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Approximation by Riemann sums in modular spaces

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Abstract. Here we give an estimation of the modular convergence of translated equidistant Riemann sums to the integral of a function belonging to a modular space. Thus we extend some previous results by Fominykh and Kaminska-Musielak.

Key words: modular spaces, translated equidistant Riemann sums, modulus of continuity.

1. Introduction

The aim of this paper is to give a modular estimation for the error of approximation of the integral of a function f belonging to a modular space, by means of the sequence of translated equidistant Riemann sums of f, in terms of its modular modulus of continuity (see [13], [3]). As a consequence we obtain a modular approximation theorem for the integral of f.

This problem was studied by M.Yu. Fominykh in [8] in $L^p[0,1]$, $1 \leq p < \infty$ and, in [10], for multivariate functions defined on the hypercube $[0,1]^n$, by A. Kaminska and J. Musielak, in Musielak-Orlicz spaces.

For a sake of simplicity our extension to modular spaces is given for functions of one variable, but the extension to multivariate case gives no further problems. Moreover here we take the interval $Q = [0,1]$ as a basic interval but the results remain valid also for a general bounded interval $[a,b]$.

In Section 4 we give various examples of modular spaces for which the theory here developed is applicable. We will also discuss examples of modulars which haven't an integral representation, thus proving that the extension we give is meaningful.

We wish to recall here that related results for integrals of L^p functions defined on the entire real line, were given by P.L. Butzer and R.L. Stens in [6], and by Butzer and A. Gessingher in [7].

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2. Notations and definitions

Let $Q = [0, 1]$ and let us denote by $L^0(Q)$ the space of all (Lebesgue) measurable functions $f : Q \to \mathbb{R}$. When it is necessary, we extend functions over Q to the real line by 1-periodicity.

Let $\rho : L^0(Q) \to [0, +\infty]$ be a modular, i.e., a functional with the following assumptions:

1. $\rho(f) = 0 \Longleftrightarrow f = 0$, a.e. in Q.
2. $\rho(-f) = \rho(f)$, for every $f \in L^0(Q)$.
3. $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$, for every $f, g \in L^0(Q)$ and $\alpha, \beta \in \mathbb{R}_0^+$ with $\alpha + \beta = 1$.

If in place of **3.** we have:

$$\rho\Big(\sum_{i=1}^{n} \alpha_i f_i\Big) \leq M \sum_{i=1}^{n} \alpha_i \rho(M f_i),$$

for every $n \in \mathbb{N}$, $\alpha_i \geq 0$ with $\alpha_1 + \cdots + \alpha_n = 1$, and $f_i \in L^0(Q)$, for every $i = 1, \ldots, n$ and for an absolute constant $M \geq 1$, we will say that the modular is **discretely quasi convex**. If moreover $M = 1$ we will say that the modular is (discretely) **convex**.

If ρ is a modular on $L^0(Q)$, we will denote by $L^\rho(Q)$ the corresponding modular space generated by ρ, i.e.

$$L^\rho(Q) = \Big\{ f \in L^0(Q) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \Big\}.$$

It is well-known that $L^\rho(Q)$ is a vector subspace of $L^0(Q)$ and it is possible to define on it the concept of "modular convergence" by the following way: we say that a sequence of functions $f_n \in L^\rho(Q)$ is modular convergent (or ρ-convergent) to a function $f \in L^\rho(Q)$, if there is a $\lambda > 0$ such that

$$\lim_{n \to +\infty} \rho(\lambda(f_n - f)) = 0.$$

This notion of convergence is weaker than the "norm-convergence" induced by the Luxemburg norm generated by the modular (see [12]). This is equivalent to say that the above limit relation is satisfied for any $\lambda > 0$. These two notions of convergence are equivalent in the special case when the modular ρ has the Δ_2 property (see [12]). For a general theory of the modular spaces we refer to [12].

