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Global existence for a class of cubic nonlinear Schrödinger equations in one space dimension

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Abstract. In this paper, we prove the global existence of a small solution to the Cauchy problem for the nonlinear Schrödinger equation with a class of cubic nonlinearities in one space dimension. Moreover, we also consider the asymptotic behavior in large time of the solution. Our results says that two cubic nonlinearities given in this paper can be considered as nonlinearities of higher degree (more precisely, of degree 5).

Key words: cubic nonlinear Schrödinger equation, asymptotically free solution.

1. Introduction and results

We consider the Cauchy problem for the nonlinear Schrödinger equation in one space dimension

$$iu_t + \frac{1}{2}u_{xx} = F(u, \bar{u}, u_x, \bar{u}_x), \tag{1.1}$$

$$u(0, x) = u_0(x). \tag{1.2}$$

Here u is a complex-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}$ and F is a smooth function on a neighborhood of the origin such that for some integer $p \geq 2$

$$F(u, \bar{u}, q, \bar{q}) = O\left(|u|^p + |q|^p\right) \quad \text{near the origin.} \tag{1.3}$$

It is known that if, in addition, F satisfies

$$\mathrm{Re}\, \frac{\partial F}{\partial q}(u, \bar{u}, q, \bar{q}) \equiv 0, \tag{1.4}$$

then the usual energy method yields the local existence. If we do not assume (1.4) on F, we meet with a difficulty so-called *loss of derivatives*. When the nonlinearity F does not neccesarily satisfy (1.4) the local existence has also been established this decade (see [7] and [9]). Concerning the global existence of solutions, Klainerman-Ponce [10] and Shatah [12] showed that if F satisfies (1.3) with $p \geq 4$ and (1.4), then (1.1)–(1.2) possesses a unique

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global solution provided that the intial data u_0 is small in a certain Sobolev space. If the nonlinearity is of lower degree (*i.e.* quadratic or cubic), it seems difficult to prove the global existence in general. This situation is explained as follows. We consider small amplitude solutions u here. Then we may expect that they behave like free solutions, which implies $\|u(t)\|_{L^2} = O(1)$ and $\|u(t)\|_{L^\infty} = O(t^{-1/2})$ as $t \to \infty$. Therefore the nonlinearity F of degree p is expected to behave like $\|F(t)\|_{L^2} = O(t^{-(p-1)/2})$. It is known that the integrability in time of $||F(t)||_{L^2}$ is almost equivalent to the global existence of solutions, and thus it is natural that there should exist global solutions of the nonlinear Schrödinger equation with a nonlinearity of degree 4 or higher (note that $-(p-1)/2 < -1 \Leftrightarrow p > 3$). On the other hand, when the nonlinearity is quadratic or cubic ($p = 2$ or 3), $||F(t)||_{L^2}$ is not integrable in time so that the global existence of solutions seems hard to prove.

In spite of this, there are not a few papers on the global existence when the nonlinearity is cubic or quadratic. In particular, in the case where the nonlinearity F is cubic and *gauge invariant*, that is, F satisfies

$$F(\omega u, \overline{\omega u}, \omega q, \overline{\omega q}) = \omega F(u, \bar{u}, q, \bar{q}) \tag{1.5}$$

for any $\omega \in \mathbf{C}(|\omega| = 1)$, $u, q \in \mathbf{C}$, much has been studied. For $F = \lambda |u|^2 u$ or $F = i\lambda \partial_x(|u|^2 u)$ with some $\lambda \in \mathbf{R} \setminus \{0\}$, the global existence is well known. Furthermore, for these nonlinearities, the asymptotic behavior of solutions is studied and the existence of modified scattering states is proved by Hayashi and Naumkin [4], [5]. They also established the asymptotic formula for large time. Katayama and Tsutsumi [8] showed that if F satisfies (1.5) and "null gauge condition of order 3" (a typical example which satisfies these conditions is $F = \partial_x(|u|^2)(\lambda u + \mu u_x)$ with $\lambda, \mu \in \mathbf{C}$) then (1.1)–(1.2) has a unique global solution for small initial data u_0 and the usual scattering state exists. Recently, Hayashi and Naumkin [6] considered nonlinear Schrödinger equations with a derivative cubic nonlinearity which does not satisfy (1.5) and proved the existence of usual or modified scattering states as well as the global existence of solutions for small initial data. However, it still remains open what kind of cubic nonlinearities assures the global existence of solutions with a free profile in large time for small initial data. In the present paper, we are interested in finding out some other nonlinearity F than those considered before such that (1.1)–(1.2) has a unique global solution which is asymptotically free. We prove the global existence of a solution to the Cauchy problem (1.1)–(1.2) in the usual Sobolev spaces for small initial data and the existence of scattering states in a usual sense if $F = cuu_x^2$ or $F = c\bar{u}\bar{u}_x^2$ with $c \in \mathbf{C}$. To treat these critical cubic nonlinearities we use the techniques which transform them into harmless ones. These were developed by Shatah [13], Cohn [1], [2] and Ozawa [11] for quadratic nonlinearity. While they discussed quadratic nonlinear Schrödinger equations in [1], [2] and [11] (quadratic nonlinear Klein-Gordon equations in [13]), a class of cubic nonlinear Schrödinger equations will be treated in the present paper. So, it should be emphasized that the transformation in the present paper will be more complicated than those for quadratic nonlinearities.

Before stating our results we give several notations.

Notation Let $[a]$ denote the largest integer less than or equal to a. Let $\hat{f}$ and $\mathcal{F}f$ denote the Fourier transform of f with respect to the space variable:

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

For $1 \le p \le \infty$ and nonnegative integers m, we put

$$L^p = L^p(\mathbf{R}) := \left\{ f \in \mathcal{S}'(\mathbf{R}) \mid \|f\|_{L^p} < \infty \right\},$$

$$W^{m,p} = W^{m,p}(\mathbf{R}) := \left\{ f \in \mathcal{S}'(\mathbf{R}) \;\middle|\; \|f\|_{W^{m,p}} := \sum_{k=0}^{m} \|\partial_x^k u\|_{L^p} < \infty \right\},$$

where

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbf{R}} |f(x)|^p dx \right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \operatorname*{ess.sup}_{x \in \mathbf{R}} |f(x)|, & p = \infty \end{cases}$$

We also use the notation $H^m := W^{m,2}$ for L^2-type Sobolev spaces. Let $C^k(I; B)$ denote the space of functions continuous with their derivatives up to k from a time interval $I \subset \mathbf{R}$ to a Banach space B, and let $C(I; B) := C^0(I; B)$. Let $U(t) = e^{\frac{it}{2}\partial_x^2}$ be the evolution operator associated with the free Schrödinger equation.

Our main results are the following. The first theorem gives a cubic nonlinear Schrödinger equation which is convertible into the free Schrödinger equation.

