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On the scaling exponents of Takagi, Lévy and Weierstrass functions

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Abstract. We study the scaling exponents of the Takagi, Lévy and Weierstrass functions. We show that their pointwise Hölder exponents coincide with their weak scaling exponents at each point of the real line. A partial result about the scaling exponent of the Lévy function is also given.

Key words: wavelets, scaling exponents, Takagi function, Lévy function, Weierstrass functions.

1. Introduction

Let s be a positive number, which is not an integer and let x_0 be a point in $\mathbf{R}^n$. Then a function f on $\mathbf{R}^n$ belongs to the pointwise Hölder space $C^s(x_0)$, if there exists a polynomial P of degree less than s such that

$$|f(x)-P(x-x_0)|\leq C|x-x_0|^s$$

in a neighborhood of x_0. $\mathcal{S}_0(\mathbf{R}^n)$ denotes the closed subspace of the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} x^\alpha\psi(x)\,dx=0$$

for any multi-index α in $\mathbf{Z}_+^n$. Then a tempered distribution f belongs to $\Gamma^s(x_0)$, if for each ψ in $\mathcal{S}_0(\mathbf{R}^n)$, there exists a constant $C(\psi)$ such that

$$\left|\int_{\mathbf{R}^n} f(x)\frac{1}{a^n}\psi\left(\frac{x-x_0}{a}\right)dx\right|\leq C(\psi)a^s,\quad 0<a\leq 1.$$

Let φ be a function in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ such that

$$\hat{\varphi}(\xi)=\begin{cases}1 & \text{on } |\xi|\leq\dfrac{1}{2}\\ 0 & \text{on } |\xi|\geq 1\end{cases},$$

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where $\hat{\varphi}$ is the Fourier transform of φ. For any non-negative integer j, we define the convolution operator $S_j(f) = f * \varphi_{\frac{1}{2^j}}$ where $\varphi_a(x) = \frac{1}{a^n}\varphi\left(\frac{x}{a}\right)$, and the difference operator $\Delta_j = S_{j+1} - S_j$. Then

$$I = S_0 + \sum_{j=0}^{\infty} \Delta_j.$$

Let $\psi = \varphi_{\frac{1}{2}} - \varphi$. Then $\psi \in \mathcal{S}_0(\mathbf{R}^n)$ and

$$\Delta_j(f) = f * \psi_{\frac{1}{2^j}}.$$

Here let us recall the definition of the two-microlocal spaces $C^{s,s'}_{x_0}$.

Let s and s' be two real numbers and x_0 a point in $\mathbf{R}^n$. Then a tempered distribution f belongs to the two-microlocal spaces $C^{s,s'}_{x_0}$, if there exists a constant C such that

$$|S_0(f)(x)| \leq C(1 + |x - x_0|)^{-s'}$$

and

$$|\Delta_j(f)(x)| \leq C2^{-js}(1 + 2^j|x - x_0|)^{-s'}$$

for $j \in \mathbf{Z}_+$ and $x \in \mathbf{R}^n$.

The following remarkable theorems with respect to the two-microlocal spaces $C^{s,s'}_{x_0}$ and $\Gamma^s(x_0)$ were given in [3].

Theorem A [3, Theorem 1.8] *Let s and s' be two real numbers and x_0 a point in $\mathbf{R}^n$ and let us assume two non-negative integers r and N satisfying*

$$r + s + \inf(s', n) > 0$$

and

$$N > \sup(s, s + s').$$

Let ψ be a function such that

$$|\partial^\alpha \psi(x)| \leq C(q)(1 + |x|)^{-q}, \quad |\alpha| \leq r, \quad q \geq 1$$

and

$$\int_{\mathbf{R}^n} x^\beta \psi(x)\, dx = 0, \quad |\beta| \leq N - 1.$$

If a function or a distribution f belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$, then we have

$$|W(a,b)| \leq Ca^s \left(1 + \frac{|b - x_0|}{a}\right)^{-s'}, \quad 0 < a \leq 1, \quad |b - x_0| \leq 1,$$

where $W(a,b) = \langle f, \psi_{(a,b)} \rangle$ and $\psi_{(a,b)}(x) = \frac{1}{a^n}\psi\left(\frac{x-b}{a}\right)$.

Theorem B [3, Theorem 1.2] *Let s be a real number and f a function or a distribution defined on a neighborhood V of x_0.*

Then f locally belongs to $\Gamma^s(x_0)$ if and only if f locally belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$ for some s'.