We will also need of the following definitions, concerning modular functionals.

a. We say that a modular ρ is **monotone** if $\rho(f) \leq \rho(g)$ whenever $|f| \leq |g|$.

b. Let (Ω, Σ, μ) be a measure space and let $L^0(\Omega)$ be the space of all measurable functions, finite μ-a.e. on Ω. A modular ρ on $L^0(\Omega)$ will be called **quasi-convex**, if there exists a constant $M \geq 1$ such that

$$\rho\left(\int_\Omega p(t)h(t,\cdot)d\mu(t)\right) \leq M \int_\Omega p(t)\rho(Mh(t,\cdot))d\mu(t)$$

for $p \in L^1(\Omega)$, $p(t) \geq 0$, $\int_\Omega p(t)d\mu(t) = 1$ and for $h(\cdot, u) \in L^0(\Omega)$ and $u > 0$. Arguing as in [5], it is easily shown that if μ is atomless and ρ is quasi-convex, then the modular ρ is discretely quasi-convex, with the same constant $M \geq 1$.

c. The modular ρ is **finite** if the characteristic function χ_A of a measurable set A of finite Lebesgue measure, belongs to the modular space $L^\rho(Q)$.

d. The modular ρ is **absolutely finite** if it is finite and moreover, for every $\varepsilon > 0$ and every $\lambda_0 > 0$ there is a $\delta > 0$ for which, for every measurable subset $A \subset Q$ with $|A| < \delta$, we have $\rho(\lambda_0 \chi_A) < \varepsilon$.

e. The modular ρ is **absolutely continuous** if there is an $\alpha > 0$ such that for every $f \in L^0(Q)$ with $\rho(f) < +\infty$ the following condition holds:

- For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ for all measurable sets $B \subset Q$ with $|B| < \delta$.

f. The modular ρ is **τ-bounded** if there are a constant $C \geq 1$ and a measurable nonnegative essentially bounded function $h : \mathbb{R} \to \mathbb{R}$ such that:

$$\rho(f(\cdot - t)) \leq \rho(Cf) + h(t),$$

for a.e. $t \in \mathbb{R}$ and for every $f \in L^\rho(Q)$.

For the above concepts we refer to [13], [1], [2], [3].

We remark that the concept of quasi convexity for modulars is related to the notion of quasi convexity for functions (see [9], [11], [4]). We recall that a function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is **quasi convex** if there is a constant $M \geq 1$

such that, for every $f \in L^1(Q)$, we have:

$$\varphi\left(\int_Q |f(x)|dx\right) \leq M \int_Q \varphi(M|f(x)|)dx.$$

An analogous definition is applied to functions φ depending on parameters. In this case we will say that such a function is quasi convex with constant $M \geq 1$, if it is so for all values of the parameters, and with the same constant M.

Note that also for functions it is known a concept of **discrete quasi convexity**, with a constant $M \geq 1$, and it is defined in an obvious way. In what follows the following assumption is of importance:

we will say that a modular ρ satisfies condition $(*)$ if there is a constant $C' \geq 1$, such that for every $n \in \mathbb{N}$, $g \in L^\rho(Q)$, we have:

$$\sum_{k=0}^{n-1} \rho\left[g\left(\frac{\cdot + k}{n}\right)\right] \leq n\rho(C'g) + \varepsilon_n, \tag{1}$$

where $\{\varepsilon_n\}$ is a sequence of nonnegative real numbers.

We define the ρ-modulus of continuity of a function $f \in L^\rho(Q)$ by the following functional:

$$\omega_\rho(f, \delta) = \sup_{|s| \leq \delta} \rho[f(\cdot + s) - f(\cdot)],$$

for $\delta > 0$.

Finally we will need of the following notation:

for every $y \in Q$ and $f \in L^0(Q)$ we denote by $R_n(f, y)$ the translated equidistant Riemann sums of f, i.e. we put:

$$R_n(f, y) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{y+k}{n}\right), \quad y \in Q.$$

3. A modular estimation for approximation by Riemann sums

In order to state that $R_n(f, \cdot) \in L^\rho(Q)$, for every $f \in L^1(Q) \cap L^\rho(Q)$, we give the following proposition.