Theorem 1 *Let m be an integer with $m \geq 1$ and let $F = cuu_x^2$ where c is a complex constant. We put*

$$\varphi(u) = \varphi(u; c) = \sum_{k=0}^{\infty} \frac{(-c)^k}{(2k+1)k!} u^{2k+1} \quad \left(= \int_0^u e^{-cz^2} dz \right).$$

Then there exists $\varepsilon_0 > 0$ such that for any $u_0 \in H^m$ with $\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon_0$ the Cauchy problem (1.1)–(1.2) has a unique global solution $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$. Moreover, the solution u is given explicitly by $u(t) = \varphi^{-1}\big(U(t)\varphi(u_0)\big)$. If in addition $u_0 \in L^1$, then

$$\|u(t)\|_{L^\infty} = O(|t|^{-\frac{1}{2}}) \quad as \quad t \to \pm\infty \tag{1.6}$$

and there exists a unique $\phi \in H^m \cap L^1$ such that

$$\|u(t) - U(t)\phi\|_{H^m} = O(|t|^{-1}) \quad as \quad t \to \pm\infty. \tag{1.7}$$

Furthermore, ϕ is given explicitly by $\phi = \varphi(u_0)$.

Remark (i) The assumption $\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon_0$ is fulfilled if $\|u_0\|_{H^1}$ is sufficiently small, since

$$\begin{aligned}
\|\mathcal{F}\varphi(u_0)\|_{L^1} &= \left\| \sum_{k=0}^{\infty} \frac{(-c)^k}{(2k+1)k!} \mathcal{F}(u_0^{2k+1}) \right\|_{L^1} \\
&= \left\| \sum_{k=0}^{\infty} \frac{(-c)^k}{(2k+1)k!} \left(\frac{1}{\sqrt{2\pi}} \right)^{2k} \overbrace{\widehat{u_0} * \cdots * \widehat{u_0}}^{2k+1 \text{ times}} \right\|_{L^1} \\
&\leq \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{|c|^k}{(2k+1)k!} \left(\frac{1}{\sqrt{2\pi}} \|\widehat{u_0}\|_{L^1} \right)^{2k+1}
\end{aligned}$$

and $\|\widehat{u_0}\|_{L^1} \leq \|(1+\xi^2)^{1/2}\widehat{u_0}\|_{L^2} \|(1+\xi^2)^{-1/2}\|_{L^2} = \sqrt{\pi}\|u_0\|_{H^1}$.

(ii) For ε_0 in Theorem 1, we can take the radius of convergence of the Taylor expansion at the origin of the inverse function of φ.

(iii) It is φ given in Theorem 1 which converts a solution of the original cubic nonlinear Schrödinger equation into a solution of the free Schrödinger equation (see Lemma 2.3 in Section 2).

By modifying the proof of Theorem 1 slightly, we can also prove the following theorem for the nonlinear Schrödinger equation with more general nonlinearity.

Theorem 1′ *Let m be an integer with $m \geq 1$ and let $F = f(u)u_x^2$ where $f(u)$ is a holomorphic function on a neighborhood of the origin. We put*

$$\varphi(u) = \int_0^u e^{-\int_0^z 2f(w)dw} dz.$$

Then there exist $\varepsilon_0, \varepsilon_1 > 0$ such that for any $u_0 \in H^m$ with $\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon_0$ and $\|u_0\|_{L^\infty} < \varepsilon_1$, the Cauchy problem (1.1)–(1.2) has a unique global solution $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$. Moreover, the solution u is given explicitly by $u(t) = \varphi^{-1}\big(U(t)\varphi(u_0)\big)$. If in addition $u_0 \in L^1$, then

$$\|u(t)\|_{L^\infty} = O(|t|^{-\frac{1}{2}}) \quad as \quad t \to \pm\infty$$

and there exists a unique $\phi \in H^m \cap L^1$ such that

$$\|u(t) - U(t)\phi\|_{H^m} = O(|t|^{-\frac{1}{2}}) \quad as \quad t \to \pm\infty.$$

Furthermore, ϕ is given explicitly by $\phi = \varphi(u_0)$.

We next state the theorem concerning a cubic nonlinear Schrödinger equation to which the normal form argument by Shatah [13] is applicable.

Theorem 2 *Let m be an integer with $m \geq 4$ and let $F = c\bar{u}\bar{u}_x^2$ where c is a complex constant. Then there exists $\varepsilon_0 > 0$ such that for any $u_0 \in H^m \cap W^{[(m+5)/2],1}$ with $\max\{\|u_0\|_{H^m}, \|u_0\|_{W^{[(m+5)/2],1}}\} < \varepsilon_0$ the Cauchy problem (1.1)–(1.2) has a unique global solution u satisfying*

$$u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2}),$$

and

$$\|u(t)\|_{H^m} = O(1), \quad \|u(t)\|_{W^{[(m+1)/2],\infty}} = O(|t|^{-\frac{1}{2}}) \quad as \quad t \to \pm\infty. \tag{1.8}$$

Moreover, there exist a unique $\phi_+ \in H^m$ and a unique $\phi_- \in H^m$ such that

$$\|u(t) - U(t)\phi_+\|_{H^m} = O(|t|^{-1}) \quad as \quad t \to +\infty,$$
$$\|u(t) - U(t)\phi_-\|_{H^m} = O(|t|^{-1}) \quad as \quad t \to -\infty. \tag{1.9}$$

We state here an outline of proofs of Theorems 1 and 2 and how to organize this paper.

In Section 2, we give a proof of Theorem 1. Since the nonlinearity $F = cuu_x^2$ is cubic, we cannot directly derive sufficient a priori estimates to prove a global existence result. The reason why we can prove the global existence of solutions (and in addition, existence of the scattering operator) neverthless is that a favorable transformation exists. In the first half of Section 2, we prove that φ given in Theorem 1 converts a solution of the original nonlinear Schrödinger equation into a solution of the free Schrödinger equation. In addition, we show that the transformation φ is invertible under a certain smallness condition. In the second half, we prove that the function obtained from a solution of the free Schrödinger equation via the inverse transformation φ^{-1} solves the original nonlinear Schrödinger equation and behaves asymptotically like a free solution. We remark that this technique was used in [11] to prove the global existence and the asymptotic behavior of solutions to the nonlinear Schrödinger equation with $F = cu_x^2$.

In Section 3, we give a proof of Theorem 2. The crucial part of proof is to establish a priori estimates of the solution to (1.1)–(1.2). The global existence result is obtained by combining a local existence theory and a priori estimates. Since the nonlinearity $F = c\bar{u}\bar{u}_x^2$ satisfies (1.4), the local existence is an immediate consequence of the usual energy method. But we cannot derive sufficient time decay estimates to prove the global existence directly from the original equation since F is cubic. In order to obtain good a priori estimates, we use the argument of normal forms introduced by Shatah (see [1], [2], [13]). In the first half of Section 3, we prove the existence of a transformation to convert the cubic nonlinearity into the one of higher degree and also prove some lemmas saying that the obtained transformation is regular in the space where we consider the Cauchy problem. The results on Fourier multipliers due to Coifman and Meyer ([3]) play important roles when we verify the regularity of the transformation. After that, we establish a priori estimates via the transformed equation implying global existence of small solutions. This completes the proof of Theorem 3.

2. Proof of Theorem 1

In this section, we prove Theorem 1 concerning the global existence of a solution to the Cauchy problem (1.1)–(1.2) with $F = cuu_x^2$.

2.1. Transformation of the unknown and its regularity

To prove Theorem 1, we will make use of the following complex function

$$\varphi(u) = \varphi(u; c) := \sum_{k=0}^{\infty} \frac{(-c)^k}{(2k+1)k!} u^{2k+1} \quad \left(= \int_0^u e^{-cz^2} dz \right).$$

For $r > 0$, we put $B_r = \{ z \in \mathbf{C} \mid |z| < r \}$.