The pointwise Hölder exponent of a function f at a point x_0 in $\mathbf{R}^n$ is defined as

$$H(f, x_0) = \sup\{s > 0;\ f \in C^s(x_0)\}.$$

The weak scaling exponent of a function f at a point x_0 in $\mathbf{R}^n$ is defined as

$$\beta(f, x_0) = \sup\{s \in \mathbf{R};\ f \text{ locally belongs to } \Gamma^s(x_0)\}.$$

Since it is known that the pointwise Hölder space $C^s(x_0)$ is contained in local $\Gamma^s(x_0)$, it is obvious that

$$H(f, x_0) \leq \beta(f, x_0).$$

Using the exponents, Meyer defined two types of singularities of functions as follows [3]: a point x_0 in $\mathbf{R}^n$ is called a cusp singularity of a function f, when

$$H(f, x_0) = \beta(f, x_0) < \infty,$$

while a point x_0 in $\mathbf{R}^n$ is called an oscillating singularity of a function f, when

$$H(f, x_0) < \beta(f, x_0).$$

From now on, in the following two sections, we study what kind of singularities have Takagi, Lévy and Weierstrass functions.

2. Takagi and Lévy Functions

Let (x) and $|(x)|$ be the 1-periodic functions such that

$$(x) = \begin{cases} x & \text{if } |x| < \dfrac{1}{2} \\ 0 & \text{if } x = \dfrac{1}{2} \end{cases} \quad \text{and} \quad |(x)| = |x| \quad \text{if} \quad -\frac{1}{2} < x \leq \frac{1}{2}.$$

Using these notations, the Takagi and Lévy functions are defined as follows:

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \frac{|(2^n x)|}{2^n} \qquad \text{(Takagi function)}$$

$$\mathcal{L}(x) = \sum_{n=0}^{\infty} \frac{(2^n x)}{2^n} \qquad \text{(Lévy function).}$$

It is known that $H(\mathcal{T}, x_0) = 1$ at each point x_0 in $\mathbf{R}$ and that $H(\mathcal{L}, x_0) = -\liminf_{j\to\infty} \frac{j}{\log_2 \Delta_j(x_0)}$ at each nondyadic point x_0 [2], where

$$\Delta_n(x) = \operatorname{dist}\left(x, \frac{\mathbf{Z}}{2^n}\right).$$

Theorem 1 *Let f be a function such that $f(x) = (x)$ or $f(x) = |(x)|$. We define a function F by*

$$F(x) = \sum_{n=0}^{\infty} \frac{f(2^n x)}{2^n}.$$

Then the weak scaling exponent $\beta(F, x_0)$ at each point x_0 in $\mathbf{R}$ *satisfies*

$$\beta(F, x_0) \leq 1.$$

Proof. Let us assume F locally belongs to $\Gamma^s(x_0)$. Then by Theorem B, F locally belongs to $C_{x_0}^{s,s'}$ for some $s' < 0$. Let N be an integer greater than s. Let ψ be a function supported on $[0,1]$, has $N-1$ vanishing moments. Since F locally belongs to $C_{x_0}^{s,s'}$, by Theorem A, there exist $C > 0$ and $\delta \in (0,1]$ such that

$$\left| \int_{\mathbf{R}} F(x) \frac{1}{a} \psi\left(\frac{x-b}{a}\right) dx \right| \leq C a^s \left(1 + \frac{|b - x_0|}{a}\right)^{-s'},$$
$$0 < a \leq \delta, \quad |b - x_0| \leq \delta. \qquad (1)$$

Let j_0 be a non-negative integer such that $\frac{1}{2^{j_0}} \leq \delta$. For each $j \geq j_0$, there exists $k_j \in \mathbf{Z}$ such that $\frac{k_j}{2^j} \leq x_0 < \frac{k_j+1}{2^j}$ and we define a_j and b_j by $a_j = \frac{1}{2^j}$ and $b_j = \frac{k_j}{2^j}$. Then $|b_j - x_0| < a_j$ and we have from (1)

$$\left| \int_{\mathbf{R}} F(x) 2^j \psi(2^j x - k_j)\, dx \right| < \frac{C 2^{-s'}}{2^{js}}, \quad j \geq j_0. \tag{2}$$

