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A note on non-classical eigenvalue asymptotics

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Abstract. The purpose of this paper is an extension of the results of Simon [13] on the asymptotic behavior of the trace of the heat kernel for the Schrödinger operator. We discuss the case where the operator has compact resolvents in spite of the fact that the electric potential is degenerate on some submanifold. According to the degree of the degeneracy, we obtain non-classical asymptotics.

Key words: eigenvalue asymptotics, Schrödinger operator, heat kernel.

1. Introduction

In this note, we shall extend a part of Simon's paper [13]. To be more precise, we consider the Schrödinger operator on $\mathbb{R}^d$ with the electric scalar potential $V(z)$:

$$H_0 = -\frac{1}{2}\Delta_z + V(z). \tag{1.1}$$

Assume that H_0 be essentially self-adjoint in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$ and denote the unique self-adjoint extension by H. It is well known that if

$$\lim_{|z|\to\infty} V(z) = +\infty, \tag{1.2}$$

H has compact resolvents (cf. for example, Reed and Simon [9]). However, (1.2) is not a necessary condition in order that H has compact resolvents. In spite of the lack of (1.2), there are some cases where H has compact resolvents. For every $t > 0$, we define

$$Z_{\mathrm{cl}}(t) = (2\pi)^{-d} \iint e^{-t(|\zeta|^2/2+V(z))} dz\, d\zeta.$$

Then it follows from the Golden-Thompson theorem that if $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and bounded from below, we have $\mathrm{Tr}[\exp(-tH)] \le Z_{\mathrm{cl}}(t)$ for $t > 0$ (cf. Golden [5], Thompson [15] and Simon [12]). We concentrate on the case

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where $V(z)$ is of the form

$$V(z) = V(x,y) = |x|^{2p}|y|^{2q} \quad (p, q > 0),$$
$$z = (x,y) \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m. \tag{1.3}$$

Then it holds that H_0 is essentially self-adjoint in $L^2(\mathbb{R}^d)$ (cf. Schechter [11]), H has compact resolvents and the trace of the heat kernel $\mathrm{Tr}[\exp(-tH)]$ is finite for every $t > 0$, but $Z_{\mathrm{cl}}(t) \equiv \infty$.

When $n = m = 1$, Simon [13] succeeded in showing the asymptotics of $\mathrm{Tr}[\exp(-tH)]$ as $t \downarrow 0$. Moreover, by the Karamata Tauberian theorem, he obtained the asymptotics of the counting function $N(\lambda)$ of eigenvalues of H as $\lambda \to \infty$. In this paper, we shall give an extension of the results for general dimension n, m. Robert [10] and Aramaki [2], [3] considered a slightly different potential $V(x,y) = (1 + |x|^2)^p|y|^{2q}$. In this case, the classical and non-classical results can occur. That is to say, $Z_{\mathrm{cl}}(t)$ is finite in the case $pm > qn$ and infinite in the case $pm \le qn$. In our case, however, we have only the non-classical results.

In order get the upper bound of $\mathrm{Tr}[\exp(-tH)]$, we shall apply the sliced Golden-Thompson inequality and the sliced bread inequality developped in [13]. On the other hand, for the lower bound, we shall use the probabilistic approach which is slightly more convenient than that of [13].

The plan of this paper is as follows. In §2, we give the main theorem and a corollary. Section 3 is devoted to preliminary remarks for the proof of the main theorem. In §4, we give the proof of the main theorem in the case where $pm \ne qn$ and in §5, we prove the case where $pm = qn$.

2. Main results

Let $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$ and we write a variable z in $\mathbb{R}^d$ by $z = (x,y) \in \mathbb{R}^n_x \times \mathbb{R}^m_y$. We consider the operator:

$$H_0 = -\frac{1}{2}\Delta_{(x,y)} + V(x,y) \tag{2.1}$$

where $\Delta_{(x,y)}$ denotes the Laplacian operator on $\mathbb{R}^n \times \mathbb{R}^m$, the potential $V(x,y)$ is of the form:

$$V(x,y) = |x|^{2p}|y|^{2q}, \quad (p, q > 0) \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m \tag{2.2}$$

and write the unique self-adjoint extension of H_0 in $L^2(\mathbb{R}^d)$ by H.

Then we state the main theorem on the asymptotics for the trace of the heat kernel of H.

Theorem 2.1 *For the above operator H with the potential of the form (2.2), there are the following three cases.*

(i) *If $pm > qn$, we have*

$$\mathrm{Tr}[e^{-tH}] = a_1 t^{-m(1+p+q)/(2q)}(1+o(1)) \quad as \quad t \downarrow 0.$$

where

$$a_1 = \frac{(p+1)\Gamma((p+1)m/(2q))}{2^{m/2}q\Gamma(m/2)} \mathrm{Tr}\big[A^{-(p+1)m/(2q)}\big]$$

and A is the self-adjoint extension in $L^2(\mathbb{R}^n)$ starting from $C_0^\infty(\mathbb{R}^n)$ of $A_0 = -\frac{1}{2}\Delta_x + |x|^{2p}$.

(ii) *If $pm < qn$, we have*

$$\mathrm{Tr}[e^{-tH}] = a_2 t^{-n(1+p+q)/(2p)}(1+o(1)) \quad as \quad t \downarrow 0$$

where

$$a_2 = \frac{(q+1)\Gamma((q+1)n/(2p))}{2^{n/2}p\Gamma(n/2)} \mathrm{Tr}\big[B^{-(q+1)n/(2p)}\big]$$

and B is the self-adjoint extension in $L^2(\mathbb{R}^m)$ starting from $C_0^\infty(\mathbb{R}^m)$ of $B_0 = -\frac{1}{2}\Delta_y + |y|^{2q}$.

(iii) *If $pm = qn$, we have*

$$\mathrm{Tr}[e^{-tH}] = a_3 t^{-n(1+p+q)/(2p)} \log t^{-1}(1+o(1)) \quad as \quad t \downarrow 0$$

where

$$a_3 = \frac{(p+q+1)\Gamma(n/(2p))}{2^{d/2}pq\Gamma(n/2)\,\Gamma(m/2)}.$$

Remark 2.2 Thanks to Aramaki [1], we see that $A^{-(p+1)m/(2q)}$ and $B^{-(q+1)n/(2p)}$ are of trace class in the case $pm > qn$ and $pm < qn$, respectively.