Proposition 1 *Let ρ be a quasi convex modular on $L^0(Q)$ with a constant $M \geq 1$ satisfying $(*)$ and let $\lambda > 0$ be so small that $\rho(\lambda C' M f) < +\infty$. Then $\rho(\lambda R_n(f, \cdot)) < +\infty$ for every $n \in \mathbb{N}$. Moreover if ρ is finite, then for every*

$n \in \mathbb{N}$ it results $R_n(f,\cdot) - \int_Q f(x)dx \in L^\rho(Q)$.

Proof. For the given $\lambda > 0$, by quasi convexity of the modular ρ in its discrete form, we have for $n = 1, 2, \ldots$

$$\rho(\lambda R_n(f,\cdot)) = \rho\left(\frac{\lambda}{n}\sum_{k=0}^{n-1} f\left(\frac{\cdot + k}{n}\right)\right)$$
$$\leq \frac{M}{n}\sum_{k=0}^{n-1}\rho\left(\lambda M f\left(\frac{\cdot + k}{n}\right)\right)$$

Thus by applying condition $(*)$ to the function $g(t) = \lambda M f(t)$ we obtain

$$\rho(\lambda R_n(f,\cdot)) \leq M\rho(\lambda C' M f) + \frac{M}{n}\varepsilon_n.$$

Finally by finiteness of ρ, the second part of the proposition easily follows.

□

Now in order to state the main result of this paper we give the following:

Proposition 2 *For $f \in L^1(Q)$, we have:*

$$R_n(f,y) - \int_Q f(x)dx = \frac{1}{n}\sum_{k=0}^{n-1} g\left(\frac{y+k}{n}\right),$$

for any $y \in Q$, where $g : Q \to \mathbb{R}$ is defined by

$$g(t) = n\sum_{k=0}^{n-1}\chi_{Q_k}(t)\int_{Q_k}[f(t) - f(x)]dx,$$

for every $t \in Q$, and $Q_k = [k/n, (k+1)/n)$, $k = 0, 1, \ldots, n-2$, $Q_{n-1} = [(n-1)/n, 1]$.

Proof. We have:

$$R_n(f,y) - \int_Q f(x)dx = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{y+k}{n}\right) - \int_Q f(x)dx$$
$$= \sum_{k=0}^{n-1}\int_{Q_k}\left[f\left(\frac{y+k}{n}\right) - f(x)\right]dx$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\int_{Q_k} n\left[f\left(\frac{y+k}{n}\right) - f(x)\right]dx.$$

Now, for any $0 \le y \le 1$, we have $k/n \le (y+k)/n \le (k+1)/n$, and so

$$g\left(\frac{y+k}{n}\right) = n\int_{Q_k}\left[f\left(\frac{y+k}{n}\right) - f(x)\right]dx.$$

Hence the assertion follows. □

Now we are ready to prove the main result of this paper:

Theorem 1 *Under the assumptions of Proposition* 1, *if moreover* ρ *is monotone, we have*

$$\rho\left[\lambda\left(R_n(f,\cdot) - \int_Q f(x)dx\right)\right] \le M^2\omega_\rho[2\lambda C'M^2 f, 2/n] + M\frac{\varepsilon_n}{n}, \tag{2}$$

for any $\lambda > 0$.

Proof. From Proposition 2, quasi convexity (in its discrete form) and property $(*)$, we have for $\lambda > 0$,

$$\begin{aligned}
&\rho\left[\lambda\left(R_n(f,\cdot) - \int_Q f(x)dx\right)\right] \\
&\quad\le \frac{M}{n}\sum_{k=0}^{n-1}\rho[\lambda M g((\cdot+k)/n)] \le M\rho[\lambda M C' g] + \frac{M}{n}\varepsilon_n \\
&\quad= M\rho\left[\lambda M C' n\sum_{k=0}^{n-1}\chi_{Q_k}(\cdot)\int_{Q_k}(f(\cdot) - f(x))dx\right] + \frac{M}{n}\varepsilon_n.
\end{aligned}$$