Since

$$\int_{\mathbf{R}} F(x) 2^j \psi(2^j x - k_j)\, dx = \int_0^1 F\left(\frac{x+k_j}{2^j}\right) \psi(x)\, dx, \tag{3}$$

now we consider $F(\frac{x+k_j}{2^j})$. Then

$$\begin{aligned} F\left(\frac{x+k_j}{2^j}\right) &= \sum_{n=0}^{\infty} \frac{f(2^{n-j}(x+k_j))}{2^n} \\ &= \sum_{n=0}^{j-1} \frac{f(2^{n-j}(x+k_j))}{2^n} + \frac{1}{2^j} F(x) \\ &= \frac{1}{2^j} \sum_{n=1}^{j} 2^n f\left(\frac{x+k_j}{2^n}\right) + \frac{1}{2^j} F(x). \end{aligned}$$

When $1 \leq n \leq j$, we have

$$\frac{k_j}{2^n} - \left[\frac{k_j}{2^n}\right] < \frac{1}{2}, \quad 0 < x < 1 \quad \text{imply} \quad \left[\frac{k_j}{2^n}\right] < \frac{x+k_j}{2^n} < \left[\frac{k_j}{2^n}\right] + \frac{1}{2},$$

$$\frac{k_j}{2^n} - \left[\frac{k_j}{2^n}\right] \geq \frac{1}{2}, \quad 0 < x < 1 \quad \text{imply} \quad \left[\frac{k_j}{2^n}\right] + \frac{1}{2} < \frac{x+k_j}{2^n} < \left[\frac{k_j}{2^n}\right] + 1,$$

where $[x]$ denotes the largest integer not greater than x. Hence $f(\frac{x+k_j}{2^n})$ equals a polynomial of degree at most 1 on the support of ψ. Therefore

$$\frac{1}{2^j} \int_0^1 \sum_{n=1}^{j} 2^n f\left(\frac{x+k_j}{2^n}\right) \psi(x)\, dx = 0.$$

Thus

$$\left| \int_0^1 F\left(\frac{x+k_j}{2^j}\right) \psi(x)\, dx \right| = \frac{1}{2^j} \left| \int_0^1 F(x) \psi(x)\, dx \right|. \tag{4}$$

Since F is not a polynomial, we can select a wavelet ψ such that

$\int_0^1 F(x)\psi(x)\,dx = 1$.

Then from (2), (3) and (4), $F \in \Gamma^s(x_0)$ implies $\frac{1}{2^j} < \frac{C2^{-s'}}{2^{js}}$ for every $j \geq j_0$ and hence we have $s \leq 1$.

Therefore we have $\beta(F, x_0) \leq 1$. □

By Theorem 1 and the fact that $H(\mathcal{T}, x_0) = 1$ at each point x_0 in $\mathbf{R}$ and that $H(\mathcal{L}, x_0) = -\liminf_{j\to\infty} \frac{j}{\log_2 \Delta_j(x_0)}$ at each nondyadic point x_0, we have the following corollaries.

Corollary 1 *Each point in* $\mathbf{R}$ *is a cusp singularity of the Takagi function* $\mathcal{T}$.

Corollary 2 *A nondyadic point* x_0 *is a cusp singularity of the Lévy function* $\mathcal{L}$ *when the pointwise Hölder exponent* $H(\mathcal{L}, x_0)$ *equals* 1.

In particular a nondyadic rational point is a cusp singularity of Lévy function $\mathcal{L}$.

In fact, if $\frac{p}{q}$ be a nondyadic rational point, then we have

$$\left|\frac{p}{q} - \frac{k}{2^j}\right| \geq \frac{1}{q2^j}$$

for any integer k and hence we have $\Delta_j\left(\frac{p}{q}\right) \geq \frac{1}{q2^j}$. Therefore we have $H\left(\mathcal{L}, \frac{p}{q}\right) = 1$. Thus $\frac{p}{q}$ is a cusp singularity of Lévy function $\mathcal{L}$.

3. Weierstrass Functions

Let b be an integer, a and $\frac{b}{a}$ greater than 1. Then the functions which are defined by

$$\mathcal{W}_c(x) = \sum_{n=0}^{\infty} \frac{\cos b^n x}{a^n}$$

are called the Weierstrass functions. Similarly the Weierstrass functions of sine series

$$\mathcal{W}_s(x) = \sum_{n=0}^{\infty} \frac{\sin b^n x}{a^n}$$

are also defined. It is known that $H(\mathcal{W}_c, x_0) = H(\mathcal{W}_s, x_0) = \frac{\log a}{\log b}$ at each point x_0 in $\mathbf{R}$ [1].

