ_j be a C^∞-function with compact support in U_j, such that $|\xi_j|^{-3j} \le \varphi_j(\xi) \le |\xi_j|^{-2j}$ if ξ is in B_j and $0 \le \varphi_j(\xi) \le |\xi_j|^{-2j}$ if ξ is in $U_j \backslash B_j$. Let $\varphi(\xi) = \sum_{j=1}^{\infty} \varphi_j(\xi)$.

Then

$$|\mathcal{F}^{-1}(\varphi_j)(x)| = \left|\int_{U_j} e^{i<x,\xi>}\varphi_j(\xi)d\xi\right| \le \int_{U_j} |\varphi_j(\xi)|d\xi \le j^{-4j}\varepsilon_j.$$

Hence the function $\mathcal{F}^{-1}(\varphi)(x) = \sum_{j=1}^{\infty}\mathcal{F}^{-1}(\varphi_j)(x)$ satisfies the estimates

$$|\mathcal{F}^{-1}(\varphi)(x)| \le \sum_{j=1}^{\infty}|\mathcal{F}^{-1}(\varphi_j)(x)| \le \sum_{j=1}^{\infty} j^{-4j}\varepsilon_j \le \sum_{j=1}^{\infty} j^{-4j} < \infty.$$

Thus $\mathcal{F}^{-1}(\varphi)$ is a well defined function. We claim that the function $P(\xi)\varphi(\xi)$ is in L^p for any polynomial $P(\xi)$. For, there exist a positive integer k and a constant C such that $|P(\xi)| \le C(1+|\xi|)^k$. Moreover, for any $\xi \in U_j$ one has $|\xi| \le |\xi_j| + \varepsilon_j$, and

$$(1+|\xi|)^{kp} \le (1+\varepsilon_j+|\xi_j|)^{kp} \le (2+|\xi_j|)^{kp}.$$

Thus one has

$$\begin{aligned}
&\|P(\xi)\varphi_j(\xi)\|_p^p \\
&= \int_{U_j} |P(\xi)\varphi_j(\xi)|^p d\xi \le \int_{U_j} C^p(1+|\xi|)^{kp}|\varphi_j(\xi)|^p d\xi \\
&\le C^p(2+|\xi_j|)^{kp}\int_{U_j}|\varphi_j(\xi)|^p d\xi \le C^p(2+|\xi_j|)^{kp}|\xi_j|^{-2jp}\varepsilon_j^{n}. \qquad (9)
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=1}^{\infty}\|P(\xi)\varphi_j(\xi)\|_p &\le \sum_{j=1}^{\infty} C(2+|\xi_j|)^k|\xi_j|^{-2j}\,\varepsilon_j^{\frac{n}{p}} \\
&\le \sum_{j=1}^{\infty} C(2^k(2^k+|\xi_j|^k))|\xi_j|^{-2j} \qquad (10) \\
&\le C_{1,k}\sum_{j=1}^{\infty}|\xi_j|^{-2j} + C_{1,k}\sum_{j=1}^{\infty}|\xi_j|^{-(2j-k)} \qquad (11) \\
&\le C_{1,k}\left(1+\sum_{j=1}^{\infty} j^{-2(2j-k)}\right) \le C_{2,k} < \infty, \qquad (12)
\end{aligned}$$

where $C_{1,k}$ and $C_{2,k}$ are constants which depend on k (the polynomial P) only.

On the other hand, the inequality $\|f+g\|_p \le \|f\|_p + \|g\|_p$ whenever $f, g \in L^p; 1 \le p < \infty$, and induction imply that $\|\sum_{j=1}^n f_j\|_p \le \sum_{j=1}^n \|f_j\|_p$, where $f_1, f_2, \ldots, f_n$ are in L^p. Hence continuity of the norm function imply that

$$\left\|\sum_{j=1}^{\infty} f_j\right\|_p = \left\|\lim_{n\to\infty}\sum_{j=1}^{n} f_j\right\|_p = \lim_{n\to\infty}\left\|\sum_{j=1}^{n} f_j\right\|_p \le \lim_{n\to\infty}\sum_{j=1}^{n}\|f_j\|_p = \sum_{j=1}^{\infty}\|f_j\|_p. \tag{13}$$

From (11) and (12) one has

$$\|P(\xi)\varphi(\xi)\|_p = \left\|\sum_{j=1}^{\infty} P(\xi)\varphi_j(\xi)\right\|_p \le \sum_{j=1}^{\infty}\|P(\xi)\varphi_j(\xi)\|_p \le C_{2,k} < \infty. \tag{14}$$

Thus φ is in $\mathcal{F}(\mathcal{D}_{L^q})$.