Now, with the substitution $x = t + s$, we obtain:

$$\begin{aligned}
\int_{Q_k}|f(t) - f(x)|dx &= \int_{Q_k - t}|f(t) - f(t+s)|ds \\
&\le \int_{-1/n}^{1/n}|f(t+s) - f(t)|ds
\end{aligned}$$

and so by monotonicity of ρ, we have:

$$\rho\left[\lambda\left(R_n(f,\cdot)-\int_Q f(x)dx\right)\right]$$
$$\leq M\rho\left[\lambda C'Mn\sum_{k=0}^{n-1}\chi_{Q_k}(\cdot)\int_{-1/n}^{1/n}|f(\cdot+s)-f(\cdot)|ds\right]+\frac{M}{n}\varepsilon_n$$
$$=M\rho\left[\lambda C'Mn\int_{-1/n}^{1/n}|f(\cdot+s)-f(\cdot)|ds\sum_{k=0}^{n-1}\chi_{Q_k}(\cdot)\right]+\frac{M}{n}\varepsilon_n.$$

Now, since $\sum_{k=0}^{n-1}\chi_{Q_k}(t)=1$ for every $t\in Q$, we have, by quasi convexity of ρ :

$$\rho\left[\lambda\left(R_n(f,\cdot)-\int_Q f(x)dx\right)\right]$$
$$\leq M\rho\left[(n/2)\int_{-1/n}^{1/n}2\lambda C'M|f(\cdot+s)-f(\cdot)|ds\right]+\frac{M}{n}\varepsilon_n$$
$$\leq M^2(n/2)\int_{-1/n}^{1/n}\rho[2\lambda C'M^2|f(\cdot+s)-f(\cdot)|]ds+\frac{M}{n}\varepsilon_n$$
$$\leq M^2\omega_\rho[2\lambda C'M^2f,2/n]+\frac{M}{n}\varepsilon_n,$$

and so the assertion follows. □

As a consequence of Theorem 1, we give the following approximation result:

Theorem 2 *Let ρ be a quasi convex, monotone, absolutely continuous, absolutely finite and τ-bounded modular on $L^\rho(Q)$. Let us suppose that ρ satisfies $(*)$. If $\varepsilon_n/n\to 0$ as $n\to+\infty$, then for every $f\in L^1(Q)\cap L^\rho(Q)$ we have:*

$$R_n(f,\cdot)\xrightarrow{\rho}\int_Q f(x)dx,\quad n\to+\infty.$$

Proof. The result follows from Theorem 1 and from the properties of the modulus of continuity ω_ρ, (see Theorem 2 in [3]). □

4. Examples

Here we discuss some examples of modular spaces for which the previous theory is applicable.

I. *Musielak-Orlicz spaces.* Let $\varphi : Q \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function such that the following assumptions hold.

(φ.1) $t \to \varphi(t, u)$ is measurable and integrable over Q, for every $u \in \mathbb{R}_0^+$.
(φ.2) $u \to \varphi(t, u)$ is continuous and nondecreasing for every $t \in Q$, $\varphi(t, 0) = 0$ and $\varphi(t, u) > 0$, for each $u \in \mathbb{R}_0^+$.

We will denote by Φ the class of all functions φ which satisfy (φ.1) and (φ.2). If $\varphi(t, u) = \widetilde{\varphi}(u)$, for every $t \in Q$ and $u \in \mathbb{R}_0^+$, then clearly (φ.1) is satisfied, and if $\widetilde{\varphi}$ is continuous and nondecreasing and $\widetilde{\varphi}(0) = 0$, $\widetilde{\varphi}(u) > 0$, for $u > 0$, then $\varphi \in \Phi$. So we will denote again by Φ the class of functions φ depending only on u such that (φ.2) is satisfied.

For $\varphi \in \Phi$, we define the modular:

$$I_\varphi(f) = \int_Q \varphi(t, |f(t)|)dt, \quad f \in L^0(Q).$$

We will denote by $L^\varphi(Q)$ the corresponding modular space.

We will say that $\varphi \in \Phi$ is M-quasi convex, with $M \geq 1$, if for any $g \in L^1(Q)$, we have:

$$\varphi\Big(t, \int_Q |g(s)|ds\Big) \leq M \int_Q \varphi\left(t, M|g(s)|\right) ds,$$

for any $t \in Q$, (see [9], [11]).

