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The discrete and nondiscrete subgroups of $SL(2, R)$ and $SL(2, C)$*

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Abstract. It is shown in this paper that some special subsets of $\mathrm{tr}(G)=\{\mathrm{tr}(f): f\in G\}$ are sufficient to determine whether G is discrete or not when $G\subset SL(2, R)$ or $SL(2, C)$ is nonelementary. One of Beardon's open problems is affirmatively answered.

Key words: discreteness, nondiscreteness, trace.

1. Introduction

In [1], Beardon tried to determine whether a group G of $SL(2, R)$ is discrete or not by using some subset of $\mathrm{tr}(G)=\{\mathrm{tr}(f): f\in G\}$ when G is finitely generated and contains parabolic elements.

The main aim of this paper is twofold. The first is to generalize Beardon's discussion as mentioned above. Our main results are Theorems 3.1 and 3.2. The second is to give an affirmative answer to one of the open problems raised by Beardon in [1].

2. Some concepts and notations

Let G be a subgroup of $SL(2, C)$. G is called elemetary if G has a finite orbit in $\bar{H}^3$, i.e., there is $z\in\bar{H}^3$ such that $G_z=\{\widetilde{f}(z): f\in G\}$ is finite, where $\widetilde{f}$ denotes the Poincaré extension of f (cf. [2]). Otherwise G is called nonelementary.

From [7], the following is obvious.

Lemma 2.1 *Two elliptic elements f, $g\in SL(2, R)$, whose orders are not both equal to 2, generate a nonelementary group if and only if f and g have no common fixed points in $\bar{C}$.*

For any nonelementary group $G\in SL(2, R)$, let

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$$H(G,G) = \{[f,g]\colon f,\ g \text{ hyperbolic elements of } G \text{ with no common fixed points}\};$$

$$TH(G,G) = \{\mathrm{tr}(Q) - 2\colon Q \in H(G,G)\}.$$

If G contains parabolic elements, then

$$P(G,G) = \{[f,g]\colon f,g \in G \text{ parabolic elements with no common fixed point}\};$$

$$TP(G,G) = \{\mathrm{tr}(Q) - 2\colon Q \in P(G,G)\}.$$

If G contains elliptic elements, then

$$E(G,G) = \{[f,g]\colon f,g \in G \text{ elliptic elements with } \langle f,g\rangle \text{ being nonelementary}\};$$

$$TE(G,G) = \{\mathrm{tr}(Q) - 2\colon Q \in E(G,G)\}.$$

By a sequence in G we will always mean an infinite sequence of distinct elements of G.

For $f \in SL(2,R)$, $\mathrm{fix}(f) = \{x \in \bar{C} : f(x) = x\}$; $\mathrm{ord}(f) =$ the order of f when f is regarded as a Möbius transformation.

3. Discrete and nondiscrete subgroups of $SL(2,R)$

In [1], Beardon proved

Theorem A *Let $G \subset SL(2,R)$ be nonelementary and finitely generated. If G contains parabolic elements, then*
(1) *G is discrete if and only if $TP(G,G)$ is a discrete subset of $[1,\infty)$;*
(2) *G is nondiscrete if and only if $TP(G,G)$ is dense in $[1,\infty)$.*

We will prove

Theorem 3.1 *Let $G \subset SL(2,R)$ be nonelementary. Then*
(1) *G is discrete if and only if either $E(G,G) = \emptyset$ or $\inf\{TE(G,G)\} \geq c_0$, where $c_0 = 2 - 2\cos\frac{\pi}{7}$;*
(2) *G is not discrete if and only if $TE(G,G)$ is dense in $[0,\infty)$.*

Proof. The necessity of (1) follows from [6–7]. For the proof of the sufficiency, by [4], we may assume that $E(G,G) \neq \emptyset$, i.e., G contains some elliptic elements of order at least 3, cf. [7, Corollary 2.3].