Finally using (4), the definition of T, and the definition of the functions φ_j one has

$$\begin{aligned}\left\langle \frac{T}{|\widehat{S}|}, \varphi\right\rangle &= \sum_{j=1}^{\infty}\int_{U_j}\frac{T(\xi)}{|\widehat{S}(\xi)|}\varphi_j(\xi)d\xi \\ &\ge \sum_{j=1}^{\infty}\int_{B_j}\varepsilon_j^{-n}(1+|\xi|)^{-j}2^{j^3}(1+|\xi_j|)^{5j}|\xi_j|^{-3j}d\xi \\ &\ge \sum_{j=1}^{\infty}\varepsilon_j^{-n}2^{j^3}|\xi_j|^{-3j}(1+|\xi_j|)^{3j}\left(\frac{\varepsilon_j}{2}\right)^n \\ &\ge \left(\frac{1}{2}\right)^n\sum_{j=1}^{\infty}2^{j^3} = \infty. \end{aligned} \tag{15}$$

Therefore $\frac{T}{|\widehat{S}|}$ is not in $\mathcal{F}(\mathcal{D}'_{L^p})$. This implies that $\frac{T}{\widehat{S}}$ is not in $\mathcal{F}(\mathcal{D}'_{L^p})$. Indeed, if $\frac{T}{\widehat{S}}$ is in $\mathcal{F}(\mathcal{D}'_{L^p})$ where $\widehat{S}(\xi) = S_1(\xi) + iS_2(\xi)$, then $\frac{T(\xi)S_1(\xi)}{|\widehat{S}(\xi)|^2} - i\frac{T(\xi)S_2(\xi)}{|\widehat{S}(\xi)|^2} \in \mathcal{F}(\mathcal{D}'_{L^p})$. Hence $\frac{T(\xi)S_1(\xi)}{|\widehat{S}(\xi)|^2}$ and $\frac{T(\xi)S_2(\xi)}{|\widehat{S}(\xi)|^2}$ are in $\mathcal{F}(\mathcal{D}'_{L^p})$. Inparticular, $\left\langle\frac{T(\xi)S_1(\xi)}{|\widehat{S}(\xi)|^2}, \varphi\right\rangle$ and $\left\langle\frac{T(\xi)S_2(\xi)}{|\widehat{S}(\xi)|^2}, \varphi\right\rangle$ are bounded, where $\varphi(\xi) = \sum_{j=1}^{\infty}\varphi_j(\xi)$

as above. On the other hand $\left\langle \frac{T}{|\widehat{S}|}, \varphi \right\rangle$ is unbounded. Thus $\frac{S_1(\xi)}{|\widehat{S}(\xi)|}$ and $\frac{S_2(\xi)}{|\widehat{S}(\xi)|}$ must be very small in absolute value, which contradicts the fact that the modulus of $\frac{\widehat{S(\xi)}}{|\widehat{S}(\xi)|}$ is 1. The contradiction shows that $\frac{T}{\widehat{S}}$ is not in $\mathcal{F}(\mathcal{D}'_{L^p})$. Thus the convolution equation $S * u = \mathcal{F}^{-1}(T)$ does not have a solution in $\mathcal{D}'_{L^p}$.This contradicts the hypothesis and completes the proof of the theorem. □

The next result provides sufficient conditions for solvabilty of the convolution equation in $\mathcal{D}'_{L^2}$. This result covers a wider set of convolution operators than the corresponding theorem of Pilipović for the space $\mathcal{D}'^{(M_p)}_{L^2}$ (see Proposition 8 of [4]). In our result we did not assume that $\widehat{S}$ has analytic continuation onto C^n. As well, our proof is different from that of Pilipović. We recall that the Fourier transformation is a topological isomorphism from L^2 onto itself. Since $\mathcal{D}_{L^2}$ is a subspace of L^2 it follows that $\mathcal{F}(\mathcal{D}_{L^2})$ is a subspace of L^2. We provide $\mathcal{F}(\mathcal{D}_{L^2})$ with the L^2 norm. If S is a convolution operator on $\mathcal{D}'_{L^2}$ we provide the space $S * \mathcal{D}_{L^2}$ with the topology induced by $\mathcal{D}_{L^2}$. The following lemma follows from the above cited fact that the Fourier transformation is a topological isomorphism from L^2 onto itself. We provide its proof for the sake of completeness.

Lemma 2 *The Fourier transform is a topological isomorphism of $\mathcal{D}_{L^2}$ onto $\mathcal{F}(\mathcal{D}_{L^2})$.*

Proof. Let φ be any element in $\mathcal{D}_{L^2}$. Let $k \geq 1$ be any integer. From continuity of the Fourier transform on L^2 and contiuity of the differential operator from $\mathcal{D}_{L^2}$ into itself, it folows that

$$\|\widehat{\varphi}\|_2 \leq \|(1 + |\xi|^2)^k \widehat{\varphi}\|_2 \leq C_1 \|P(D)\varphi\|_2 \leq C_1 C_2 \|\varphi\|_{2,m};$$

where $P(D) = (1 + D_1^2 + \cdots + D_n^2)^2$, C_1, C_2 are positive constants and m is a positive integer. This takes care of contiuity of the Fourier transform. To establish continuity of the inverse Fourier transform, let k be any positive integer. From continuity of the differential operator from $\mathcal{D}_{L^2}$ into itself, and contiuity of the inverse Fourier transform from L^2 onto itself one has for any positive integer k,

$$\|\varphi\|_{2,k}^2 = \sum_{|\beta| \leq k} \|D^\beta \varphi\|_2^2 \leq \sum_{|\beta| \leq k} C_\beta \|\varphi\|_2^2 \leq C_k \|\widehat{\varphi}\|_2^2,$$

where C_k is a constant which is independent of φ. This takes care of continuity of the inverse Fourier transform. □