We begin with the following lemma.

Lemma 2.1 (a) *φ is an entire function on the whole complex plane.*

(b) *There exist a constant $\varepsilon > 0$ and a holomorphic function $\psi : B_\varepsilon \to \varphi^{-1}(B_\varepsilon)$ such that $\varphi \circ \psi = \mathrm{id}_{B_\varepsilon}$, $\psi \circ \varphi = \mathrm{id}_{\varphi^{-1}(B_\varepsilon)}$ and $\varphi^{-1}(B_\varepsilon)$ is bounded.*

Proof. (a) The fact that

$$\lim_{k \to \infty} \sqrt[k]{\frac{|c|^k}{(2k+1)k!}} = 0$$

leads to (a).

(b) We put $D_1 = \{ z \in \mathbf{C} \mid \Re \varphi'(z) > 0 \}$. Then D_1 is open and includes 0 since $\Re \varphi'(0) = 1 > 0$. We can take a bounded convex open domain D_2 such that $0 \in D_2 \subset D_1$. In fact, for a sufficiently small $r > 0$, we have $B_r \subset D_1$ since D_1 is open and $0 \in D_1$. The fact that $0 \in D_2$ and $\varphi(0) = 0$ implies $0 \in \varphi(D_2)$. Since $\Re \varphi' > 0$ on a convex domain D_2, φ is a one-to-one function on D_2. Therefore $\varphi : D_2 \to \varphi(D_2)$ is a bijection. Since $0 \in \varphi(D_2)$ and $\varphi(D_2)$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon \subset \varphi(D_2)$. $\varphi^{-1}(B_\varepsilon) \subset D_2$ implies that $\varphi : \varphi^{-1}(B_\varepsilon) \to B_\varepsilon$ is a bijection, from which the existence of the inverse function $\psi : B_\varepsilon \to \varphi^{-1}(B_\varepsilon)$ of φ follows. Since φ is holomorphic on $\varphi^{-1}(B_\varepsilon)$, ψ is holomorphic on B_ε. □

The following lemma implies the regularity of φ as a transformation on H^m.

Lemma 2.2 *Let m be an integer with $m \geq 1$. For any $u_0 \in H^m$, $\varphi(u_0) \in H^m$.*

Proof. Recall that $H^m \subset L^\infty$ since $m \geq 1$.

We consider the series $\sum_{j=0}^{\infty} \frac{(-c)^j}{(2j+1)j!} u_0^{2j+1}$. The series is estimated in the L^2 norm by

$$\sum_{j=0}^{\infty} \frac{|c|^j}{(2j+1)j!} \|u_0^{2j+1}\|_{L^2} \leq \sum_{j=0}^{\infty} \frac{|c|^j}{(2j+1)j!} \|u_0\|_{L^\infty}^{2j} \|u_0\|_{L^2}$$

$$\leq \sum_{j=0}^{\infty} \frac{|c|^j}{(2j+1)j!}(C\|u_0\|_{H^1})^{2j+1}$$

so that the series converges absolutely in L^2 and $\varphi(u_0) \in L^2$. By the chain rule with respect to the distributional derivatives,

$$\partial_x \varphi(u_0) = \varphi'(u_0)\partial_x u_0 = e^{-cu_0^2}\partial_x u_0 \in L^2,$$

since $\|e^{-cu_0^2}\|_{L^\infty} \leq e^{|c|\|u_0\|_{L^\infty}^2} \leq e^{|c|(C\|u_0\|_{H^1})^2}$ and $\partial_x u_0 \in L^2$. Similarly, for any k with $2 \leq k \leq m$, we have $\partial_x^k \varphi(u_0) \in L^2$ from the identity

$$\partial_x^k \varphi(u_0) = e^{-cu_0^2}\left(\partial_x^k u_0 + \sum_{l=2}^{k} \sum_{\substack{k_1+\cdots+k_{2l-1}=k \\ \max\{k_j\}\leq k-1}} C(k,l,\{k_j\}) \prod_{j=1}^{2l-1} \partial_x^{k_j} u_0\right).$$

Here $e^{-cu_0^2} \in L^\infty$, $\partial_x^k u_0 \in L^2$ and every term in the summation is in L^2, since the Hölder inequality and the Gagliardo-Nirenberg inequality give

$$\left\|\prod_{j=1}^{2l-1} \partial_x^{k_j} u_0\right\|_{L^2} \leq \prod_{j=1}^{2l-1} \|\partial_x^{k_j} u_0\|_{L^{2k/k_j}}$$
$$\leq \prod_{j=1}^{2l-1} C\|\partial_x^m u_0\|_{L^2}^{\frac{1-2k}{1-2m}\frac{k_j}{k}} \|u_0\|_{L^\infty}^{1-\frac{1-2k}{1-2m}\frac{k_j}{k}} \leq C\|u_0\|_{H^m}^{2l-1}.$$

This comletes the proof. □

The following lemma shows that φ transforms a solution of the Cauchy problem (1.1)-(1.2) with $F = cuu_x^2$ into a solution of the linear Schrödinger equation.

Lemma 2.3 *Let m be an integer with $m \geq 1$. Let $u_0 \in H^m$ and let $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$ satisfy (1.1)-(1.2) with $F = cuu_x^2$. Then $\varphi(u(\cdot)) \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$ and*

$$\varphi(u(t)) = U(t)\varphi(u_0). \tag{2.1}$$

Proof. Let $v = \varphi(u)$ and let $v_k = \sum_{j=0}^{k} \frac{(-c)^j}{(2j+1)j!} u^{2j+1}$. We have

$$\|v_k(t) - v_l(t)\|_{L^2} \leq \sum_{j=l+1}^{k} \frac{|c|^j}{(2j+1)j!} \|u(t)^{2j+1}\|_{L^2}$$

$$\leq \sum_{j=l+1}^{k} \frac{|c|^j}{(2j+1)j!}(C\|u(t)\|_{H^1})^{2j+1} \to 0 \qquad \text{as} \quad k, l \to \infty,$$

where the convergence is uniform on every compact set of $\mathbf{R}$, since $\|u(\cdot)\|_{H^1}$ is locally bounded on $\mathbf{R}$. This implies $v \in C(\mathbf{R}; L^2)$.

From the chain rule, we have for any $t \in \mathbf{R}$, $\partial_x v = e^{-cu^2}\partial_x u \in L^2$, and therefore $\partial_x v \in C(\mathbf{R}; L^2)$ since $u \in C(\mathbf{R}; H^1 \cap L^\infty)$. Thus $\partial_x^2 v \in C(\mathbf{R}; H^{-1})$. Another application of the chain rule gives

$$\partial_x^2 v = e^{-cu^2}(u_{xx} - 2cuu_x^2), \tag{2.2}$$

where $u_{xx}, uu_x^2 \in C(\mathbf{R}; H^{-1})$ since $\|uu_x^2\|_{H^{-1}} \leq C\|u\|_{L^\infty}\|u_x\|_{L^2}^2$. Here we remark that the last inequality follows from

$$\|fg\|_{H^{-1}} \leq C\|f\|_{L^2}\|g\|_{L^2}$$

which is proved by the (H^1, H^{-1}) duality and the embedding $H^1 \subset L^\infty$.