Assume that $\inf\{TE(G,G)\} \geq c_0$, but G is not discrete. Then by [3, Corollary p.199], there is a sequence $\{h_n\}$ of G such that each h_n is elliptic and

$$h_n \to I \ (n \to \infty),$$

where I denotes the unit element. Then there are an elliptic element g_1 of order at least 3 and a suitable subsequence of $\{h_n\}$ (denoted by the same manner) (cf. [7]) in G such that

$$\operatorname{fix}(g_1) \cap \operatorname{fix}(h_n) = \emptyset.$$

The assumptions imply that for large n, $\langle g_1, h_n \rangle$ is elementary. By Lemma 2.1, it is impossible. So G is discrete.

The proof of (1) is completed.

The sufficiency of (2) follows from (1). For the proof of the necessity, since G is nonelementary, by [3], we can find two sequences $\{f_n\}$ and $\{g_n\} \subset G$ such that each f_n is elliptic, each g_n is hyperbolic and

$$f_n \to I, \ g_n \to I \ (n \to \infty).$$

Then for any $r_1, r_2 \colon 0 < r_1 < r_2$, there is a hyperbolic element $f \in G$ with the multiplier r^2 which satisfies

$$1 < r^2 < 2\beta(\alpha+\beta)^{-1},$$

where $\alpha = \sqrt{[(p+1)r_1 + r_2]p^{-1}}$, $\beta = \sqrt{[(p-1)r_2 - r_1]p^{-1}}$, $p > \frac{2(r_1+r_2)}{r_2-r_1}$.

Without loss of generality, suppose that

$$f = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}.$$

By passing to a subsequence we assume that $\operatorname{fix}(f_n)$ tends in the Hausdorff metric toward a one or two point set X. Since G is nonelementary, we may further assume that for large n and t ($t > T_0$),

$$\operatorname{fix}(f_n) \cap \operatorname{fix}(f^t f_n f^{-t}) = \emptyset.$$

Let

$$f_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Then

$$a_n, d_n \to 1,\ b_n, c_n \to 0\ (n \to \infty).$$

An elementary calculation shows that for each $t \in N$,

$$\begin{aligned}\operatorname{tr}[f_n, f^t f_n f^{-t}] - 2 \\ = [b_n^2 c_n^2 + (a_n - d_n)^2 b_n c_n (r^t + r^{-t})^{-2}](r^{2t} - r^{-2t})^2.\end{aligned}$$

Let $r^{2t} - r^{-2t} = s_t$, $(r^t + r^{-t})^2 = q_t$. Then for large n and t ($t > T_0$),

$$\operatorname{tr}[f_n, f^t f_n f^{-t}] - 2 = [b_n^2 c_n^2 + (a_n - d_n)^2 b_n c_n q_t^{-1}] s_t^2 \in TE(G, G).$$

Since $a_n, d_n \to 1$ ($n \to \infty$), there is $M > 0$ such that for all $n > M$,

$$(a_n - d_n)^2 < (r_1 + r_2) p^{-1} \beta^{-1}.$$

For large enough T, the union of intervals

$$\left(\frac{\alpha}{s_T}, \frac{\beta}{s_T}\right) \cup \left(\frac{\alpha}{s_{T+1}}, \frac{\beta}{s_{T+1}}\right) \cup \ldots$$

is connected and so contains some interval of the form $(0, q)$, where $q > 0$.

Since $b_n c_n \to 0$ ($n \to \infty$), for large n ($n > M$), $|b_n c_n| \in (0, q)$. Then there is t ($t \geq T_0$) such that

$$\frac{\alpha}{s_t} < |b_n c_n| < \frac{\beta}{s_t}.$$

Since $n > M$, we have

$$\frac{r_1}{s_t^2} < b_n^2 c_n^2 + (a_n - d_n)^2 b_n c_n q_t^{-1} < \frac{r_2}{s_t^2},$$

i.e.,

$$\operatorname{tr}[f_n, f^t f_n f^{-t}] - 2 \in (r_1, r_2).$$

Since $[f_n, f^t f_n f^{-t}] \in E(G, G)$, by the arbitrariness of r_1, r_2 ($0 < r_1 < r_2$), we know $TE(G, G)$ is dense in $[0, \infty)$. □

By [5] and Theorem 3.1, the following is obvious.