Using the Karamata Tauberian theorem, we can easily prove that the asymptotics of distribution function $N(\lambda)$ of eigenvalues of H.

Corollary 2.3 *We have the following.*

(i) *If* $pm > qn$, *we have*

$$N(\lambda) = b_1 \lambda^{m(1+p+q)/(2q)}(1 + o(1)) \quad as \ \ \lambda \to \infty$$

where $b_1 = a_1/\Gamma(1 + m(1 + p + q)/(2q))$.
(ii) *If* $pm < qn$, *we have*

$$N(\lambda) = b_2 \lambda^{n(1+p+q)/(2p)}(1 + o(1)) \quad as \ \ \lambda \to \infty$$

where $b_2 = a_2/\Gamma(1 + n(1 + p + q)/(2p))$.
(iii) *If* $pm = qn$, *we have*

$$N(\lambda) = b_3 \lambda^{n(1+q+p)/(2p)} \log \lambda (1 + o(1)) \quad as \ \ \lambda \to \infty$$

where $b_3 = a_3/\Gamma(1 + n(1 + p + q)/(2p))$.

3. Preliminaries

In this section, we summarize some basic facts required to get the upper bounds of $\mathrm{Tr}[\exp(-tH)]$ in the main theorem. All the facts in this section can be found in Simon [13]. We consider the Schrödinger operator with a general potential $V(x, y)$ which is bounded from below:

$$\begin{aligned} &H_0 = -\frac{1}{2}\Delta_z + V(z) = -\frac{1}{2}\Delta_{(x,y)} + V(x, y), \\ &z = (x, y) \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m \end{aligned} \tag{3.1}$$

and assume that H_0 is essentially self-adjoint in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$ and the unique self-adjoint extension H of H_0 has compact resolvents. Define

$$Z_{\rm cl}(t) = (2\pi)^{-d} \iint e^{-t(|\zeta|^2/2 + V(z))} dz\, d\zeta. \tag{3.2}$$

Then the Golden-Thompson inequality says that if $V \in L^2_{\rm loc}(\mathbb{R}^d)$ and bounded from below, then it holds that

$$\mathrm{Tr}[e^{-tH}] \le Z_{\rm cl}(t) \quad \text{for} \ \ t > 0. \tag{3.3}$$

This inequality was proved in [5] and [15] by using abstract operator inequality or in [12] by using the probalilistic representation of the heat kernel of H.

Note that in our case where $V(x, y) = |x|^{2p}|y|^{2q}$, $\mathrm{Tr}[e^{-tH}]$ is finite but $Z_{\rm cl}(t) = \infty$, so this inequality is not so useful.

Now, for fixed $x \in \mathbb{R}^n$, assume that $H_{0,x} = -\frac{1}{2}\Delta_y + V(x,y)$ is essentially self-adjoint in $L^2(\mathbb{R}^m)$ starting from $C_0^\infty(\mathbb{R}^m)$ and the unique self-adjoint extension H_x has compact resolvents. Define

$$Z_{\mathrm{SGT}}(t) = (2\pi)^{-n} \iint e^{-t|\xi|^2/2} \,\mathrm{Tr}_{L^2(\mathbb{R}^m)}\left[e^{-t(-\frac{1}{2}\Delta_y + V(x,y))}\right] dx\, d\xi. \tag{3.4}$$

Then the sliced Golden-Thompson inequality says that

$$\mathrm{Tr}[e^{-tH}] \le Z_{\mathrm{SGT}}(t) \ \left(\le Z_{\mathrm{cl}}(t)\right) \quad \text{for} \quad t > 0. \tag{3.5}$$

The type of this inequality is sufficient for the proof of (i) and (ii) in Theorem 2.1 but not for (iii).

Next, let $\epsilon_1(x) \le \epsilon_2(x) \le \cdots$ be the eigenvalues of H_x according to multiplicities. Define

$$Z_{\mathrm{SB}}(t) = \sum_{k=1}^{\infty} \mathrm{Tr}_{L^2(\mathbb{R}^n)}\left[e^{-t(-\frac{1}{2}\Delta_x + \epsilon_k(x))}\right] \quad \text{for} \quad t > 0.$$

Then Simon [13] succeeded in showing the sliced bread inequality:

$$\mathrm{Tr}[e^{-tH}] \le Z_{\mathrm{SB}}(t) \quad \text{for} \quad t > 0. \tag{3.6}$$

We will see that (3.6) is sufficient for the proof of (iii) in Theorem 2.1.

If we apply (3.3) to $\mathrm{Tr}_{L^2(\mathbb{R}^n)}[e^{-t(-\frac{1}{2}\Delta_x + \epsilon_k(x))}]$ and use $\sum_{k=1}^{\infty} e^{-t\epsilon_k} = \mathrm{Tr}_{L^2(\mathbb{R}^m)}[e^{-tH_x}]$, we obtain

$$Z_{\mathrm{SB}}(t) \le Z_{\mathrm{SGT}}(t).$$

Therefore we reach the string of inequalities:

$$\mathrm{Tr}[e^{-tH}] \le Z_{\mathrm{SB}}(t) \le Z_{\mathrm{SGT}}(t) \le Z_{\mathrm{cl}}(t) \quad \text{for} \quad t > 0.$$

4. Proof of the main theorem in the case where $pm \neq qn$

In this section, we shall prove Theorem 2.1 in the case where $pm \neq qn$. However, by symmetry if $pm > qn$, we need only interchange p and q, n and m, so we may assume $pm < qn$ i.e., it suffices to prove (ii) of Theorem 2.1.

From now, we denote various positive constants independent of $t > 0$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ by the same notations C, C_j $(j = 1, 2, \ldots)$ etc.