We will assume the following assumption: there is a constant $C' \geq 1$ such that

$$\varphi(nt - k, u) \leq \varphi(t, C'u) + \widetilde{\varepsilon}_{k,n}(t), \tag{3}$$

for every $n \in \mathbb{N}$, $k = 0, 1, \ldots, n-1$, and where $\widetilde{\varepsilon}_{k,n}$ is an integrable function, for every $n \in \mathbb{N}$, $k = 0, 1, \ldots, n-1$, (here we extend $\varphi(\cdot, u)$ 1-periodically outside Q).

If φ satisfies (3) we will denote by $\widetilde{\varepsilon}_n(t)$ the function

$$\widetilde{\varepsilon}_n(t) = n \sum_{k=0}^{n-1} \widetilde{\varepsilon}_{k,n}(t), \quad t \in Q.$$

If $\varphi \in \Phi$ satisfies (3), the corresponding modular I_φ satisfies condition ($*$). Indeed, with the notations of Section 2, we have

$$\begin{aligned}\sum_{k=0}^{n-1} I_\varphi\left[f\left(\frac{\cdot+k}{n}\right)\right] &= \sum_{k=0}^{n-1}\int_Q \varphi\left(y, \left|f\left(\frac{y+k}{n}\right)\right|\right)dy \\ &= \sum_{k=0}^{n-1} n\int_{Q_k} \varphi(nt-k, |f(t)|)dt \\ &\leq n\sum_{k=0}^{n-1}\int_{Q_k} \varphi(t, C'|f(t)|)dt + n\sum_{k=0}^{n-1}\int_{Q_k} \widetilde{\varepsilon}_{k,n}(t)dt \\ &= nI_\varphi(C'f) + \int_Q \widetilde{\varepsilon}_n(t)dt = nI\varphi(C'f) + \varepsilon_n,\end{aligned}$$

where $\varepsilon_n := \int_Q \widetilde{\varepsilon}_n(t)dt$.

Thus Theorem 1 can be applied to the modular I_φ whenever $\varphi \in \Phi$ is M-quasi convex and satisfies (3).

Let us assume now that φ is τ-bounded, i.e. there is $C \geq 1$ and a measurable funtion $F : Q \times Q \to \mathbb{R}_0^+$, such that:

$$\varphi(t-v, u) \leq \varphi(t, Cu) + F(t, v)$$

for every $t, v \in Q$, $u \in \mathbb{R}_0^+$ and F is such that $h(v) := \int_Q F(t,v)dt$, $v \in Q$, is a bounded function and $h(v) \to 0$ as $v \to 0^+$. Here φ and f are extended 1-periodically outside Q. If φ is τ-bounded, then the corresponding modular I_φ is also τ-bounded, according to the definition given in Section 2.

Hence if $\varphi \in \Phi$ is M-quasi convex, τ-bounded and satisfies (3), then we can apply Theorem 2 in order to obtain:

$$I_\varphi\left(\lambda\left(R_n(f,\cdot) - \int_Q f(x)dx\right)\right) \to 0, \quad n \to +\infty,$$

for a suitable $\lambda > 0$.

In particular if $\varphi(t,u) = \widetilde{\varphi}(u)$, φ is obviously τ-bounded, with constant $C = 1$ and $F \equiv 0$, and satisfies (3) with $C' = 1$ and $\widetilde{\varepsilon}_{k,n} \equiv 0$.

Thus, if $\varphi \in \Phi$ and $\varphi(t,u) = \widetilde{\varphi}(u)$ is M-quasi convex, then the previous theory can be also applied for classical Orlicz spaces.