Theorem 2 *Let w be a function such that $w(x) = \cos x$ or $w(x) = \sin x$. We define the Weierstrass functions $\mathcal{W}$ by*

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \frac{w(b^n x)}{a^n}.$$

Then the weak scaling exponents $\beta(\mathcal{W}, x_0)$ of the Weierstrass functions $\mathcal{W}$ at each point x_0 in $\mathbf{R}$ satisfy

$$\beta(\mathcal{W}, x_0) \leq \frac{\log a}{\log b}.$$

In order to prove this fact, we first show the following lemmas.

Lemma 1 *Let l be a non-negative integer. Then*

$$\text{(A)} \qquad \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathcal{W}_c(x) \cos^{2l+1} x\, dx \geq \frac{2(2l+1)!!\pi}{(2l+2)!!}$$

$$\text{(B)} \qquad \int_0^{2\pi} \mathcal{W}_s(x) \sin^{2l+1} x\, dx > \frac{2(2l+1)!!\pi}{(2l+2)!!} \left(1 - \frac{1}{a}\right).$$

Proof. We use the following formulas.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1} x \cos(2m+1)x\, dx = \begin{cases} \dfrac{\pi}{2^{2n+1}} \dbinom{2n+1}{n-m} & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}, \tag{5}$$

$$\int_0^{\pi} \sin^{2n+1} x \sin(2m+1)x\, dx = \begin{cases} \dfrac{(-1)^m \pi}{2^{2n+1}} \dbinom{2n+1}{n-m} & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}, \tag{6}$$

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^{2n+1} x \cos 2mx\, dx = 0, \tag{7}$$

$$\int_0^{\pi} \sin^n x \sin 2mx\, dx = 0 \tag{8}$$

for any $n, m \in \mathbf{Z}_+$. When $b \in 2\mathbf{Z} + 1$, we have by (5) and (6)

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathcal{W}_c(x)\cos^{2l+1}x\,dx &= 2\sum_{n=0}^{\infty}\frac{1}{a^n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(b^n x)\cos^{2l+1}x\,dx\\ &= 2\sum_{n=0}^{\left[\frac{\log(2l+1)}{\log b}\right]}\frac{1}{a^n}\frac{\pi}{2^{2l+1}}\binom{2l+1}{\frac{2l+1-b^n}{2}}\\ &\geq \frac{2\pi}{2^{2l+1}}\binom{2l+1}{l}\\ &= \frac{2(2l+1)!!\pi}{(2l+2)!!}\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi}\mathcal{W}_s(x)\sin^{2l+1}x\,dx &= 2\sum_{n=0}^{\infty}\frac{1}{a^n}\int_0^{\pi}\sin(b^n x)\sin^{2l+1}x\,dx\\ &= 2\sum_{n=0}^{\left[\frac{\log(2l+1)}{\log b}\right]}\frac{1}{a^n}\frac{(-1)^{\frac{b^n-1}{2}}\pi}{2^{2l+1}}\binom{2l+1}{\frac{2l+1-b^n}{2}}\\ &\geq 2\sum_{n=0}^{\left[\frac{\log(2l+1)}{\log b}\right]}\left(-\frac{1}{a}\right)^n\frac{\pi}{2^{2l+1}}\binom{2l+1}{\frac{2l+1-b^n}{2}}\\ &> \frac{2\pi}{2^{2l+1}}\binom{2l+1}{l}\left(1-\frac{1}{a}\right)\\ &= \frac{2(2l+1)!!\pi}{(2l+2)!!}\left(1-\frac{1}{a}\right).\end{aligned}$$

When $b \in 2\mathbf{Z}$, we have by (7) and (8)

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathcal{W}_c(x)\cos^{2l+1}x\,dx &= \sum_{n=0}^{\infty}\frac{1}{a^n}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos(b^n x)\cos^{2l+1}x\,dx\\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{2l+2}x\,dx\\ &= \frac{2(2l+1)!!\pi}{(2l+2)!!}\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi} \mathcal{W}_s(x)\sin^{2l+1} x\,dx &= 2\sum_{n=0}^{\infty}\frac{1}{a^n}\int_0^{\pi}\sin(b^n x)\sin^{2l+1} x\,dx\\ &= 2\int_0^{\pi}\sin^{2l+2} x\,dx\\ &= \frac{2(2l+1)!!\pi}{(2l+2)!!}.\end{aligned}$$