Corollary 3.1 *Let $G \subset SL(2, R)$ be nonelementary and finitely generated. Then*

(1) *G is discrete if and only if either $TE(G, G) = \emptyset$ or $TE(G, G)$ is discrete in $[c_0, \infty)$;*

(2) *G is nondiscrete if and only if $TE(G,G)$ is dense in $[0,\infty)$.*

Theorem 3.2 *Let $G \subset SL(2,R)$ be nonelementary. Then*
(1) *G is discrete if and only if* $\inf\{|A| \colon A \in TH(G,G)\} \geq c_0$;
(2) *G is nondiscrete if and only if $TH(G,G)$ is dense in R.*

Proof. By [6], the necessity of (1) is obvious. For the sufficiency, we suppose that $\inf\{|A| \colon A \in TH(G,G)\} \geq c_0$, but G is not discrete.

Let h_j $(j = 1,2,3) \in G$ be hyperbolic such that

$$\mathrm{fix}(h_i) \cap \mathrm{fix}(h_j) = \emptyset \ (i \neq j,\ i,j = 1,2,3).$$

By [3], there is a sequence $\{g_n\}$ of G satisfying that each g_n is hyperbolic and

$$g_n \to I \ (n \to \infty).$$

Then for large enough n, $\langle h_j, g_n \rangle$ $(j = 1,2,3)$ are elementary by the assumptions. It is impossible. So G is discrete.

The sufficiency of (2) follows from (1). For the necessity of (2), let $f \in G$ be hyperbolic and

$$f = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \ (r > 1). \tag{3.1}$$

Then for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have

$$\mathrm{tr}[f,g] - 2 = -(r - r^{-1})^2 bc.$$

Since G is nondiscrete and nonelementary, there is a sequence $\{f_n\}$ of G such that each f_n is hyperbolic,

$$f_n \to I \ (n \to \infty)$$

and for each n,

$$\mathrm{fix}(f) \cap \mathrm{fix}(f_n) = \emptyset.$$

Then $[f, f_n] \in H(G,G)$. Hence there is a sequence $\{x_n\} \in TH(G,G)$ such that

$$x_n \to 0 \ (n \to \infty). \tag{3.2}$$

For every $t \in N$, if $f_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, then

$$\mathrm{tr}[f^{t+1}, f_n] - 2 = -(r - r^{-1})^2 b_n c_n s_t,$$

where $s_t = (r^{t+1} - r^{-(t+1)})^2 (r - r^{-1})^{-2}$.

Obviously

$$s_t \to \infty \ (t \to \infty).$$

Thus if $x_n \in TH(G,G)$, then $s_t x_n \in TH(G,G)$ for each $t \in N$. (3.3)

For any v_1, v_2 $(0 < v_1 < v_2)$, by [3], we may assume in (3.1) that

$$1 < r^4 < (p_2 p_1^{-1})^{\frac{1}{4}},$$

where $p_1 = \frac{2v_1+v_2}{3}$, $p_2 = \frac{v_1+2v_2}{3}$.

Then there is T_1 such that for all $T > T_1$, the unions of intervals

$$\left(\frac{v_1}{s_T}, \frac{v_2}{s_T}\right) \cup \left(\frac{v_1}{s_{T+1}}, \frac{v_2}{s_{T+1}}\right) \cup \ldots$$

and

$$\left(\frac{v_1}{s_T^2}, \frac{v_2}{s_T^2}\right) \cup \left(\frac{v_1}{s_{T+1}^2}, \frac{v_2}{s_{T+1}^2}\right) \cup \ldots$$

are connected and so contains intervals with the forms $(0,p)$ $(p > 0)$.