Theorem 3 *Let S be a convolution operator on $\mathcal{D}'_{L^2}$. If $\widehat{S}(\xi) \neq 0$ for all $\xi \in \Re^n$ and $|\widehat{S}(\xi)| \geq c(1+|\xi|)^{-d}$ whenever $|\xi| \geq k$ for some positive constants c, d, and k, then the convolution equation*

$$S * u = v \tag{16}$$

has a solution u in $\mathcal{D}'_{L^2}$ for every v in $\mathcal{D}'_{L^2}$.

Proof. Using the Hahn-Banach theorem, it suffices to show that the map $S * \varphi \longrightarrow \varphi$ from $S * \mathcal{D}_{L^2}$ into $\mathcal{D}_{L^2}$ is continuous, where we assumed without loss of generality that $S = \check{S}$ the symmetry of S with respect to the origin. From Lemma 2 it suffices to show that the map $\widehat{S}\widehat{\varphi} \longrightarrow \widehat{\varphi}$ from $\mathcal{F}(\mathcal{D}_{L^2})$ into itself is continuous in the L^2 norm. We consider two cases:

Case I: If the support of $\widehat{\varphi}$ is contained in the closed ball $\overline{B(0,k)}$. Then

$$\begin{aligned} \|\widehat{\varphi}\|_2^2 = \left\|\frac{\widehat{S}\widehat{\varphi}}{\widehat{S}}\right\|_2^2 &= \int |\widehat{S}(\xi)\widehat{\varphi}(\xi)|^2 \frac{1}{|\widehat{S}(\xi)|^2} d\xi \\ &\leq \sup_{\xi \in \overline{B(0,k)}} \frac{1}{|\widehat{S}(\xi)|} \int |\widehat{S}(\xi)\widehat{\varphi}(\xi)|^2 d\xi \leq C\|\widehat{S}\widehat{\varphi}\|_2^2. \end{aligned} \tag{17}$$

Case II: If the support of $\widehat{\varphi}$ is not contained in the closed ball $\overline{B(0,k)}$. Then from condition (2) and continuity of the differential operator on $\mathcal{D}_{L^2}$ one has,

$$\begin{aligned} \|\widehat{\varphi}\|_2^2 &= \left\|\frac{\widehat{S}\widehat{\varphi}}{\widehat{S}}\right\|_2^2 = \int |\widehat{S}(\xi)\widehat{\varphi}(\xi)|^2 \frac{1}{|\widehat{S}(\xi)|^2} d\xi \\ &\leq C_1 \int |\widehat{S}(\xi)\widehat{\varphi}(\xi)|^2 |P(\xi)|^2 d\xi \\ &\leq C_1 \int |P(D)\widehat{(S*\varphi)}(\xi)|^2 d\xi \\ &\leq C_1 \|P(D)\widehat{(S*\varphi)}\|_2^2 \leq C\|\widehat{S*\varphi}\|_2^2, \end{aligned} \tag{18}$$

for some polynomial $P(\xi)$ and constants C_1, C independent of φ. Thus the map $S * \varphi \longrightarrow \varphi$ from $S*\ \mathcal{D}_{L^2}$ into $\mathcal{D}_{L^2}$ is continuous. □

Remark 1 The additional assumption $\widehat{S}(\xi) \neq 0$ for all ξ in $\Re^n$, was used in the proof of Theorem 3 in a very essential way. Thus it is not expected that the neccessary condition for solvability to be sufficient. Moreover, the proof of Theorem 3 does not work for the general case p, $2 < p < \infty$. This is because, in the general case, the Fourier transform does not have continuous inverse.

Remark 2 We leave the conjecture that Theorem 3 is true for general $p > 2$ un answered. To prove the conjecture one needs to study carefully the relation between the topologies of the space $\mathcal{F}(\mathcal{D}_{L^q})$

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References

[1] Abdullah S., *On Convolution Operators and Multipliers of Distributions of L^p-Growth.* Jou. Math. Anal. & Appl. **183**, No.1, (1994), 196–207.

[2] Barros-Neto J., *An Introduction to the Theory of Distributions.* Marcel Dekker, New York, 1973.

[3] Pahk D., *On the Convolution Equations in the Space of Distributions of L^p-Growth.* Proc. Amer. Math. Soc. **94**, No.1, (1985), 81–86.

[4] Pilipović S., *On the Convolution in the Space $\mathcal{D}'^{(M_p)}_{L^2}$.* Rend. Sem. Mat. Univ. Padova **79** (1988), 25–36.

[5] Schwartz L., *Theorie des Distributions.* Hermann, Paris, 1973.

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