We next prove that $v \in C^1(\mathbf{R}; H^{-1})$ and

$$\partial_t v = \varphi'(u)\partial_t u = e^{-cu^2}u_t. \tag{2.3}$$

Some calculations give

$$\begin{aligned}
&h^{-1}(v(t+h) - v(t)) - \varphi'(u(t))u_t(t) \\
&\quad = \int_0^1 \varphi'\big(\lambda u(t+h) + (1-\lambda)u(t)\big)d\lambda \\
&\qquad\qquad \cdot h^{-1}(u(t+h) - u(t)) - \varphi'(u(t))u_t(t) \\
&\quad = \left\{\int_0^1 \Big(\varphi'\big(\lambda u(t+h) + (1-\lambda)u(t)\big) - 1\Big)d\lambda + 1\right\} \\
&\qquad\qquad\qquad \Big\{h^{-1}\big(u(t+h) - u(t)\big) - u_t(t)\Big\} \\
&\qquad + \int_0^1\int_0^1 \varphi''\big(\lambda\mu u(t+h) + (1-\lambda\mu)u(t)\big)\lambda\, d\mu\, d\lambda \\
&\qquad\qquad \cdot \big(u(t+h) - u(t)\big)u_t(t)
\end{aligned} \tag{2.4}$$

so that we have

$$\begin{aligned}
&\|h^{-1}(v(t+h) - v(t)) - \varphi'(u(t))u_t(t)\|_{H^{-1}} \\
&\quad \leq \Big(C \sup_{\lambda \in [0,1]} \|\varphi'(\lambda u(t+h) + (1-\lambda)u(t))\|_{H^1} + 1\Big)
\end{aligned}$$

$$\begin{aligned} &\|h^{-1}(u(t+h)-u(t))-u_t(t)\|_{H^{-1}} \\ +C\sup_{\lambda\in[0,1]}&\|\varphi''(\lambda u(t+h)+(1-\lambda)u(t))\|_{H^1}\|u(t+h)-u(t)\|_{H^1} \\ &\|u_t(t)\|_{H^{-1}}\to 0 \quad \text{as } h\to 0, \end{aligned}$$

where we have used $\|fgh\|_{H^{-1}} \le C\|f\|_{H^1}\|g\|_{H^1}\|h\|_{H^{-1}}$ to estimate the second term in the right hand side of the last equality of (2.4). We have thus proved that (2.3) holds in H^{-1}. The continuity in t of the right hand side of (2.3) follows similarly from the inequality $\|fg\|_{H^{-1}} \le C\|f\|_{H^1}\|g\|_{H^{-1}}$.

We have proved $v \in C(\mathbf{R}; H^1) \cap C^1(\mathbf{R}; H^{-1})$ and moreover, $i\partial_t v + \frac{1}{2}\partial_x^2 v = 0$ by (1.1) with $F = cuu_x^2$, (2.2) and (2.3). Accordingly we have $v(t) = U(t)v(0)$, which is exactly (2.1). Lemma 2.2 and (2.1) imply $\varphi(u(\cdot)) \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$. □

2.2. Proof of Theorem 1

We denote by ε the constant obtained in Lemma 2.1 (b). Let $\varepsilon_0 = \sqrt{2\pi}\varepsilon$.

First, we prove that under the assumption $\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon_0$, the function

$$u(t) = \varphi^{-1}(U(t)\varphi(u_0)) \tag{2.5}$$

provides a solution of the Cauchy problem (1.1)–(1.2) with $F = cuu_x^2$ and is in $C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$.

From Lemma 2.1, we have the expansion

$$\varphi^{-1}(z) = \sum_{j=0}^{\infty} a_j z^j$$

with the radius of convergence larger than or equal to ε. We easily see that $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = c/3$ since $\varphi(0) = 0$, $\varphi'(0) = 1$, $\varphi''(0) = 0$, $\varphi'''(0) = -2c$.

By the assumption $\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon_0 = \sqrt{2\pi}\varepsilon$, we have

$$\begin{aligned} \sup_{t\in\mathbf{R}}\|U(t)\varphi(u_0)\|_{L^\infty} &= \sup_{t\in\mathbf{R}}\|\mathcal{F}^{-1}e^{-it|\cdot|^2}\mathcal{F}\varphi(u_0)\|_{L^\infty} \\ &\le \frac{1}{\sqrt{2\pi}}\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon. \end{aligned} \tag{2.6}$$

We consider the series

$$U(t)\varphi(u_0)+\sum_{j=3}^{\infty}a_j(U(t)\varphi(u_0))^j. \tag{2.7}$$

By (2.6) and the unitarity of $U(t)$ in L^2, the series (2.7) converges absolutely in L^2 and its L^2 norm is estimated by

$$\|\varphi(u_0)\|_{L^2}\left(1+\sum_{j=3}^{\infty}|a_j|\left(\frac{1}{\sqrt{2\pi}}\|\mathcal{F}\varphi(u_0)\|_{L^1}\right)^{j-1}\right).$$

This proves that u defined in (2.5) makes sense and is in $C(\mathbf{R};L^2)$. By the chain rule, we have for any $t\in\mathbf{R}$

$$\partial_x u(t)=\frac{d\varphi^{-1}}{dz}\big(U(t)\varphi(u_0)\big)U(t)\partial_x\varphi(u_0) \tag{2.8}$$

so that $\partial_x u(\cdot)\in C(\mathbf{R};L^2)$ since $U(t)\partial_x\varphi(u_0)\in L^2$ by Lemma 2.2 and

$$\begin{aligned}\frac{d\varphi^{-1}}{dz}\big(U(t)\varphi(u_0)\big)&=\left[\varphi'(\varphi^{-1}(U(t)\varphi(u_0)))\right]^{-1}\\&=e^{c\{\varphi^{-1}(U(t)\varphi(u_0))\}^2}\in C(\mathbf{R};L^\infty),\end{aligned}$$

which follows from (2.6), the boundedness of $\varphi^{-1}(U(t)\varphi(u_0))$ and $U(t)\varphi(u_0)\in C(\mathbf{R};H^1\cap L^\infty)$. Similarly, for any k with $2\le k\le m$, we have

$$\partial_x^k u=\frac{\partial_x^k w}{\varphi'(\varphi^{-1}(w))}+\sum_{l=2}^{k}\sum_{\substack{m_1+\cdots+m_l=k\\ \min\{m_j\}\ge1}}\sum_{\substack{n_1+\cdots+n_{l-1}=2l-2\\ \min\{n_j\}\ge1}}C(k,l,\{m_j\},\{n_j\})\frac{\prod_{j=1}^{l-1}\varphi^{(n_j)}(\varphi^{-1}(w))}{\varphi'(\varphi^{-1}(w))^{2l-1}}\prod_{j=1}^{l}\partial_x^{m_j}w \tag{2.9}$$

and $\partial_x^k u\in C(\mathbf{R};L^2)$ since $\partial_x^k w$, $\prod_{j=1}^{l}\partial_x^{m_j}w\in C(\mathbf{R};L^2)$ and

$$\frac{\prod_{j=1}^{l-1}\varphi^{(n_j)}(\varphi^{-1}(w))}{\varphi'(\varphi^{-1}(w))^{2l-1}}\in C(\mathbf{R};L^\infty)$$