If there is a subsequence of $\{f_n\}$ (still denoted by the same manner) such that for each n, $b_n c_n < 0$, then, by (3.2), there is M_2 $(> M_1)$ such that for all $n > M_2$,

$$-(r - r^{-1})^2 b_n c_n \in (0,p).$$

Hence, there is $t \in N$ such that

$$\frac{v_1}{s_t} < -(r - r^{-1})^2 b_n c_n < \frac{v_2}{s_t}.$$

Then

$$v_1 < -(r - r^{-1})^2 b_n c_n s_t < v_2.$$

(3.3) implies that there is $Q \in H(G,G)$ such that

$$\mathrm{tr}(Q) - 2 \in (v_1, v_2).$$

Thus we assume that for each n, $b_n c_n > 0$, then there is M_3 ($> M_2$) such that for all $n > M_3$,

$$(r - r^{-1})^4 b_n c_n (1 + b_n c_n) \in (0, p).$$

Hence there is $t \in N$ such that

$$\frac{v_1}{s_t^2} < (r - r^{-1})^4 b_n c_n (1 + b_n c_n) < \frac{v_2}{s_t^2}.$$

Then

$$v_1 < (r - r^{-1})^4 b_n c_n (1 + b_n c_n) s_t^2 < v_2.$$

Since for large n, $[f, f_n] \in H(G,G)$ if and only if $[f, f_n f f_n^{-1}] \in H(G,G)$, this implies that there is $Q \in H(G,G)$ such that

$$\mathrm{tr}(Q) - 2 \in (v_1, v_2).$$

Now we prove that $TH(G,G) \cap [0, \infty)$ is dense in $[0, \infty)$.

In the same way as above we can show that $TH(G,G) \cap (-\infty, 0]$ is dense in $(-\infty, 0]$.

Hence $TH(G,G)$ is dense in R. □

Corollary 3.2 *Let G be nonelementary and finitely generated, then*
(1) *G is discrete if and only if $TH(G,G)$ is discrete in $(-\infty, -c_0] \cup [c_0, \infty)$;*
(2) *G is nondiscrete if and only if $TH(G,G)$ is dense in R.*

For any nonelementary subgroup $G \subset SL(2,R)$ and nontrivial $f \in G$, let

$$G_f(G,G) = \{[f, gfg^{-1}] \colon \langle f, gfg^{-1} \rangle \text{ nonelementary for } f, g \in G\};$$

$$TG_f(G,G) = \{\mathrm{tr}(Q) - 2 \colon Q \in G_f(G,G)\}.$$

Then, by the proofs of Theorems 3.1, 3.2 and [7], we get

Theorem 3.3 *Let G be nonelementary.*
(1) *If G contains an elliptic element f of order at least 3, then G is discrete if and only if $\inf\{TG_f(G,G)\} \geq c_0$.*
(2) *G is discrete if and only if for any fixed hyperbolic element f, $\inf\{|A| \colon A \in TG_f(G,G)\} \geq c_0$.*

We also can get the following form of (1) of Theorem A (i.e., Theorem 4.1 in [1]) as follows.

Theorem 3.4 *Let $G \subset SL(2,R)$ be nonelementary and $f \in G$ be parabolic. Then*
(1) *G is discrete if and only if* $\inf\{TG_f(G,G)\} \geq 1$;
Furthermore, if G is finitely generated, then
(2) *G is discrete if and and if $TG_f(G,G)$ is discrete in* $[1,\infty)$.

Proof. The proof of (2) follows from (1) and [5], we need only prove (1). The necessity of (1) is obvious. For the sufficiency, we suppose that $\inf\{TG_f(G,G)\} \geq 1$, but G is not discrete, contrary to our assertion.

Without loss of generality, let

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since G is nonelementary, there is a hyperbolic element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $c \neq 0$. Let G_∞ be the stabilizer of ∞ of G. Then for every parabolic element $q \in G_\infty$,

$$q = \begin{pmatrix} \epsilon & s \\ 0 & \epsilon \end{pmatrix} \ (s \neq 0,\ \epsilon = \pm 1).$$