Let $H_{|x|}$ be the self-adjoint extension in $L^2(\mathbb{R}^m)$ of $H_{0,|x|} = -\frac{1}{2}\Delta_y +$

$|x|^{2p}|y|^{2q}$ and define

$$F(t;|x|) = \mathrm{Tr}_{L^2(\mathbb{R}^m)}[e^{-tH_{|x|}}]. \tag{4.1}$$

At first, we estimate $\mathrm{Tr}[e^{-tH}]$ from below. By the celebrated Feynman-Kac formula, we can write

$$\mathrm{Tr}[e^{-tH}] = (2\pi t)^{-d/2} \iint E_{0,(0,0)}^{1,(0,0)}\left[e^{-t\int_0^1 |x+\sqrt{t}X_s|^{2p}|y+\sqrt{t}Y_s|^{2q}ds}\right]dx\,dy \tag{4.2}$$

where

$$\{Z_s\}_{0\le s\le 1} = \{(X_s,Y_s)\}_{0\le s\le 1} = \{(X_s^{(1)},\dots,X_s^{(n)},Y_s^{(1)},\dots,Y_s^{(m)})\}_{0\le s\le 1}$$

are $d = n+m$ dimensional pinned Brownian motion such that $Z_0 = (X_0,Y_0) = 0 = (0,0)$, $Z_1 = (X_1,Y_1) = 0 = (0,0)$ and $E_{0,(0,0)}^{1,(0,0)}[\cdot]$ denotes the expectation with respect to $\{Z_s\}_{0\le s\le 1} = \{(X_s,Y_s)\}_{0\le s\le 1}$. From now, we simply write $E_{0,(0,0)}^{1,(0,0)}[\cdot]$ by $E_Z[\cdot]$ and the expectation $E_{0,0}^{1,0}[\cdot]$ with respect to $\{X_s\}_{0\le s\le 1}$ and $\{Y_s\}_{0\le s\le 1}$ by $E_X[\cdot]$ and $E_Y[\cdot]$, respectively. For such probabilistic theory, see Itô and Mckean [6].

The following lemma is essentially due to the result of P. Lévy on the joint distribution of the position and the maximum of Brownian motion (cf. [6; p.27]).

Lemma 4.1 *For every $R > 0$, we have*

$$P_X\left(\left\{\sup_{0\le s\le 1}|X_s| \ge R\right\}\right) \le 2ne^{-2R^2/n}$$

where P_X denotes the probability law of $\{X_s\}_{0\le s\le 1}$.

For the precise proof, see Simon [14] or Matsumoto [7; Lemma 1].

Let χ be the characteristic function of the set $\{\sup_{0\le s\le 1}|X_s| \le 1/\sqrt{t}\}$. This lemma implies that $E_X[\chi] \ge 1-\rho(t)$ where $\rho(t) = 2ne^{-2/(nt)} \to 0$ as $t \downarrow 0$. Since $|x+\sqrt{t}X_s| \le 1+|x|$ on $\mathrm{supp}\,\chi$, we have

$$\begin{aligned}\mathrm{Tr}[e^{-tH}] &\ge (2\pi t)^{-d/2}\iint E_Z\left[e^{-t\int_0^1(1+|x|)^{2p}|y+\sqrt{t}Y_s|^{2q}ds}\chi\right]dx\,dy\\ &\ge (2\pi t)^{-d/2}(1-\rho(t))\iint E_Y\left[e^{-t\int_0^1(1+|x|)^{2p}|y+\sqrt{t}Y_s|^{2q}ds}\right]dy\,dx.\end{aligned}$$

Thus we get

$$\mathrm{Tr}[e^{-tH}] \geq (2\pi t)^{-n/2}(1-\rho(t)) \int F(t; 1+|x|)dx. \tag{4.3}$$

The second lemma is concerned in the estimate of $F(t;|x|)$.

Lemma 4.2 (i) *For every* $\lambda > 0$,

$$F(\lambda^{-2}t; |\lambda^{(1+q)/p}x|) = F(t;|x|) \quad \textit{for} \ \ t>0, \quad x \in \mathbb{R}^n.$$

(ii) *For any* $R > 0$, *there exist* $C_R, C_R' > 0$ *such that*

$$F(1;|x|) \leq \begin{cases} C_R|x|^{-pm/q}, & \textit{for} \ \ |x| \leq R \\ C_R e^{-C_R'|x|^{2p/(q+1)}}, & \textit{for} \ \ |x| \geq R. \end{cases}$$

(iii) *There exists* $C > 0$ *such that* $F(t;1) \leq Ct^{-(1+q)m/(2q)}$ *for any* $t \in (0,1]$.

Proof. (i) We can write $F(t;|x|) = \int J(t;|x|,y)dy$ where

$$J(t;|x|,y) = (2\pi t)^{-m/2} E_Y\left[e^{-t\int_0^1 |x|^{2p}|y+\sqrt{t}Y_s|^{2q}ds}\right].$$

Then, for $\lambda > 0$,

$$\begin{aligned} &J(t;|x|,\lambda y) \\ &= \lambda^{-m}(2\pi\lambda^{-2}t)^{-m/2}E_Y\left[e^{-\lambda^{-2}t\int_0^1 |\lambda^{(1+q)/p}x|^{2p}|y+\sqrt{\lambda^{-2}t}Y_s|^{2q}ds}\right] \\ &= \lambda^{-m}J(\lambda^{-2}t; |\lambda^{(1+q)/p}x|, y). \end{aligned}$$

Thus we get

$$\begin{aligned} F(t;|x|) &= \lambda^{-m}\int J(\lambda^{-2}t; |\lambda^{(1+q)/p}x|, \lambda^{-1}y)dy \\ &= \int J(\lambda^{-2}t; |\lambda^{(1+q)/p}x|, y)dy \\ &= F(\lambda^{-2}t; |\lambda^{(1+q)/p}x|). \end{aligned}$$

(ii) Since $J(t;1,y)$ is the restriction to the diagonal of the heat kernel of B on $L^2(\mathbb{R}^m)$ which is introduced in the statement of Theorem 2.1 (ii), there exist $C_j > 0$ $(j=1,2,3)$ such that

$$J(t;1,y) \leq C_1 t^{-m/2}\left(e^{-C_2 t|y|^{2q}} + e^{-C_3|y|^2/t}\right).$$

(cf. Matsumoto [8; Lemma 3.1] and Aramaki [4]). By using (i) with $\lambda = |x|^{-p/(1+q)}$ and then changing of variable, we have

$$\begin{aligned}
F(1;|x|) &= F(|x|^{2p/(1+q)};1) \\
&= \int J(|x|^{2p/(1+q)};1,y)dy \\
&\le C_1|x|^{-pm/(1+q)}\int \big(e^{-C_2|x|^{2p/(1+q)}|y|^{2q}} + e^{-C_3|y|^2/|x|^{2p/(1+q)}}\big)dy \\
&\le C_1|x|^{-pm/q}\int e^{-C_2|y|^{2q}}dy + C_1\int e^{-C_3|y|^2}dy \\
&\le C_4|x|^{-pm/q}
\end{aligned}$$

for $|x| \le R$.