Therefore the lemma is proved. □

Lemma 2 *Let l and p be non-negative integers such that*

$$\frac{2(2l)!!\pi}{(2l+1)!!}\sum_{n=p+1}^{\infty}\left(\frac{a}{b}\right)^n \le \frac{(2l+1)!!\pi}{(2l+2)!!}\left(1-\frac{1}{a}\right). \tag{9}$$

(A) *If k is an integer such that*

$$\sin\left(\frac{(4k+1)\pi}{2b^n}\right) \le 0, \quad 1\le n\le p, \tag{10}$$

then

$$\begin{aligned}&\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\sum_{n=1}^{j} a^n\cos\left(\frac{x+2k\pi}{b^n}\right)\cos^{2l+1} x\,dx\\ &\qquad \ge -\frac{(2l+1)!!\pi}{(2l+2)!!}\left(1-\frac{1}{a}\right), \quad j\ge p+1.\end{aligned}$$

(B) *If k is an integer such that*

$$\cos\left(\frac{(2k+1)\pi}{b^n}\right) \le 0, \quad 1\le n\le p, \tag{11}$$

then

$$\begin{aligned}&\int_0^{2\pi}\sum_{n=1}^{j} a^n\sin\left(\frac{x+2k\pi}{b^n}\right)\sin^{2l+1} x\,dx\\ &\qquad \ge -\frac{(2l+1)!!\pi}{(2l+2)!!}\left(1-\frac{1}{a}\right), \quad j\ge p+1.\end{aligned}$$

Proof. (A) Since

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\left(\frac{x+2k\pi}{b^n}\right)\cos^{2(l-m)+1} x\,dx$$
$$= \frac{2(l-m)(2(l-m)+1)}{(2(l-m)+1)^2-\frac{1}{b^{2n}}}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\left(\frac{x+2k\pi}{b^n}\right)\cos^{2(l-m)-1} x\,dx,$$
$$0 \le m \le l-1$$

and

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\left(\frac{x+2k\pi}{b^n}\right)\cos x\,dx = -\frac{2\sin\left(\frac{(4k+1)\pi}{2b^n}\right)\sin\left(\frac{\pi}{b^n}\right)}{1-\frac{1}{b^{2n}}},$$

we have

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \sum_{n=1}^{j} a^n \cos\left(\frac{x+2k\pi}{b^n}\right)\cos^{2l+1} x\,dx$$
$$= -\sum_{n=1}^{j} \frac{2(2l+1)!a^n \sin\left(\frac{(4k+1)\pi}{2b^n}\right)\sin\left(\frac{\pi}{b^n}\right)}{\prod_{m=0}^{l}\left((2m+1)^2-\frac{1}{b^{2n}}\right)}$$
$$= -\sum_{n=1}^{j} \frac{4(2l)!!a^n \sin\left(\frac{(4k+1)\pi}{2b^n}\right)\sin\left(\frac{\pi}{2b^n}\right)\cos\left(\frac{\pi}{2b^n}\right)}{(2l+1)!!\prod_{m=0}^{l}\left(1-\frac{1}{(2m+1)^2b^{2n}}\right)}$$
$$= -\frac{2(2l)!!\pi}{(2l+1)!!}\sum_{n=1}^{j}\sin\left(\frac{(4k+1)\pi}{2b^n}\right)\frac{\sin\left(\frac{\pi}{2b^n}\right)}{\frac{\pi}{2b^n}}$$
$$\frac{\cos\left(\frac{\pi}{2b^n}\right)}{\prod_{m=0}^{l}\left(1-\frac{1}{(2m+1)^2b^{2n}}\right)}\left(\frac{a}{b}\right)^n$$
$$\ge -\frac{2(2l)!!\pi}{(2l+1)!!}\sum_{n=p+1}^{j}\sin\left(\frac{(4k+1)\pi}{2b^n}\right)\frac{\sin\left(\frac{\pi}{2b^n}\right)}{\frac{\pi}{2b^n}}$$
$$\frac{\cos\left(\frac{\pi}{2b^n}\right)}{\prod_{m=0}^{l}\left(1-\frac{1}{(2m+1)^2b^{2n}}\right)}\left(\frac{a}{b}\right)^n, \quad (12)$$

where the last inequality is due to (10). Since $\prod_{n=0}^{\infty}\left(1-\frac{x^2}{(2n+1)^2}\right) = \cos\left(\frac{\pi x}{2}\right)$,

$$0 < \frac{\cos\left(\frac{\pi}{2b^n}\right)}{\prod_{m=0}^{l}\left(1 - \frac{1}{(2m+1)^2 b^{2n}}\right)} \le 1. \tag{13}$$