by (2.6), where $w(t)=U(t)\varphi(u_0)$. In a way similar to the proof of (2.3), we obtain

$$\partial_t u=\frac{\partial_t w}{\varphi'(\varphi^{-1}(w))}=\frac{\frac{i}{2}\partial_x^2 w}{\varphi'(\varphi^{-1}(w))} \tag{2.10}$$

and $\partial_t u \in C(\mathbf{R}; H^{-1})$. As in the derivation of (2.9) from (2.8), the formula (2.10) yields $\partial_t u \in C(\mathbf{R}; H^{m-2})$. This proves $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$, which implies $uu_x^2 \in C(\mathbf{R}; H^{m-2})$. A simple calculation shows

$$\partial_x^2 u = \frac{\partial_x^2 w}{\varphi'(\varphi^{-1}(w))} - \frac{\varphi''(\varphi^{-1}(w))}{\varphi'(\varphi^{-1}(w))^3}(\partial_x w)^2 = -2i\partial_t u + 2cuu_x^2,$$

which is exactly (1.1) with $F = cuu_x^2$. This completes the proof of the existence of a global solution to (1.1)–(1.2) with $F = cuu_x^2$.

We next prove the uniqueness of the solution. Assume that u and v are solutions to (1.1)–(1.2) with $F = cuu_x^2$. Then we have by Lemma 2.3

$$\varphi(u(t)) = \varphi(v(t)) = U(t)\varphi(u_0). \tag{2.11}$$

By assumption on the initial data, it follows from (2.6) that $\|\varphi(u(t))\|_{L^\infty} = \|\varphi(v(t))\|_{L^\infty} = \|U(t)\varphi(u_0)\|_{L^\infty} < \varepsilon$, which, combined with Lemma 2.1 (b) and (2.11), leads to $u(t) = v(t)$.

We proceed to the decay estimate (1.6) of the solution. Note that for $m \geq 1$, $\phi := \varphi(u_0) \in H^m \cap L^1$ if $u_0 \in H^m \cap L^1$. Estimating (2.7) in the L^∞ norm and using the standard L^∞-decay estimate of the fundamental solution, we obtain

$$\begin{aligned}\|u(t)\|_{L^\infty} &\leq \|U(t)\phi\|_{L^\infty} + \sum_{j=3}^{\infty} |a_j| \|(U(t)\phi)^j\|_{L^\infty} \\ &\leq \frac{\|\phi\|_{L^1}}{(2\pi|t|)^{1/2}}\left(1 + \sum_{j=3}^{\infty} |a_j| \left(\frac{\|\mathcal{F}\phi\|_{L^1}}{\sqrt{2\pi}}\right)^{j-1}\right).\end{aligned}$$

Here the last summation converges since $\|\mathcal{F}\phi\|_{L^1}/\sqrt{2\pi} < \varepsilon$.

We finally prove (1.7). In the same way as above, we obtain

$$\begin{aligned}\|u(t) - U(t)\phi\|_{L^2} &\leq \sum_{j=3}^{\infty} |a_j| \|(U(t)\phi)^j\|_{L^2} \\ &\leq \frac{\|\phi\|_{L^1}^2 \|\phi\|_{L^2}}{2\pi|t|} \sum_{j=3}^{\infty} |a_j| \left(\frac{\|\mathcal{F}\phi\|_{L^1}}{\sqrt{2\pi}}\right)^{j-3}.\end{aligned}$$

For any k with $1 \leq k \leq m$, a somewhat complicated calculation gives

$$\|\partial_x^k(u(t) - U(t)\phi)\|_{L^2} \leq \frac{C(\|\mathcal{F}\phi\|_{L^1})}{|t|} \|\phi\|_{L^1}^2 \|\phi\|_{H^k},$$

which proves (1.7), that is, the existence of a free profile in H^m. The uniqueness of the free profile follows from (1.7) and the unitarity of $U(t)$ on H^m.

3. Proof of Theorem 2

In this section, we prove Theorem 2 concerning the global existence of a solution to the Cauchy problem (1.1)–(1.2) with $F = c\bar{u}\bar{u}_x^2$.

3.1. Transformation of the unknown and its regularity

In the first half of this subsection, we obatin the transformation which converts the cubic nonlinearity into the one of higher degree. Following Shatah [13], we introduce a new unknown function v:

$$v = u + K(\bar{u}, \bar{u}, \bar{u}), \tag{3.1}$$

where K is thought of as a distribution and the representaion of the cubic term is given by

$$K(f, g, h)(x) = \int_{\mathbf{R}^3} K(x-y, x-z, x-w) f(y) g(z) h(w) dy\, dz\, dw. \tag{3.2}$$

After some calculations, we obtain

$$\begin{aligned} &K(f, g, h)(x) \\ &\quad = (2\pi)^{3/2} \int_{\mathbf{R}^3} \widehat{K}(p, q, r) \widehat{f}(p) \widehat{g}(q) \widehat{h}(r) e^{ix(p+q+r)} dp\, dq\, dr, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &i\partial_t v + \frac{1}{2}\partial_x^2 v \\ &\quad = c\bar{u}\bar{u}_x^2 + \Big[\big(\partial_y^2 + \partial_z^2 + \partial_w^2 + \partial_y\partial_z + \partial_z\partial_w + \partial_w\partial_y\big) K\Big](\bar{u}, \bar{u}, \bar{u}) \\ &\qquad - K(\bar{c}uu_x^2, \bar{u}, \bar{u}) - K(\bar{u}, \bar{c}uu_x^2, \bar{u}) - K(\bar{u}, \bar{u}, \bar{c}uu_x^2). \end{aligned} \tag{3.4}$$

All cubic terms in (3.4) cancel out, when we take K as follows:

$$\widehat{K}(p, q, r) = -\frac{c}{3} \frac{pq + qr + rp}{p^2 + q^2 + r^2 + pq + qr + rp}.$$

Then the function v defined by the transformation (3.1) satisfies

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \frac{|c|^2}{3}\Big(\Omega(uu_x^2, \bar{u}, \bar{u}) + \Omega(\bar{u}, uu_x^2, \bar{u}) + \Omega(\bar{u}, \bar{u}, uu_x^2)\Big), \tag{3.5}$$

where we put $\Omega = -\frac{3}{c}K$. We remark that the nonlinear term in the right hand side of (3.5) is of degree five, which is of higher degree than the original nonlinearity $F = c\bar{u}\bar{u}_x^2$.

The following lemma due to Coifman and Meyer is needed when we prove the regularity of Ω.

Lemma 3.1 *Let*

$$\Lambda(f, g, h)(x) = \int_{\mathbf{R}^3} \lambda(p, q, r)\widehat{f}(p)\widehat{g}(q)\widehat{h}(r)e^{ix(p+q+r)}dp\,dq\,dr,$$

and let

$$|\partial_p^j \partial_q^k \partial_r^l \lambda(p, q, r)| \leq C_{j,k,l}(|p| + |q| + |r|)^{-(j+k+l)} \tag{3.6}$$

for all nonnegative integers j, k, l such that $0 \leq j + k + l \leq 1$. Then

$$\|\Lambda(f, g, h)\|_{L^p} \leq C_{p_1,p_2,p_3}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}}$$

where

$$\frac{1}{p} = \sum_{j=1}^{3}\frac{1}{p_j}, \quad 1 < p_j \leq \infty \quad (j = 1, 2) \quad \textit{and} \quad 1 < p_3 < \infty.$$

For the proof, see [3].