On the contrary, for $|x| \ge R$, since B is positive definite, there exists $C_5 > 0$ such that

$$J(|x|^{2p/(1+q)};1,y) \le C_1 e^{-C_5|x|^{2p/(1+q)}}\big(e^{-C_2'|y|^{2q}} + e^{-C_3'|y|^2}\big).$$

From this, it is easily seen that $F(1;|x|) \le C_6 e^{-C_5|x|^{2p/(1+q)}}$.

(iii) It suffices to note that

$$\begin{aligned}
F(t;1) &= \int J(t;1,y)dy \\
&\le Ct^{-m/2}\int \big(e^{-C_1t|y|^{2q}} + e^{-C_3|y|^2/t}\big)dy \\
&\le Ct^{-m(1+q)/(2q)} \quad \text{for} \quad t \in (0,1].
\end{aligned}$$

□

If we apply Lemma 4.2 (i) and (ii), we see that

$$\begin{aligned}
F(t;|x|) &= F(1;|t^{(1+q)/(2p)}x|) \\
&\le \begin{cases} C|x|^{-pm/q}t^{-(1+q)m/(2q)}, & \text{if } |t^{(1+q)/(2p)}x| \le 1, \\ Ce^{-C_1t|x|^{2p/(1+q)}}, & \text{if } |t^{(1+q)/(2p)}x| \ge 1. \end{cases}
\end{aligned}$$

Next, we examine the integral of $F(t;1+|x|)$ in (4.3).

Lemma 4.3 *For $k = 0, 1, \dots, n$, we have*

$$\int_1^\infty F(t;r)r^{k-1}dr = \begin{cases} O(t^{-(1+q)k/(2p)}) & \text{if } pm \neq qk \\ O(t^{-(1+q)k/(2p)}\log t^{-1}) & \text{if } pm = qk \end{cases} \tag{4.4}$$

as $t \downarrow 0$.

Proof. By the above estimate and the elementary calculation,

$$\begin{aligned}
\int_1^\infty F(t;r)r^{k-1}dr &= \int_1^{t^{-(1+q)/(2p)}} F(t;r)r^{k-1}dr \\
&\quad + \int_{t^{-(1+q)/(2p)}}^\infty F(t;r)r^{k-1}dr \\
&\le C_1 t^{-(1+q)m/(2q)} \int_1^{t^{-(1+q)/(2p)}} r^{-pm/q+k-1}dr \\
&\quad + C_2 \int_{t^{-(1+q)/(2p)}}^\infty e^{-C_2 tr^{2p/(1+q)}} r^{k-1}dr \\
&= C_1 t^{-(1+q)m/(2q)} \int_1^{t^{-(1+q)/(2p)}} r^{-pm/q+k-1}dr \\
&\quad + C_2 t^{-(1+q)n/(2p)} \int_1^\infty e^{-C_2 r^{2p/(1+q)}} r^{k-1}dr.
\end{aligned}$$

When $pm \neq qk$, we have

$$\begin{aligned}
&t^{-(1+q)m/(2q)} \int_1^{t^{-(1+q)/(2p)}} r^{-pm/q+k-1}dr \\
&\quad \le C_3 t^{-(1+q)m/(2q)}(t^{-((1+q)/(2p))\cdot(k-pm/q)} - 1) \\
&\quad \le C_3 t^{-(1+q)k/(2p)}.
\end{aligned}$$

When $pm = qk$, we have

$$\begin{aligned}
&t^{-(1+q)m/(2q)} \int_1^{t^{-(1+q)/(2p)}} r^{-pm/q+k-1}dr \\
&\quad = C_4 t^{-(1+q)m/(2q)} \log t^{-1} = C_4 t^{-(1+q)k/(2p)} \log t^{-1}.
\end{aligned}$$

Thus (4.4) holds. □

Now we return to (4.3). By Lemma 4.3, we see that

$$\begin{aligned}\int F(t;1+|x|)dx &= |S^{n-1}|\int_0^\infty F(t;1+r)r^{n-1}dr\\ &= |S^{n-1}|\int_1^\infty F(t;r)(r-1)^{n-1}dr\\ &= |S^{n-1}|\int_1^\infty F(t;r)r^{n-1}dr\\ &\quad + |S^{n-1}|\sum_{k=1}^{n-1}\binom{n-1}{k}(-1)^k\int_1^\infty F(t;r)r^{n-1-k}dr\\ &= \int_{|x|\geq 1} F(t;|x|)dx + o(t^{-(1+q)n/(2p)})\end{aligned}$$

as $t \downarrow 0$ where $|S^{n-1}|$ denotes the surface area of the unit sphere in $\mathbb{R}^n$ i.e., $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$. Since $qn - pm > 0$, it follows that

$$\begin{aligned}\int_{|x|\leq 1} F(t;|x|)dx &\leq Ct^{-(1+q)m/(2q)}\int_{|x|\leq 1}|x|^{-pm/q}dx\\ &= o(t^{-(1+q)n/(2p)}) \qquad (4.5)\end{aligned}$$

as $t \downarrow 0$. Moreover, since

$$\begin{aligned}\int F(t;|x|)dx &= \int F(1;|t^{-(1+q)/(2p)}x|)dx\\ &= t^{-(1+q)n/(2p)}\int F(1;|x|)dx, \qquad (4.6)\end{aligned}$$

if we show that

$$(2\pi)^{-n/2}\int F(1;|x|)dx = a_2, \qquad (4.7)$$

it follows from (4.2) and (4.3) that

$$\liminf_{t\downarrow 0} t^{(1+p+q)n/(2p)}\,\mathrm{Tr}[e^{-tH}] \geq a_2. \qquad (4.8)$$

For the estimate of $\mathrm{Tr}[e^{-tH}]$ from above, we use the sliced Golden-Thompson inequality (3.5). Taking the definition (4.1) of $F(t;|x|)$ into consideration, the inequality shows that

$$\mathrm{Tr}[e^{-tH}] \leq (2\pi t)^{-n/2}\int F(t;|x|)dx.$$

Thus if (4.7) holds, we have

$$\limsup_{t\downarrow 0} t^{(1+p+q)n/(2p)}\,\mathrm{Tr}[e^{-tH}] \le a_2, \tag{4.9}$$

using the relation (4.6) again.