Then we have from (12), (13) and (9)

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \sum_{n=1}^{j} a^n \cos\left(\frac{x+2k\pi}{b^n}\right)\cos^{2l+1} x\,dx &\ge -\frac{2(2l)!!\pi}{(2l+1)!!}\sum_{n=p+1}^{\infty}\left(\frac{a}{b}\right)^n \\ &\ge -\frac{(2l+1)!!\pi}{(2l+2)!!}\left(1-\frac{1}{a}\right).\end{aligned}$$

(B) Since

$$\begin{aligned}&\int_0^{2\pi} \sin\left(\frac{x+2k\pi}{b^n}\right)\sin^{2(l-m)+1} x\,dx \\ &= \frac{2(l-m)(2(l-m)+1)}{(2(l-m)+1)^2 - \frac{1}{b^{2n}}}\int_0^{2\pi}\sin\left(\frac{x+2k\pi}{b^n}\right)\sin^{2(l-m)-1} x\,dx, \\ &\qquad 0 \le m \le l-1\end{aligned}$$

and

$$\int_0^{2\pi}\sin\left(\frac{x+2k\pi}{b^n}\right)\sin x\,dx = -\frac{2\cos\left(\frac{(2k+1)\pi}{b^n}\right)\sin\left(\frac{\pi}{b^n}\right)}{1-\frac{1}{b^{2n}}},$$

we have

$$\begin{aligned}&\int_0^{2\pi}\sum_{n=1}^{j} a^n \sin\left(\frac{x+2k\pi}{b^n}\right)\sin^{2l+1} x\,dx \\ &= -\sum_{n=1}^{j}\frac{2(2l+1)!a^n\cos\left(\frac{(2k+1)\pi}{b^n}\right)\sin\left(\frac{\pi}{b^n}\right)}{\prod_{m=0}^{l}\left((2m+1)^2 - \frac{1}{b^{2n}}\right)} \\ &= -\sum_{n=1}^{j}\frac{4(2l)!!a^n\cos\left(\frac{(2k+1)\pi}{b^n}\right)\sin\left(\frac{\pi}{2b^n}\right)\cos\left(\frac{\pi}{2b^n}\right)}{(2l+1)!!\prod_{m=0}^{l}\left(1-\frac{1}{(2m+1)^2 b^{2n}}\right)} \\ &= -\frac{2(2l)!!\pi}{(2l+1)!!}\sum_{n=1}^{j}\cos\left(\frac{(2k+1)\pi}{b^n}\right)\frac{\sin\left(\frac{\pi}{2b^n}\right)}{\frac{\pi}{2b^n}} \\ &\qquad \frac{\cos\left(\frac{\pi}{2b^n}\right)}{\prod_{m=0}^{l}\left(1-\frac{1}{(2m+1)^2 b^{2n}}\right)}\left(\frac{a}{b}\right)^n\end{aligned}$$

$$\geq -\frac{2(2l)!!\pi}{(2l+1)!!} \sum_{n=p+1}^{j} \cos\left(\frac{(2k+1)\pi}{b^n}\right) \frac{\sin\left(\frac{\pi}{2b^n}\right)}{\frac{\pi}{2b^n}} \frac{\cos\left(\frac{\pi}{2b^n}\right)}{\prod_{m=0}^{l}\left(1-\frac{1}{(2m+1)^2 b^{2n}}\right)} \left(\frac{a}{b}\right)^n, \quad (14)$$

where the last inequality is due to (11). Then we have from (14), (13) and (9)

$$\int_0^{2\pi} \sum_{n=1}^{j} a^n \sin\left(\frac{x+2k\pi}{b^n}\right) \sin^{2l+1} x\, dx \geq -\frac{2(2l)!!\pi}{(2l+1)!!} \sum_{n=p+1}^{\infty} \left(\frac{a}{b}\right)^n$$
$$\geq -\frac{(2l+1)!!\pi}{(2l+2)!!}\left(1-\frac{1}{a}\right).$$

□

Proof of Theorem 2. Since $\mathcal{W}$ is 2π-periodic, we only need to prove at $x_0 \in [0, 2\pi)$.