II. Here we discuss a modular functional which hasn't an integral representation.

Let (Ω, Σ, μ) be a measure space and let $\varphi : \Omega \times Q \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a globally measurable function such that for any fixed $\xi \in \Omega$, the function $\varphi(\xi, \cdot, \cdot) \in \Phi$. We will suppose that the function $\sup_{\xi\in\Omega} \varphi(\xi, t, \cdot)$ is continuous at 0 for every $t \in Q$ and there is $\lambda_0 > 0$ such that $\sup\text{ess}_{\xi\in\Omega}\varphi(\xi,\cdot,\lambda_0) \in L^1(Q)$. By means of the previous assumptions, the functional

$$J_\varphi(f) = \sup\text{ess}_{\xi\in\Omega} \int_Q \varphi(\xi, t, |f(t)|)dt, \quad f \in L^0(Q),$$

is a finite, monotone modular on $L^0(Q)$.

Next we will assume that $\varphi(\xi, t, \cdot)$ is M-quasi convex for every $\xi \in \Omega$, $t \in Q$ and putting for any $k = 0, 1, \ldots, n-1, n \in \mathbb{N}$, $\eta \in \Omega$,

$$\varepsilon_{k,n}(\eta) = \sup\text{ess}_{\xi\in\Omega}\sup\text{ess}_{t\in Q}\sup_{u\geq 0}[\varphi(\xi, nt-k, u) - \varphi(\eta, t, C'u)],$$

for an absolute constant $C' \geq 1$, we suppose that the function $\varepsilon_{k,n}$ belongs to the space $L^\infty(\Omega)$. Under these assumptions we obtain, in particular, that:

$$\varphi(\xi, nt-k, u) \leq \varphi(\eta, t, C'u) + \varepsilon_{k,n}(\eta), \tag{4}$$

for every $\xi, \eta \in \Omega$, $n \in \mathbb{N}$, $k = 0, 1, \ldots, n-1$, $u \in \mathbb{R}_0^+$ and $t \in Q$. Then for $g \in L^0(Q)$,

$$\begin{aligned}\sum_{k=0}^{n-1} J_\varphi\left[g\left(\frac{\cdot+k}{n}\right)\right] &= \sum_{k=0}^{n-1}\sup\text{ess}_{\xi\in\Omega}\int_Q \varphi\left(\xi, t, \left|g\left(\frac{t+k}{n}\right)\right|\right)dt \\ &= \sum_{k=0}^{n-1}\sup\text{ess}_{\xi\in\Omega} n\int_{Q_k}\varphi(\xi, nt-k, |g(t)|)dt \\ &\leq n\sum_{k=0}^{n-1}\int_{Q_k}\varphi(\eta, t, C'|g(t)|)dt + n\sum_{k=0}^{n-1}\int_{Q_k}\varepsilon_{k,n}(\eta)dt \\ &\leq nJ_\varphi(C'g) + \sum_{k=0}^{n-1}\varepsilon_{k,n}(\eta).\end{aligned}$$

So, putting $\varepsilon_n := \sup\text{ess}_{\eta\in\Omega}\sum_{k=0}^{n-1}\varepsilon_{k,n}(\eta)$, we obtain condition $(*)$, and Theorem 1 is now applicable.

Now we will introduce some sufficient conditions in order to give a modular approximation result, by applying Theorem 2.

In order to do that, let $\varphi : \Omega \times Q \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with the assumptions introduced above. If, moreover, $\sup\text{ess}_{\xi\in\Omega}\varphi(\xi,\cdot,\lambda) \in L^1(Q)$ for any $\lambda > 0$

and for every $f \in L^0(Q)$ for which $J_\varphi(f) < +\infty$, we have that the function

$$H(\cdot) = \sup \mathrm{ess}_{\xi \in \Omega} \varphi(\xi, \cdot, |f(\cdot)|), \tag{5}$$

belongs to $L^1(Q)$, then J_φ is absolutely continuous and absolutely finite.

Next, let us suppose that

$$\varphi(\xi, t - v, u) \leq \varphi(\xi, t, Cu) + F(\xi, t, v) \tag{6}$$

for an absolute constant $C \geq 1$, for every $\xi \in \Omega$, $t, v \in Q$ and $u \in \mathbb{R}_0^+$, and where $F(\xi, \cdot, \cdot)$ is globally measurable and the function

$$h(v) = \sup \mathrm{ess}_{\xi \in \Omega} \int_Q F(\xi, t, v) dt, \quad v \in Q,$$

is in $L^\infty(Q)$ and $h(v) \to 0$ as $v \to 0^+$.