Therefore, it only remains to show that (4.7) holds. In order to do so, we use the polar coordinate system and apply Lemma 4.2 (i). Then we have

$$(2\pi)^{-n/2}\int F(1;|x|)dx = (2\pi)^{-n/2}|S^{n-1}|\int_0^\infty F(r^{2p/(1+q)};1)r^{n-1}dr.$$

Putting $s = r^{2p/(1+q)}$, the last integral is equal to

$$\begin{aligned}&(2\pi)^{-n/2}|S^{n-1}|\frac{1+q}{2p}\int_0^\infty F(s;1)s^{-1+(1+q)n/(2p)}ds\\ &= (2\pi)^{-n/2}|S^{n-1}|\frac{1+q}{2p}\,\mathrm{Tr}\Big[\int_0^\infty s^{-1+(1+q)n/(2p)}e^{-sB}ds\Big]\\ &= (2\pi)^{-n/2}|S^{n-1}|\frac{1+q}{2p}\Gamma\Big(\frac{(1+q)n}{2p}\Big)\,\mathrm{Tr}\big[B^{-(1+q)n/(2p)}\big].\end{aligned}$$

Thus (4.7) holds. By (4.8) and (4.9), we complete the proof of Theorem 2.1 (ii).

5. Proof of the main theorem in the case where $pm = qn$

In this section, we shall prove Theorem 2.1 (iii) whose proof is more delicate than the case $pm \neq qn$.

At first, we estimate $\mathrm{Tr}[e^{-tH}]$ from below. Let χ and η be the characteristic functions of the sets $\{\sup_{0\le s\le 1}|X_s| \le |\log t|\}$ and $\{\sup_{0\le s\le 1}|Y_s| \le |\log t|\}$, respectively. Then by Lemma 4.1, we have $E_X[\chi] \ge 1-\rho(t)$ and $E_Y[\eta] \ge 1-\rho(t)$ where

$$\rho(t) = \max\{2ne^{-2|\log t|^2/n}, 2me^{-2|\log t|^2/m}\} \to 0 \quad \text{as} \quad t\downarrow 0.$$

Using the representation (4.2) of $\mathrm{Tr}[e^{-tH}]$, we see

$$\begin{aligned}&\mathrm{Tr}[e^{-tH}]\\ &\ge (2\pi t)^{-d/2}\iint_{\substack{|x|\ge\sqrt{t}(\log t)^2\\ |y|\ge\sqrt{t}(\log t)^2}} E_Z\big[e^{-t\int_0^1 |x+\sqrt{t}X_s|^{2p}|y+\sqrt{t}Y_s|^{2q}ds}\chi\eta\big]\,dx\,dy.\end{aligned} \tag{5.1}$$

In the integral domain and $\mathrm{supp}(\chi\eta)$, there exists a small constant $c > 0$

such that

$$|x+\sqrt{t}X_s| \ge |x| - \sqrt{t}|\log t| \ge \sqrt{t}(|\log t|^2 - |\log t|) > 0,$$
$$|y+\sqrt{t}Y_s| \ge \sqrt{t}(|\log t|^2 - |\log t|) > 0 \quad \text{for} \quad t \in (0,c).$$

From this and the mean value theorem, it is easily seen that

$$\big|\log |x+\sqrt{t}X_s|^{2p}|y+\sqrt{t}Y_s|^{2q} - \log |x|^{2p}|y|^{2q}\big| \le C/|\log t|$$

for $t \in (0,c)$. So if we put $k(t) = e^{C/|\log t|}$, we get

$$|x+\sqrt{t}X_s|^{2p}|y+\sqrt{t}Y_s|^{2q} \le k(t)|x|^{2p}|y|^{2q}.$$

Thus it follows from (5.1) that

$$\begin{aligned}\mathrm{Tr}[e^{-tH}] &\ge (2\pi t)^{-d/2}(1-\rho(t))^2 \iint_{\substack{|x|\ge\sqrt{t}(\log t)^2\\ |y|\ge\sqrt{t}(\log t)^2}} e^{-tk(t)|x|^{2p}|y|^{2q}}dx\,dy \\ &= (2\pi t)^{-d/2}(1-\rho(t))^2|S^{n-1}||S^{m-1}|I(t)\end{aligned}$$

where

$$I(t) = \int_{\sqrt{t}(\log t)^2}^{\infty} s^{m-1}ds \int_{\sqrt{t}(\log t)^2}^{\infty} e^{-tk(t)s^{2q}r^{2p}}r^{n-1}dr.$$

Change of variable: $s^{q/p}r \to r$ and the assumption $pm = qn$ lead to

$$\begin{aligned}I(t) &= \int_{\sqrt{t}(\log t)^2}^{\infty} s^{-1}ds \int_{s^{q/p}\sqrt{t}(\log t)^2}^{\infty} e^{-tk(t)r^{2p}}r^{n-1}dr \\ &= \int_{t^{(p+q)/(2p)}(\log t)^{2(p+q)/p}}^{\infty} e^{-tk(t)r^{2p}}r^{n-1}dr \int_{\sqrt{t}(\log t)^2}^{(\sqrt{t}(\log t)^2)^{-p/q}r^{p/q}} s^{-1}ds \\ &= \int_{t^{(p+q)/(2p)}(\log t)^{2(p+q)/p}}^{\infty} \log\frac{r^{p/q}}{(\sqrt{t}(\log t)^2)^{(p+q)/q}}e^{-tk(t)r^{2p}}r^{n-1}dr.\end{aligned}$$