The following estimate for $K(\cdot,\cdot,\cdot)$ defined by (3.2) follows immediately from (3.3) and Lemma 3.1. We will make use of this estimate to prove the a priori energy estimate later.

Lemma 3.2 *Let p, p_j $(j = 1, 2, 3)$ satisfy $\frac{1}{p} = \sum_{j=1}^{3}\frac{1}{p_j}$, $1 < p_j \leq \infty$ $(j = 1, 2)$ and $1 < p_3 < \infty$. If $\widehat{K}$ is a Coifman-Meyer kernel (that is, $\lambda = \widehat{K}$ satisfies (3.6)), then*

$$\|K(f, g, h)\|_{L^p} \leq C_{p_1,p_2,p_3}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}}.$$

The following lemma gives several formulas which are useful to simplify the representation of nonlinear terms.

Lemma 3.3 (a) $\Omega(f,g,h)=\Omega(f,h,g)=\Omega(g,f,h)$.

(b) $\partial_x\Omega(f,g,g)=M(f,g,g_x)$, *where* $\widehat{M}$ *is a Coifman-Meyer kernel.*

Proof. (a) $\widehat{\Omega}(p,q,r)=\widehat{\Omega}(p,r,q)=\widehat{\Omega}(q,p,r)$ and so the representation (3.3) imply (a).

(b) Let $\widehat{M}=(2p^2+3pq+2qr+2rp)/D$, where $D=p^2+q^2+r^2+pq+qr+rp$.

We have by (a)

$$\begin{aligned}\partial_x\Omega(f,g,g)&=\Omega(f_x,g,g)+\Omega(f,g_x,g)+\Omega(f,g,g_x)\\&=\Omega(f_x,g,g)+2\Omega(f,g,g_x).\end{aligned}\tag{3.7}$$

The representation (3.3) and properties of the Fourier transform imply

$$\begin{aligned}&\Omega(f_x,g,g)\\&=(2\pi)^{3/2}\int_{R^3}\widehat{\Omega}(p,q,r)ip\widehat{f}(p)\widehat{g}(q)\widehat{g}(r)e^{ix(p+q+r)}dp\,dq\,dr\\&=(2\pi)^{3/2}\int_{R^3}\frac{1}{D}\left\{\left(p^2+\frac{1}{2}pq\right)ir+\left(p^2+\frac{1}{2}pr\right)iq\right\}\\&\qquad\qquad\widehat{f}(p)\widehat{g}(q)\widehat{g}(r)e^{ix(p+q+r)}dp\,dq\,dr\\&=(2\pi)^{3/2}\int_{R^3}\frac{1}{D}(2p^2+pq)\widehat{f}(p)\widehat{g}(q)ir\widehat{g}(r)e^{ix(p+q+r)}dp\,dq\,dr.\end{aligned}\tag{3.8}$$

Therefore we have by (3.7) and (3.8)

$$\begin{aligned}\partial_x\Omega(f,g,g)&=(2\pi)^{3/2}\int_{R^3}\left\{\frac{1}{D}(2p^2+pq)+2\widehat{\Omega}(p,q,r)\right\}\\&\qquad\qquad\widehat{f}(p)\widehat{g}(q)ir\widehat{g}(r)e^{ix(p+q+r)}dp\,dq\,dr\\&=M(f,g,g_x).\end{aligned}$$

A simple calculation shows that $\widehat{M}$ is a Coifman-Meyer kernel. □

In the following lemma, we collect the inequalities which we will use to estimate nonlinear terms when we derive a priori estimates.

Lemma 3.4 *The estimates* (a) *and* (b) *hold for* $m\geq 1$, *and the others hold for* $m\geq 0$.

(a) $\|\Omega(f,g,g)\|_{H^m}$
$$\leq C\Big(\|f\|_{H^{m-1}}\|g\|^2_{W^{[(m+1)/2],\infty}}+\|f\|_{W^{[(m-1)/2],\infty}}\|g\|_{H^m}\|g\|_{W^{[(m+1)/2],\infty}}\Big),$$

(b) $\|\Omega(f,g,g)\|_{W^{m,1}}$
$\leq C\Big(\|f\|_{H^{m-1}}\|g\|_{H^m}\|g\|_{W^{[(m-1)/3]+1,\infty}} + \|f\|_{W^{[(m-1)/3],\infty}}\|g\|^2_{H^m}\Big),$

(c) $\|\Omega(f,f,f)\|_{H^m} \leq C\|f\|_{H^m}\|f\|^2_{W^{[m/2],\infty}},$

(d) $\|fg\|_{H^m} \leq C\Big(\|f\|_{H^m}\|g\|_{W^{[m/2],\infty}} + \|f\|_{W^{[m/2],\infty}}\|g\|_{H^m}\Big),$

(e) $\|fg\|_{W^{m,1}} \leq C\|f\|_{H^m}\|g\|_{H^m},$

(f) $\|fg\|_{W^{m,\infty}} \leq C\|f\|_{W^{m,\infty}}\|g\|_{W^{m,\infty}}.$

Proof. We first prove (a). Since $\widehat{\Omega}$ is a Coifman-Meyer kernel, we have by Lemma 3.2

$$\|\Omega(f,g,g)\|_{L^2} \leq C\|f\|_{L^2}\|g\|^2_{L^\infty}. \tag{3.9}$$

Let $1 \leq k \leq m$. From Lemma 3.3 (b) and the chain rule, it follows that

$$\begin{aligned}\partial_x^k\Omega(f,g,g) &= \partial_x^{k-1}M(f,g,g_x)\\ &= \sum_{k_1+k_2+k_3=k-1}\frac{(k-1)!}{k_1!k_2!k_3!}M(\partial_x^{k_1}f,\partial_x^{k_2}g,\partial_x^{k_3+1}g).\end{aligned}$$

We estimate this in the L^2 norm to obtain

$$\begin{aligned}&\|\partial_x^k\Omega(f,g,g)\|_{L^2}\\ &\leq C\sum_{k_1+k_2+k_3=k-1}\|M(\partial_x^{k_1}f,\partial_x^{k_2}g,\partial_x^{k_3+1}g)\|_{L^2}\\ &\leq C\Big(\|f\|_{H^{k-1}}\|g\|^2_{W^{[(k+1)/2],\infty}} + \|f\|_{W^{[(k-1)/2],\infty}}\|g\|_{H^k}\|g\|_{W^{[(k+1)/2],\infty}}\Big),\end{aligned} \tag{3.10}$$

where we have applied Lemma 3.2 to M and have used the fact that at most one of k_1, k_2 and k_3 will be greater than $(k-1)/2$. Summing (3.9) and (3.10) with $1 \leq k \leq m$, we obtain (a).

We can prove (b), (c) in the same way, so we omit their proof.

(d), (e) and (f) follow from the Hölder inequality and the same argument as in the proof of (a). □

3.2. Proof of Theorem 2

In this subsection, we describe the proof of Theorem 2. The proof consists of the local existence theorem and a priori energy and decay estimates of solutions.