Let us assume $\mathcal{W}$ locally belongs to $\Gamma^s(x_0)$ with $s < 1$. Then by Theorem B, $\mathcal{W}$ locally belongs to $C^{s,s'}_{x_0}$ for some $s' < 0$. When $w(x)$ is $\cos x$, ψ denotes the function on $\mathbf{R}$ defined by

$$\psi(x) = \begin{cases} \cos^{2l+1} x & \text{if } -\dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \\ 0 & \text{otherwise} \end{cases},$$

where $l \in \mathbf{Z}_+$ with $2l+1+s+s' > 0$. On the other hand, when $w(x)$ is $\sin x$, ψ is the function on $\mathbf{R}$ defined by

$$\psi(x) = \begin{cases} \sin^{2l+1} x & \text{if } 0 \leq x \leq 2\pi \\ 0 & \text{otherwise} \end{cases},$$

where $l \in \mathbf{Z}_+$ with $2l+1+s+s' > 0$. Then we can apply Theorem A to $\mathcal{W}$ and ψ with $r = 2l+1$ and $N = 1$. Since $\mathcal{W}$ locally belongs to $C^{s,s'}_{x_0}$, by Theorem A, there exist $C > 0$ and $\delta \in (0, 1]$ such that

$$\left|\int_{\mathbf{R}} \mathcal{W}(x) \frac{1}{\alpha} \psi\left(\frac{x-\beta}{\alpha}\right) dx\right| \leq C\alpha^s \left(1 + \frac{|\beta - x_0|}{\alpha}\right)^{-s'},$$
$$0 < \alpha \leq \delta, \quad |\beta - x_0| \leq \delta. \quad (15)$$

Let p and j_0 be positive integers such that

$$\frac{2(2l)!!\pi}{(2l+1)!!}\sum_{n=p+1}^{\infty}\left(\frac{a}{b}\right)^n \le \frac{(2l+1)!!\pi}{(2l+2)!!}\left(1-\frac{1}{a}\right) \tag{16}$$

and

$$\frac{2\pi}{b^{j_0-p}} \le \delta. \tag{17}$$

For each $j \in \mathbf{Z}_+$ there exists $k_j \in \mathbf{Z}_+$ such that $\frac{2k_j\pi}{b^j} \le x_0 < \frac{2(k_j+1)\pi}{b^j}$. We can represent $k_j = \varepsilon_0 + \cdots + \varepsilon_{j-1}b^{j-1}$ with $\varepsilon_i \in \{0,1,\ldots,b-1\}$.

First, we treat the case where $w(x) = \cos x$. We first choose $\varepsilon_0' \in \{0,\ldots,b-1\}$ such that

$$\sin\left(\frac{(4\varepsilon_0'+1)\pi}{2b}\right) \le 0.$$

Suppose we have chosen $\varepsilon_0',\ldots,\varepsilon_q'$ with $q \le p-2$ and $\varepsilon_i' \in \{0,\ldots,b-1\}$ such that

$$\sin\left(\frac{(4\varepsilon_0'+\cdots+4\varepsilon_{n-1}'b^{n-1}+1)\pi}{2b^n}\right) \le 0, \quad 1 \le n \le q+1.$$

Then we can choose $\varepsilon_{q+1}' \in \{0,\ldots,b-1\}$ such that

$$\sin\left(\frac{(4\varepsilon_0'+\cdots+4\varepsilon_{q+1}'b^{q+1}+1)\pi}{2b^{q+2}}\right) \le 0.$$

By induction we can choose $\varepsilon_0',\ldots,\varepsilon_{p-1}'$ such that

$$\sin\left(\frac{(4\varepsilon_0'+\cdots+4\varepsilon_{n-1}'b^{n-1}+1)\pi}{2b^n}\right) \le 0, \quad 1 \le n \le p. \tag{18}$$

Define $k_j' = \varepsilon_0' + \cdots + \varepsilon_{p-1}'b^{p-1} + \varepsilon_p b^p + \cdots + \varepsilon_{j-1}b^{j-1}$ and put $\beta_j = \frac{2k_j'\pi}{b^j}$. Then we have by (18)

$$\sin\left(\frac{(4k_j'+1)\pi}{2b^n}\right) \le 0, \quad 1 \le n \le p. \tag{19}$$