If we again use a change of variable: $(tk(t))^{1/(2p)}r \to r$, we have

$$\begin{aligned}I(t) &= (tk(t))^{-n/(2p)} \\ &\quad\times \int_{r(t)}^{\infty} \log\frac{r^{p/q}}{t^{(p+q+1)/(2q)}(\log t)^{2(p+q)/q}k(t)^{1/(2q)}}e^{-r^{2p}}r^{n-1}dr\end{aligned}$$

where $r(t) = (tk(t))^{1/(2p)}t^{(p+q)/(2p)}(\log t)^{2(p+q)/p}$. Thus we obtain

$$\mathrm{Tr}[e^{-tH}] \ge t^{-(p+q+1)n/(2p)}(2\pi)^{-d/2}|S^{n-1}||S^{m-1}|(1-\rho(t))^2k(t)^{-n/(2p)}$$

$$\times \int_{r(t)}^{\infty} \Big\{ \frac{p}{q} \log r + \frac{p+q+1}{2q} \log t^{-1} \\ - \log(\log t)^{2(p+q)/q} - \frac{1}{2q} \log k(t) \Big\} e^{-r^{2p}} r^{n-1} dr.$$

Since $\rho(t) \to 0$, $k(t) \to 1$, $r(t) \to 0$ as $t \downarrow 0$, we get

$$\liminf_{t \downarrow 0} t^{(p+q+1)n/(2p)} (\log t^{-1})^{-1} \operatorname{Tr}[e^{-tH}] \\ \geq \frac{p+q+1}{2q} (2\pi)^{-d/2} |S^{n-1}||S^{m-1}| \int_0^\infty e^{-r^{2p}} r^{n-1} dr = a_3. \quad (5.2)$$

For the estimate of $\operatorname{Tr}[e^{-tH}]$ from above, we use the sliced bread inequality (3.6). In order to do so, let A_g be the unique self-adjoint extension in $L^2(\mathbb{R}^n)$ starting from $C_0^\infty(\mathbb{R}^n)$ of $A_{g,0} = -\frac{1}{2}\Delta_x + g|x|^{2p}$ $(g > 0,\ p > 0)$ and define $F_g(t) = \operatorname{Tr}[e^{-tA_g}]$ for $t > 0$ and $N_g(\lambda)$ is the dimension of the spectral projection of A_g on $[0, \lambda]$. For brevity of the notations, we write A_1, F_1, N_1 by A, F, N. Then we need the following lemma.

Lemma 5.1 *Under the above situation, we have*

(i) $F_g(t) = F(g^{1/(p+1)}t)$ *for* $g, t > 0$.

(ii) $N_g(\lambda) = N(g^{-1/(p+1)}\lambda)$ *for* $g, \lambda > 0$.

(iii) $\lim_{t \downarrow 0} t^{n(1+p)/(2p)} F(t) = 2^{-n/2} \dfrac{\Gamma(n/(2p))}{p\Gamma(n/2)}$.

(iv) $\lim_{\lambda \to \infty} \lambda^{-n(1+p)/(2p)} N(\lambda) = 2^{-n/2} \dfrac{\Gamma(n/(2p))}{p\Gamma(n/2)\Gamma(1 + n(1+p)/(2p))}$.

(v) $\lim_{t \downarrow 0} t^{1+n(1+p)/(2p)} F'(t) = -2^{-n/2} \dfrac{n(p+1)\Gamma(n/(2p))}{2p^2\Gamma(n/2)}$

where F' denotes the derivative of F.

Proof. If we define an operator U:

$$(Uf)(x) = g^{-n/4(p+1)} f(g^{1/2(p+1)}x), \quad f \in L^2(\mathbb{R}^n),$$

it is clear that U is a unitary operator on $L^2(\mathbb{R}^n)$ and $U^* A_g U = g^{1/(p+1)} A$. Thus (i) and (ii) are clear.

If we define

$$Z_{\mathrm{cl}}^{(1)}(t) = (2\pi)^{-n} \iint e^{-t(|\xi|^2/2 + |x|^{2p})} dx\, d\xi,$$

it follows from the elementary calculation that

$$Z_{\mathrm{cl}}^{(1)}(t) = \frac{\Gamma(n/(2p))}{2^{n/2}p\Gamma(n/2)} t^{-n(1+p)/(2p)}$$

Therefore, (iii) follows from the fact $\lim_{t\downarrow 0} F(t)/Z_{\mathrm{cl}}^{(1)}(t) = 1$.

(iv) follows from the Karamata Tauberian theorem.

(v) Since

$$F(t) = \int_0^\infty e^{-t\lambda} dN(\lambda),$$

using the integration by parts and a change of variable, we see

$$\begin{aligned} -F'(t) &= \int_0^\infty \lambda e^{-t\lambda} dN(\lambda) \\ &= \int_0^\infty (\lambda t - 1) e^{-t\lambda} N(\lambda) d\lambda \\ &= t^{-1} \int_0^\infty e^{-\mu}(\mu - 1) N\Big(\frac{\mu}{t}\Big) d\mu \\ &= t^{-1} \int_0^\infty e^{-\mu}(\mu - 1) \Big(\frac{\mu}{t}\Big)^{n(1+p)/(2p)} \Big(\frac{\mu}{t}\Big)^{-n(1+p)/(2p)} N\Big(\frac{\mu}{t}\Big) d\mu. \end{aligned}$$

Therefore, (iv) implies that

$$\begin{aligned} &\lim_{t\downarrow 0} t^{1+n(1+p)/(2p)} F_1'(t) \\ &= -\frac{\Gamma(n/(2p))}{2^{n/2}p\Gamma(n/2)\,\Gamma(1+n(1+p)/(2p))} \int_0^\infty e^{-\mu}(\mu-1)\mu^{n(1+p)/(2p)} d\mu. \end{aligned}$$

Since the last integral is equal to

$$\begin{aligned} &\Gamma(2 + n(1+p)/(2p)) - \Gamma(1 + n(1+p)/(2p)) \\ &\qquad = \frac{n(1+p)}{2p} \Gamma(1 + n(1+p)/(2p)), \end{aligned}$$

(v) holds. □

Let $\epsilon_j(|x|)$ be the j-th eigenvalue of $H_{|x|}$ on $L^2(\mathbb{R}^m)$ which is obtained from $H_{0,|x|} = -\frac{1}{2}\Delta_y + |x|^{2p}|y|^{2q}$. Define a unitary operator on $L^2(\mathbb{R}^m)$ by

$$(Uf)(y) = |x|^{pm/2(q+1)} f(|x|^{p/(q+1)}y), \quad f \in L^2(\mathbb{R}^m).$$

Since $U^* H_{|x|} U = |x|^{2p/(1+q)} H_1$, we see that $\epsilon_j(|x|) = |x|^{2p/(q+1)}\epsilon_j$ where