For $m \geq 7$ and $T > 0$, we define

$$\|u\|_{m,T} = \sup_{t\in[0,T]} \Big(\|u(t)\|_{H^m} + (1+t)^{\frac{1}{2}} \|u(t)\|_{W^{[\frac{m+1}{2}],\infty}} \Big).$$

Step 1. Local Existence

We start with stating the following lemma concerning the local existence of a solution to (1.1)–(1.2) with $F = c\bar{u}\bar{u}_x^2$.

Lemma 3.5 *Let $m \geq 4$ and let $u_0 \in H^m$. Then there exists $T > 0$ such that the Cauchy problem (1.1)–(1.2) with $F = c\bar{u}\bar{u}_x^2$ has a unique solution in $C([0,T], H^m) \cap C^1([0,T], H^{m-2})$.*

For the proof, see, e.g., [1].

Step 2. A priori energy estimate

In this step, we prove the following lemma concerning the a priori energy estimate of solutions.

Lemma 3.6 *Let $m \geq 4$ and let $u_0 \in H^m$. Assume that the initial value problem (1.1)–(1.2) with $F = c\bar{u}\bar{u}_x^2$ has a solution $u \in C([0,T]; H^m) \cap C^1([0,T]; H^{m-2})$. Then the following inequality holds for any $t \in [0,T]$:*

$$\|u(t)\|_{H^m} \leq C\Big(\|u_0\|_{H^m} + \|u_0\|_{H^m}^3 + \|u\|_{m,T}^3 + \|u\|_{m,T}^5 \Big),$$

where C is independent of T and u_0.

Proof. We first evaluate the solution v of (3.5) before estimating u.

By Lemma 3.3 (a), (3.5) can be rewritten as follows:

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = |c|^2 \Omega(uu_x^2, \bar{u}, \bar{u}). \tag{3.11}$$

This yields the following inequality for any nonnegative integer k,

$$\frac{d}{dt}\|\partial_x^k v(t)\|_{L^2} \leq |c|^2 \|\partial_x^k \Omega(uu_x^2, \bar{u}, \bar{u})(t)\|_{L^2},$$

so that we have

$$\frac{d}{dt}\|v(t)\|_{H^m} \leq C\|\Omega(uu_x^2, \bar{u}, \bar{u})(t)\|_{H^m}. \tag{3.12}$$

This, combined with Lemma 3.4 (a) and (d), leads to

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_{H^m} &\le C\Big(\|uu_x^2\|_{H^{m-1}}\|u\|^2_{W^{[(m+1)/2],\infty}} \\ &\qquad + \|uu_x^2\|_{W^{[(m-1)/2],\infty}}\|u\|_{H^m}\|u\|_{W^{[(m+1)/2],\infty}}\Big) \\ &\le C\|u(t)\|_{H^m}\|u(t)\|^4_{W^{[(m+1)/2],\infty}}.\end{aligned}$$

Thus we have for any $t \in [0,T]$

$$\begin{aligned}\|v(t)\|_{H^m} &\le \|v(0)\|_{H^m} + C\int_0^t \|u(\tau)\|_{H^m}\|u(\tau)\|^4_{W^{[(m+1)/2],\infty}}d\tau \\ &\le \|v(0)\|_{H^m} + C\|u\|^5_{m,T}\int_0^t \frac{d\tau}{(1+\tau)^2} \\ &\le \|v(0)\|_{H^m} + C\|u\|^5_{m,T}. \qquad (3.13)\end{aligned}$$

We next estimate u. From (3.1) and Lemma 3.4 (a), we have for any $t \in [0,T]$

$$\begin{aligned}\|u(t)\|_{H^m} &\le \|v(t)\|_{H^m} + \frac{c}{3}\|\Omega(\bar{u},\bar{u},\bar{u})(t)\|_{H^m} \\ &\le \|v(t)\|_{H^m} + C\|u(t)\|_{H^m}\|u(t)\|^2_{W^{[(m+1)/2],\infty}} \\ &\le \|v(t)\|_{H^m} + C\|u\|^3_{m,T}, \qquad (3.14)\end{aligned}$$

and

$$\begin{aligned}\|v(0)\|_{H^m} &\le \|u(0)\|_{H^m} + \frac{c}{3}\|\Omega(\bar{u},\bar{u},\bar{u})(0)\|_{H^m} \\ &\le \|u_0\|_{H^m} + C\|u_0\|^3_{H^m}, \qquad (3.15)\end{aligned}$$

where we have used the Sobolev embedding $H^m \subset W^{[(m+1)/2],\infty}$ since $m \ge [(m+1)/2]+1$ by $m \ge 4$. Combining (3.13), (3.14) and (3.15), we obtain the desired inequality. □

Step 3. A priori decay estimate

In this step, we prove the following lemma concerning the a priori decay estimate of solutions.

Lemma 3.7 *Let $m \ge 4$ and let $u_0 \in H^m \cap W^{[(m+5)/2],1}$. Assume that the Cauchy problem (1.1)–(1.2) with $F = c\bar{u}\bar{u}_x^2$ has a solution $u \in C([0,T];H^m)$. Then the following inequality holds for any $t \in [0,T]$:*

$$\begin{aligned}&(1+t)^{1/2}\|u(t)\|_{W^{[(m+1)/2],\infty}} \\ &\quad \le C\Big(\|u_0\|_{W^{[(m+1)/2]+2,1}} + \|u_0\|^3_{H^m} + \|u\|^3_{m,T} + \|u\|^5_{m,T}\Big),\end{aligned}$$

where C is independent of T and u_0.

Proof. We first estimate the decay of v. From (3.11) and Duhamel's principle, we have

$$v(t) = U(t)v(0) - i|c|^2 \int_0^t U(t-\tau)\Omega(uu_x^2, \bar{u}, \bar{u})(\tau)d\tau.$$

By the standard L^∞-decay estimate of the fundamental solution,

$$\|v(t)\|_{W^{[(m+1)/2],\infty}} \le \frac{C\|v(0)\|_{W^{[(m+1)/2]+2,1}}}{(1+t)^{1/2}} + C\int_0^t \frac{\|\Omega(uu_x^2, \bar{u}, \bar{u})(\tau)\|_{W^{[(m+1)/2]+2,1}}}{(1+t-\tau)^{1/2}}d\tau. \tag{3.16}$$

By Lemma 3.4 (b), (d) and (f),

$$\begin{aligned}&\|\Omega(uu_x^2, \bar{u}, \bar{u})\|_{W^{[(m+1)/2]+2,1}}\\ &\le C\|u\|^2_{H^{[(m+1)/2]+2}}\|u\|_{W^{[([(m+1)/2]+1)/2]+1,\infty}}\\ &\le C\|u\|^2_{H^m}\|u\|^3_{W^{[(m+1)/2],\infty}},\end{aligned} \tag{3.17}$$

where at the last inequality we have used the fact that $[(m+1)/2]+2 \le m$ and $[([(m+1)/2]+1)/2]+1 \le [(m+1)/2]$ for $m \ge 4$. From (3.1), Lemma 3.4 (b) and the Sobolev embedding, it follows that

$$\|v(0)\|_{W^{[(m+1)/2]+2,1}} \le \|u_0\|_{W^{[(m+1)/2]+2,1}} + C\|u_0\|^3_{H^m}. \tag{3.18}$$