Further by (17)

$$
\begin{aligned}
|\beta_j - x_0| &\le \left|\frac{2k'_j\pi}{b^j} - \frac{2k_j\pi}{b^j}\right| + \left|\frac{2k_j\pi}{b^j} - x_0\right| \\
&< \frac{2(b^p-1)\pi}{b^j} + \frac{2\pi}{b^j} \\
&= \frac{2\pi}{b^{j-p}} \le \delta, \quad j \ge j_0.
\end{aligned} \tag{20}
$$

Second, we treat the case where $w(x) = \sin x$. In the same way as above, we can choose $k'_j \in \mathbf{Z}_+$ such that

$$
\cos\left(\frac{(2k'_j+1)\pi}{b^n}\right) \le 0, \quad 1 \le n \le p \tag{21}
$$

and

$$
|\beta_j - x_0| < \delta, \quad j \ge j_0, \tag{22}
$$

where $\beta_j = \frac{2k'_j\pi}{b^j}$.

Therefore we have from (15), (20) and (22) with $\alpha_j = \frac{1}{b^j}$

$$
\left|\int_{\mathbf{R}} \mathcal{W}(x) b^j \psi(b^j x - 2k'_j\pi)\, dx\right| < \frac{C(1+2b^p\pi)^{-s'}}{b^{js}}, \quad j \ge j_0. \tag{23}
$$

Since

$$
\int_{\mathbf{R}} \mathcal{W}(x) b^j \psi(b^j x - 2k'_j\pi)\, dx = \int_{\mathbf{R}} \mathcal{W}\left(\frac{x + 2k'_j\pi}{b^j}\right) \psi(x)\, dx, \tag{24}
$$

now we consider $\mathcal{W}\left(\frac{x+2k'_j\pi}{b^j}\right)$. Then

$$
\begin{aligned}
\mathcal{W}\left(\frac{x+2k'_j\pi}{b^j}\right) &= \sum_{n=0}^{\infty} \frac{w(b^{n-j}(x+2k'_j\pi))}{a^n} \\
&= \sum_{n=0}^{j-1} \frac{w(b^{n-j}(x+2k'_j\pi))}{a^n} + \frac{1}{a^j}\mathcal{W}(x) \\
&= \frac{1}{a^j}\sum_{n=1}^{j} a^n w\left(\frac{x+2k'_j\pi}{b^n}\right) + \frac{1}{a^j}\mathcal{W}(x).
\end{aligned} \tag{25}
$$

When $w(x) = \cos x$, from (16) and (19), we can use the result of Lemma 2.

From (25), Lemmas 1 and 2, we have

$$\int_{\mathbf{R}} \mathcal{W}\left(\frac{x+2k_j'\pi}{b^j}\right)\psi(x)\,dx$$
$$= \frac{1}{a^j}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \sum_{n=1}^{j} a^n \cos\left(\frac{x+2k_j'\pi}{b^n}\right)\cos^{2l+1} x\,dx$$
$$+ \frac{1}{a^j}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathcal{W}_c(x)\cos^{2l+1} x\,dx$$
$$\geq \frac{(2l+1)!!\pi}{(2l+2)!!a^j}\left(1+\frac{1}{a}\right), \quad j \geq j_0. \tag{26}$$

When $w(x) = \sin x$, from (16) and (21), we can use the result of Lemma 2. Similarly from (25), Lemmas 1 and 2, we have

$$\int_{\mathbf{R}} \mathcal{W}\left(\frac{x+2k_j'\pi}{b^j}\right)\psi(x)\,dx > \frac{(2l+1)!!\pi}{(2l+2)!!a^j}\left(1-\frac{1}{a}\right), \quad j \geq j_0. \tag{27}$$

Then from (23), (24), (26) and (27), $\mathcal{W} \in \Gamma^s(x_0)$ implies $\frac{(2l+1)!!\pi}{(2l+2)!!a^j}\left(1-\frac{1}{a}\right) < \frac{C(1+2b^p\pi)^{-s'}}{b^{js}}$ for every $j \geq j_0$ and hence we have $s \leq \frac{\log a}{\log b}$.
Therefore we have $\beta(\mathcal{W}, x_0) \leq \frac{\log a}{\log b}$. □

By Theorem 2 and the fact that $H(\mathcal{W}, x_0) = \frac{\log a}{\log b}$ at each point x_0 in $\mathbf{R}$, we have the following corollary.

Corollary 3 *Each point in* $\mathbf{R}$ *is a cusp singularity of the Weierstrass functions* $\mathcal{W}$.

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