$\epsilon_j = \epsilon_j(1)$. Therefore, by Lemma 5.1 (i),

$$\begin{aligned} Z_{\mathrm{SB}}(t) &= \sum_{j=1}^{\infty} \mathrm{Tr}\big[e^{-t(-\frac{1}{2}\Delta_x + \epsilon_j(|x|))}\big] \\ &= \sum_{j=1}^{\infty} \mathrm{Tr}\big[e^{-t\epsilon_j^{(q+1)/(p+q+1)}(-\frac{1}{2}\Delta_x + |x|^{2p/(q+1)})}\big] \\ &= \sum_{j=1}^{\infty} \widetilde{F}_{2p/(q+1)}(t\epsilon_j^{(q+1)/(p+q+1)}) \\ &= \int \widetilde{F}_{2p/(q+1)}(t\lambda^{(q+1)/(p+q+1)}) dN(\lambda) \end{aligned}$$

where $\widetilde{F}_\gamma(t) = \mathrm{Tr}[e^{-t(-\frac{1}{2}\Delta_x + |x|^\gamma)}]$ and $N(\lambda) = \#\{j; \epsilon_j \leq \lambda\}$. By the integration by parts, we have

$$Z_{\mathrm{SB}}(t) = -\frac{(q+1)t}{p+q+1} \int_0^{\infty} \lambda^{-p/(p+q+1)} \widetilde{F}'_{2p/(q+1)}(t\lambda^{(q+1)/(p+q+1)}) N(\lambda) d\lambda. \tag{5.3}$$

Since $N(\lambda) = 0$ for small $\lambda > 0$, we may assume that the integral domain is equal to $[\lambda_0, \infty)$ for some $\lambda_0 > 0$. Choose numbers λ_1, λ_2 so that $\lambda_0 < \lambda_1 < \lambda_2 < \infty$ and $\lambda_1^{(q+1)/(p+q+1)} t = |\log t|^{-1}$, $\lambda_2^{(q+1)/(p+q+1)} t = 1$ and then decompose the integral of the right hand side of (5.3) as $Z_{\mathrm{SB}}(t) = \sum_{i=1}^{3} Z_{\mathrm{SB}}^{(i)}(t)$ where

$$Z_{\mathrm{SB}}^{(i)}(t) = -\frac{(q+1)t}{p+q+1} \int_{\lambda_{i-1}}^{\lambda_i} \lambda^{-p/(p+q+1)} \widetilde{F}'_{2p/(q+1)}(t\lambda^{(q+1)/(p+q+1)}) N(\lambda) d\lambda$$

for $i = 1, 2, 3$ with $\lambda_3 = \infty$.

Firstly, we consider $Z_{\mathrm{SB}}^{(3)}(t)$. By Lemma 5.1 (iv), there exists $C > 0$ such that

$$N(\lambda) \leq C\lambda^{m(1+q)/(2q)} \quad \text{on} \quad [\lambda_2, \infty). \tag{5.4}$$

Moreover, we claim that for $\lambda \geq 1$, there exist C_1, $c > 0$ such that $-\widetilde{F}'_{2p/(q+1)}(\lambda) \leq C_1 e^{-c\lambda}$.

In fact, if we write the j-th eigenvalue of $-\frac{1}{2}\Delta_x + |x|^{2p/(q+1)}$ by $\widetilde{\epsilon}_j$, it follows from Aramaki [1] that $\widetilde{\epsilon}_j^N \geq j^2$ and $\widetilde{\epsilon}_j/2 \geq c$ $(j = 1, 2, \ldots)$ for some large N and small $c > 0$. Therefore,

$$\begin{aligned}-\widetilde{F}'_{2p/(q+1)}(\lambda) &= \sum_{j=1}^{\infty} \tilde{\epsilon}_j e^{-\lambda\tilde{\epsilon}_j} \\ &= \lambda^{-1}\sum_{j=1}^{\infty} \lambda\tilde{\epsilon}_j e^{-\lambda\tilde{\epsilon}_j/2} e^{-\lambda\tilde{\epsilon}_j/2} \\ &\leq \lambda^{-1}\sum_{j=1}^{\infty} (\lambda\tilde{\epsilon}_j)^{-N} e^{-c\lambda} \\ &\leq \lambda^{-(N+1)}\sum_{j=1}^{\infty} j^{-2} e^{-c\lambda}.\end{aligned}$$

By the above facts, we see that

$$\begin{aligned}Z_{\mathrm{SB}}^{(3)}(t) &\leq C_1 t \int_{\lambda_2}^{\infty} \lambda^{-p/(p+q+1)} e^{-ct\lambda^{(q+1)/(p+q+1)}} \lambda^{m(q+1)/(2q)} d\lambda \\ &\leq C_2 t^{-(p+q+1)m/(2q)} \int_1^{\infty} \mu^{(p+q+1)m/(2q)} e^{-c\mu} d\mu \\ &\leq C_3 t^{-(p+q+1)m/(2q)}.\end{aligned}$$

Thus $Z_{\mathrm{SB}}^{(3)}(t)$ is negligible.

Secondly, we consider $Z_{\mathrm{SB}}^{(2)}(t)$. We claim that there exists $C > 0$ such that

$$-\widetilde{F}'_{2p/(q+1)}(\lambda) \leq C\lambda^{-1-n(1+p+q)/(2p)} \quad \text{on } [\lambda_1, \lambda_2). \tag{5.5}$$

In fact, if we put $\widetilde{N}(\mu) = \#\{j; \tilde{\epsilon}_j \leq \mu\}$ and apply Lemma 5.1 (iv), (5.5) follows from that

$$\begin{aligned}-\widetilde{F}'_{2p/(q+1)}(\lambda) &= \int_0^{\infty} \mu e^{-\lambda\mu} d\widetilde{N}(\mu) \\ &= \int_0^{\infty} (\mu\lambda - 1) e^{-\lambda\mu} \widetilde{N}(\mu) d\mu \\ &\leq C\int_0^{\infty} \lambda e^{-\lambda\mu} \mu^{1+n(p+q+1)/(2p)} d\mu \\ &\leq C_1 \lambda^{-1-n(p+q+1)/(2p)}.\end{aligned}$$

From (5.5) and (5.4), we get

$$Z_{\mathrm{SB}}^{(2)}(t) \le C_2 t^{-(p+q+1)n/(2p)} \int_{\lambda_1}^{\lambda_2} \lambda^{-1} d\lambda.$$

By the choice of λ_1 and λ_2,

$$\int_{\lambda_1}^{\lambda_2} \lambda^{-1} d\lambda = \log \frac{\lambda_2}{\lambda_1} = \log |\log t|.$$

Therefore,

$$Z_{\mathrm{SB}}^{(2)}(t) \le C_1 t^{-(p+q+1)n/(2p)} \log |\log t|,$$

so $Z_{\mathrm{SB}}^{(2)}(t)$ is also negligible.