We deduce from $[f, gfg^{-1}] \in G_f(G,G)$ that

$$|s| \geq |c|^{-\frac{1}{2}}.$$

Fixing g and varying s, the above inequality shows that there is $s_0 \neq 0$ such that for any parabolic element $q = \begin{pmatrix} \epsilon & s \\ 0 & \epsilon \end{pmatrix} \in G_\infty$,

$$|s| \geq |s_0| > 0.$$

By our assumptions and [3], there is a squence $\{f_n\} \subset G$ such that each f_n is hyperbolic and

$$f_n \to I \ (n \to \infty).$$

Let

$$f_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

If $c_n \neq 0$, then

$$\operatorname{tr}[f, f_n f f_n^{-1}] \geq 3.$$

This implies that $c_n^4 \geq 1$ if $c_n \neq 0$. This contradiction shows that for large enough n, $\infty \in \operatorname{fix}(f_n)$. So either $[f, f_n] = I$ or $[f, f_n] \in G_\infty$ is parabolic. Hence

$$|1 - a_n^2| \geq |s_0| > 0 \quad \text{or} \quad a_n^2 = 1.$$

These are impossible. Hence G is discrete. □

Remark By using the method in the proof of theorem 3.1 in [8], we can get a different proof of theorem 3.4.

4. Discrete and nondiscrete subgroups in $SL(2, C)$

Given a point $z \in C$ and a set $F \subset \bar{C}$, we call F is quasi-dense about z if $S \cap F \neq \emptyset$ for any annulus $S = \{x \in \bar{C} \colon a < |x - z| < b\}$ $(0 < a < b)$.

By using similar discussions as in Section 3 and [1,8], we can get

Theorem 4.1 *Let $G \subset SL(2, C)$ be nonelementary. If G contains parabolic elements, then*

(1) *G is discrete if and only if* $\inf\{|A| \colon A \in TP(G, G)\} \geq 1$;

(2) *G is nondiscrete if and only if $TP(G, G)$ is quasi-dense about the point* $(0, 0)$.

5. On one of Beardon's problems

As in [1], let

$$A = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix},$$

$$G = \langle A, B \rangle,$$

where τ is an indeterminate in $\bar{C}$.

For any $t \in C$, $G_t = \langle A_t, B_t \rangle$, where $A_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $B_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

After showing when two non-commuting parabolic elements of G generate G, Beardon raised the following problem:

A and B are not conjugate in G, but can A_t and B_t ever be conjugate in G_t?

In the following, we show that the answer to this problem is affirmative. Let $\tau = i$ $(i^2 = -1)$. Then

$$A_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad B_i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}.$$

By taking $h = A_i B_i$, we know

$$A_i h = h B_i.$$

This shows that A_t and B_t are conjugate in G_t when $t = i$.
Let

$$G_{AB} = \{F \in G : AF = FB\}.$$

Obviously

$$\begin{aligned} G_{\sqrt{2}} &= \langle A_{\sqrt{2}}, B_{\sqrt{2}} \rangle \\ &= \left\{ \begin{pmatrix} a & \sqrt{2}b \\ \sqrt{2}c & d \end{pmatrix} : a, b, c, d \in Z,\ ad - 2bc = 1 \right\}. \end{aligned}$$

Then $G_{A_{\sqrt{2}} B_{\sqrt{2}}} = \emptyset$. This implies that A and B are not conjugate in G.

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References

[1] Beardon A.F., *Some remarks on nondiscrete Möbius groups.* Ann. Acad. Sci. Fenn. Math. **21** (1996), 69–79.

[2] Beardon A.F., *The geometry of discrete groups.* Springer-Verlag, 1983.

[3] Doyle C. and James D., *Discrete criteria and high order generators for subgroups of $SL(2, C)$*, Ill. J. Math. **25** (1981), 191–200.

[4] Jorgensen T., *A note on subgroups of $SL(2, C)$.* Quart. J. Math. Oxford **28** (1977), 209–212.

[5] Greenberg L., *Finiteness theorems for Fuchsian and Kleinian groups.* Discrete groups and Automorphic Functions, edited by W.J. Harvey, Academic Press, 1977.

[6] Rosenberger G., *All generating pairs of all two-generator Fuchsian groups.* Arch. Math. **46** (1986), 198–204.

[7] Tukia P. and Wang X., *Discreteness criteria of subgroups of $SL(2, C)$ with elliptic elements.* Priprint **251**, University of Helsinki, January 2000.

[8] Wang X. and Yang W., *Dense subgroups and discrete subgroups in $SL(2, C)$.* Quart. J. Math. Oxford **50** (1999), 517–521.

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