Then it is easy to show that the modular J_φ is τ-bounded. Hence under all the assumptions on φ, we can apply Theorem 2 in order to get the modular convergence of $R_n(f, \cdot)$ towards the integral $\int_Q f dt$.

Note that for functions $\varphi(\xi, t, u) \equiv \widetilde{\varphi}(\xi, u)$, condition (6) is obviously satisfied with $C = 1$ and $F \equiv 0$.

Particular cases of J_φ are modulars of the following forms:

$$J'_\varphi(f) = \sup_{m \in \mathbb{Z}} \int_Q \varphi(m, t, |f(t)|) dt$$

$$J''_\varphi(f) = \sup_{\xi \in [a,b[} \int_Q \varphi(\xi, t, |f(t)|) dt,$$

where $[a, b[\subset \mathbb{R}$ and $b \in \widetilde{\mathbb{R}}$.

III. *Modulars connected with strong summability*

Let W be an abstract set of indices and $\mathcal{W}$ be a filter of subsets of W. Let m be a measure on the interval $[a, b[\subset \mathbb{R}$, where b may also be equal to $+\infty$, defined on the σ-algebra of all Lebesgue measurable subsets of $[a, b[$. Let $\{a_w(\cdot)\}_{w \in W}$ be a family of Lebesgue measurable functions defined on $[a, b[$ with nonnegative values and such that the following conditions hold:

(a) $\int_a^b a_w(x) dm(x) \leq 1$, for $w \in W$.

(b) For any finite subset $F \subset W$, there is $\overline{w} \in W$ such that $a_{\overline{w}}(x) \geq a_w(x)$, for every $x \in [a, b[$, $w \in F$.

(c) If $0 \leq g(x) \nearrow s \in \widetilde{\mathbb{R}^+}$, then $\int_a^b a_w(s)g(s)dm \xrightarrow{\mathcal{W}} s$, where the symbol $\xrightarrow{\mathcal{W}}$ means convergence with respect to the filter $\mathcal{W}$.

(d) For every Lebesgue measurable subset $G \subset [a,b[$ of measure $m(G) > 0$, there is a measurable subset G_1, with $m(G_1) > 0$ and an index $\overline{w} \in W$ such that $a_{\overline{w}}(x) > 0$, m-almost everywhere.

Let $f \in L^0(Q)$ and let $\{\varphi(\xi,\cdot)\}_{\xi\in[a,b[}$ be a family of functions of Φ such that φ is Lebesgue measurable with respect to $\xi \in [a,b]$, for every $u \in \mathbb{R}_0^+$ and $\lim_{\xi\to b^-} \varphi(\xi,u) = \widetilde{\varphi}(u) < +\infty$, for every $u \geq 0$.

Putting

$$\mathcal{J}_\varphi(\xi,f) = \int_Q \varphi(\xi,|f(t)|)dt, \quad f \in L^0(Q),$$

we define the functional

$$\mathcal{A}_\varphi(f) = \sup_{w\in W} \int_a^b a_w(\xi)\mathcal{J}_\varphi(\xi,f)dm(\xi),$$

for any $f \in L^0(Q)$ such that $\mathcal{J}_\varphi(\cdot,f)$ is measurable in $[a,b[$. Under the above conditions on the family $\{a_w\}_{w\in W}$, $\mathcal{A}_\varphi$ is a monotone modular with assumption $(*)$. Indeed, we first remark that for any $n \in \mathbb{N}$, $g \in L^0(Q)$,

$$\begin{aligned}\sum_{k=0}^{n-1} \mathcal{J}_\varphi\left(\xi,\left|g\left(\frac{\cdot+k}{n}\right)\right|\right) &= \sum_{k=0}^{n-1} \int_Q \varphi\left(\xi,\left|g\left(\frac{t+k}{n}\right)\right|\right)dt \\ &= \sum_{k=0}^{n-1} n \int_{Q_k} \varphi(\xi,|g(t)|)dt = n\mathcal{J}_\varphi(\xi,g).\end{aligned}$$