Finally, we compute $Z_{\mathrm{SB}}^{(1)}(t)$. Note that $t\lambda^{(q+1)/(p+q+1)} \le |\log t|^{-1}$ for $\lambda \in [\lambda_0, \lambda_1)$. When $\lambda > 0$ is small, Lemma 5.1 (v) implies that

$$-\widetilde{F}'_{2p/(q+1)}(\lambda) = \gamma \lambda^{-1-(p+q+1)n/(2p)}(1 + o(1)) \quad \text{as} \quad \lambda \downarrow 0$$

where

$$\gamma = \frac{n(p+q+1)(q+1)}{2^{1+n/2}p^2} \frac{\Gamma(n(q+1)/(2p))}{\Gamma(n/2)}.$$

Therefore,

$$\begin{aligned} Z_{\mathrm{SB}}^{(1)}(t) &= \gamma t \frac{q+1}{p+q+1} \int_{\lambda_0}^{\lambda_1} \lambda^{-p/(p+q+1)} t^{-1-(p+q+1)n/(2p)} \\ &\quad \times \lambda^{-((q+1)/(p+q+1))\cdot(1+(p+q+1)n/(2p))}(1 + o(1))N(\lambda)d\lambda \\ &= \gamma \frac{q+1}{p+q+1} t^{-(p+q+1)n/(2p)} \\ &\quad \times \int_{\lambda_0}^{\lambda_1} \lambda^{-1-(q+1)n/(2p)} N(\lambda) d\lambda (1 + o(1)) \end{aligned}$$

as $t \downarrow 0$. Here it follows from Lemma 5.1 (iv) that

$$N(\lambda) = \delta \lambda^{-m(1+q)/(2q)}(1 + o(1)) \quad \text{as} \quad \lambda \to \infty$$

where

$$\delta = 2^{-m/2} \frac{\Gamma(m/(2q))}{q\Gamma(m/2)\Gamma(1 + m(1+q)/(2q))}.$$

Since $\lambda_1 \uparrow \infty$ as $t \downarrow 0$, we have

$$Z_{\mathrm{SB}}^{(1)}(t) = \gamma\delta \frac{q+1}{p+q+1} t^{-(1+p+q)n/(2p)} \int_{\lambda_0}^{\lambda_1} \lambda^{-1} \frac{N(\lambda)}{\delta\lambda^{m(1+q)/(2q)}} d\lambda$$
$$= \gamma\delta \frac{q+1}{p+q+1} t^{-(1+p+q)n/(2p)} \Big(\log \frac{\lambda_1}{\lambda_0}\Big)(1+o(1)) \quad \text{as} \quad t \downarrow 0.$$

Note that

$$\log \frac{\lambda_1}{\lambda_0} = \frac{p+q+1}{q+1} \log t^{-1} + o(\log t^{-1}) \quad \text{as} \quad t \downarrow 0.$$

A simple calculation leads to $\gamma\delta = a_3$. Therefore, we have

$$\limsup_{t\downarrow 0} t^{(p+q+1)n/(2p)} (\log t^{-1})^{-1} \operatorname{Tr}[e^{-tH}] \leq a_3. \tag{5.6}$$

The combination of (5.2) and (5.6) completes the proof of Theorem 2.1 (iii).

References

[1] Aramaki J., *On the asymptotic behaviors of the spectrum of quasi-elliptic pseudodifferential operators on* $\mathbb{R}^n$. Tokyo J. Math. **10**, No.2 (1987), 481–505.

[2] Aramaki J., *Complex powers of vector valued operators and their applications to asymptotic behavior of eigenvalues.* J. Funct. Anal. **87**, No.2 (1989), 294–320.

[3] Aramaki J., *An extension of the Ikehara Tauberian theorem and its application.* Acta Math. Hungarica **71**, No.4 (1996), 297–326.

[4] Aramaki J., *On the asymptotics of the trace of the heat kernel for the magnetic Schrödinger operator.* to appear in Pacific J. Math.

[5] Golden S., *Lower bounds for Helmholtz functions.* Phys. Rev. **137B** (1965), 1127–1128.

[6] Itô K. and Mckean H.P., Jr., *Diffusion processes and their sample paths.* Springer, Berlin, Heidelberg, New York (1974).

[7] Matsumoto H., *The short time asymptotics of the trace of the heat kernels for the magnetic Schrödinger operators.* J. Math. Soc. Japan **42**, No.4 (1990), 677–689.

[8] Matsumoto H., *Classical and non-classical eigenvalue asymptotics for the magnetic Schrödinger operators.* J. Funct. Anal. **95** (1991), 460–482.

[9] Reed M. and Simon B., *Method of Modern Mathematical Physics Vol.* IV, *Analysis of Operators.* Academic Press, New York (1977).

[10] Robert D., *Comportement asymptotique des valeurs propres d'opérateurs du type Schrödinger a potentiel dégénéré.* J. Math. pures et appl. **61** (1982), 275–300.

[11] Schechter M., *Spectra of Partial Differential Operators.* North-Holland, Amsterdam (1971).

[12] Simon B., *Functional integration and quantum physics.* Academic Press, New York, San Francisco, London (1979).

[13] Simon B., *Nonclassical eigenvalue asymptotics.* J. Funct. Anal. **53** (1983), 84–98.

[14] Simon B., *Semiclassical analysis of low lying eigenvalues* II, *Tunneling.* Ann. of Math. **120** (1984), 89–118.

[15] Thompson C., *Inequality with applications in statistical mechanics.* J. Math. Phys. **6** (1965), 472–502.

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