We have by (3.1), the Sobolev embedding and Lemma 3.4 (c)

$$\begin{aligned}&\|u(t)\|_{W^{[(m+1)/2],\infty}}\\ &\le \|v(t)\|_{W^{[(m+1)/2],\infty}} + C\|K(\bar{u}, \bar{u}, \bar{u})(t)\|_{H^{[(m+1)/2]+1}}\\ &\le \|v(t)\|_{W^{[(m+1)/2],\infty}} + C\|u(t)\|_{H^{[(m+1)/2]+1}}\|u(t)\|^2_{W^{[([(m+1)/2]+1)/2],\infty}}\\ &\le \|v(t)\|_{W^{[(m+1)/2],\infty}} + C\|u(t)\|_{H^m}\|u(t)\|^2_{W^{[(m+1)/2],\infty}},\end{aligned} \tag{3.19}$$

where we have used $[(m+1)/2]+1 \le m$ and $[([(m+1)/2]+1)/2]+1 \le [(m+1)/2]$ for $m \ge 4$. We combine (3.16)–(3.19) to obtain

$$\begin{aligned}&\|u(t)\|_{W^{[(m+1)/2],\infty}}\\ &\le \frac{1}{(1+t)^{1/2}}(\|u_0\|_{W^{[(m+1)/2]+2,1}} + C\|u_0\|^3_{H^m})\end{aligned}$$

$$
\begin{aligned}
&+ C\|u(t)\|_{H^m}\|u(t)\|^2_{W^{[(m+1)/2],\infty}}\\
&+ C\int_0^t \frac{1}{(1+t-\tau)^{1/2}}\|u(\tau)\|^2_{H^m}\|u(\tau)\|^3_{W^{[(m+1)/2],\infty}}d\tau\\
\le\ & \frac{C}{(1+t)^{1/2}}(\|u_0\|_{W^{[(m+1)/2]+2,1}} + \|u_0\|^3_{H^m} + \|u\|^3_{m,T} + \|u\|^5_{m,T})
\end{aligned}
$$

for any $t \in [0,T]$, where we have used

$$\int_0^t \frac{1}{(1+t-\tau)^{1/2}(1+\tau)^{3/2}}d\tau = \frac{2t}{(2+t)(1+t)^{1/2}} \le \frac{2}{(1+t)^{1/2}}$$

for $t \ge 0$. That is the desired inequality. □

Step 4. Conclusion of proof

We are now in a position to prove Theorem 2.

Lemmas 3.6 and 3.7 imply that if there exists a solution $u \in C([0,T],H^m)\cap C^1([0,T],H^{m-2})$, then

$$
\begin{aligned}
\|u\|_{m,T} \le C\big(\|u_0\|_{H^m} &+ \|u_0\|_{W^{[(m+1)/2]+2,1}}\\
&+ \|u_0\|^3_{H^m} + \|u\|^3_{m,T} + \|u\|^5_{m,T}\big). \quad (3.20)
\end{aligned}
$$

We put $\varepsilon = \max\{\|u_0\|_{H^m}, \|u_0\|_{W^{[(m+1)/2]+2,1}}\}$,

$$T^* = \sup\left\{ T>0 \;\middle|\; \begin{array}{l}\text{The Cauchy problem (1.1)–(1.2) with}\\ F = c\bar{u}\bar{u}_x^2 \text{ has a solution}\\ u \in C([0,T];H^m)\cap C^1([0,T];H^{m-2})\end{array}\right\}$$

and

$$T_* = \sup\Big\{T \in [0,T^*) \;\Big|\; \|u\|_{m,T} \le 4C\varepsilon\Big\},$$

where C is the constant appearing in (3.20). Assume that $T^* < \infty$. By (3.20), we have for $T \in [0,T_*)$

$$\|u\|_{m,T} \le C(3\varepsilon + (4C\varepsilon)^3 + (4C\varepsilon)^5) = C\varepsilon(3 + 64C^3\varepsilon^2 + 1024C^5\varepsilon^4)$$

if $\varepsilon \le 1$. If, in addition, $\varepsilon \le \sqrt{\frac{1}{128C^3+2048C^5}}$, then $\|u\|_{m,T} \le \frac{7}{2}C\varepsilon$ for any $T \in [0,T_*)$, which implies $\|u\|_{m,T_*} \le \frac{7}{2}C\varepsilon < 4C\varepsilon$ by continuity in t of u. This shows by the definition of T_* that $T_* = T^*$ and $\|u(T^*)\|_{H^m} < 4C\varepsilon$. The last inequality combined with Lemma 3.5 (local existence) asserts that for some $T' > T^*$ the solution u exists on $[0,T']$ and belongs to $C([0,T'];H^m)\cap$

$C^1([0,T'],H^{m-2})$, which contradicts the definition of T^*. We have thus proved that $T^*=\infty$ if

$$\varepsilon \le \varepsilon_0 := \min\left\{1, \sqrt{\frac{1}{128C^3+2048C^5}}\right\},$$

which completes the proof of the global existence of a solution. The argument above also proves (1.8).

We finally prove (1.9), that is, the existence of the free profile.

Now that we have (3.20), the following inequalities are essentially proved in the argument above:

$$\|\Omega(uu_x^2,\bar{u},\bar{u})(\tau)\|_{H^m} \le \frac{C}{(1+\tau)^2}, \tag{3.21}$$

$$\|\Omega(\bar{u},\bar{u},\bar{u})(\tau)\|_{H^m} \le \frac{C}{1+\tau}. \tag{3.22}$$

It follows from (3.11) that

$$U(-t')v(t') - U(-t)v(t) = -i|c|^2\int_t^{t'} U(-\tau)\Omega(uu_x^2,\bar{u},\bar{u})(\tau)d\tau,$$

which, combined with the unitarity of $U(t)$ in H^m and (3.21), implies

$$\begin{aligned} &\|U(-t')v(t') - U(-t)v(t)\|_{H^m} \\ &\le |c|^2\int_t^{t'} \|\Omega(uu_x^2,\bar{u},\bar{u})(\tau)\|_{H^m}d\tau \\ &\le C\left(\frac{1}{1+t}-\frac{1}{1+t'}\right) \quad \to 0 \text{ as } t,t'\to\infty. \end{aligned} \tag{3.23}$$

By the completeness of H^m, there exists $\lim_{t\to\infty} U(-t)v(t)\in H^m$. Putting $\phi_+ = \lim_{t\to\infty} U(-t)v(t)$ and letting $t'\to\infty$ in (3.23), we obtain

$$\|v(t) - U(t)\phi_+\|_{H^m} \le \frac{C}{1+t}, \tag{3.24}$$

where we have used the unitarity of $U(t)$ in H^m again. From (3.1) and (3.22), it follows that

$$\|u(t)-v(t)\|_{H^m} = \|\Omega(\bar{u},\bar{u},\bar{u})(t)\|_{H^m} \le \frac{C}{1+t}. \tag{3.25}$$

Thus we have by (3.24) and (3.25)

$$\begin{aligned}&\|u(t)-U(t)\phi_+\|_{H^m}\\&\leq\|u(t)-v(t)\|_{H^m}+\|v(t)-U(t)\phi_+\|_{H^m}\leq\frac{C}{1+t},\end{aligned}$$

which implies (1.9). We can prove the existence of a solution on $(-\infty, 0]$ and of $\phi_- \in H^m$ similarly as above. □

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