Let now $\{\widetilde{\varepsilon}_n\}_n$ be a sequence of positive numbers such that $n\widetilde{\varepsilon}_n \to 0^+$. Then there are $w_{k,n} \in W$ such that

$$\begin{aligned}J &:= \sup_{w\in W} \int_a^b a_w(\xi)\mathcal{J}_\varphi\left(\xi,\left|g\left(\frac{\cdot+k}{n}\right)\right|\right)dm(\xi) \\ &\leq \int_a^b a_{w_{k,n}}(\xi)\mathcal{J}_\varphi\left(\xi,\left|g\left(\frac{\cdot+k}{n}\right)\right|\right)dm(\xi) + \widetilde{\varepsilon}_n.\end{aligned}$$

Now from (b) there is $w_n \in W$ such that $a_{w_{k,n}}(\xi) \leq a_{w_n}(\xi)$, for every $k = 0,1,\dots n-1$, and so

$$J \leq \int_a^b a_{w_n}(\xi)\mathcal{J}_\varphi\left(\xi,\left|g\left(\frac{\cdot+k}{n}\right)\right|\right)dm(\xi) + \widetilde{\varepsilon}_n.$$

Thus

$$\begin{aligned}
&\sum_{k=0}^{n-1} \mathcal{A}_\varphi\Big(g\Big(\frac{\cdot+k}{n}\Big)\Big) \\
&\qquad \le \sum_{k=0}^{n-1} \int_a^b a_{w_n}(\xi)\mathcal{J}_\varphi\Big(\xi, \Big|g\Big(\frac{\cdot+k}{n}\Big)\Big|\Big)dm(\xi) + n\widetilde{\varepsilon}_n \\
&\qquad = \int_a^b a_{w_n}(\xi) \sum_{k=0}^{n-1} \mathcal{J}_\varphi\Big(\xi, \Big|g\Big(\frac{\cdot+k}{n}\Big)\Big|\Big)dm(\xi) + n\widetilde{\varepsilon}_n \\
&\qquad = n\int_a^b a_{w_n}(\xi)\mathcal{J}_\varphi(\xi, g)dm(\xi) + n\widetilde{\varepsilon}_n \\
&\qquad \le n\mathcal{A}_\varphi(g) + \varepsilon_n,
\end{aligned}$$

where $\varepsilon_n := n\widetilde{\varepsilon}_n$. Hence $(*)$ is satisfied with $C' = 1$. Moreover if the function $\varphi(x, \cdot)$ is quasi convex with a constant $M \geq 1$, then it is easy to show that $\mathcal{A}_\varphi$ is also a quasi convex modular, with the same constant. Thus Theorem 1 is now applicable to the modular $\mathcal{A}_\varphi$.

Now, in order to apply also Theorem 2 to $\mathcal{A}_\varphi$, we discuss the τ-boundedness, absolute finiteness and absolute continuity of the modular $\mathcal{A}_\varphi$.

At first, by extending the function g outside Q with period 1, it is easy to show that the modular $\mathcal{A}_\varphi$ is translation invariant, i.e. $\mathcal{A}_\varphi(g(\cdot + v)) = \mathcal{A}_\varphi(g)$, for every $v \in Q$; this means that $\mathcal{A}_\varphi$ is τ-bounded with $C = 1$ and $h(v) \equiv 0$.

Absolute finiteness and absolute continuity of $\mathcal{A}_\varphi$ are studied in [1]. Here we report some sufficient conditions in order to obtain these properties (for further details see [1]).

In order to do that, we write $[a, b[= [a, c[\cup [c, b[$, for $c \in]a, b[$ and we shall make different assumptions on $\varphi(x, u)$ for $x \in [a, c[$ and $x \in [c, b[$.

Let us suppose that for every function $f \in L^0(Q)$ such that $\mathcal{A}_\varphi(f) < +\infty$ the function:

$$H(\cdot) = \sup_{y\in[a,c[} \varphi(y, |f(\cdot)|)$$

is integrable over Q. Moreover we assume that φ is of monotone type in $[c, b[$ i.e. there are two disjoint sets $R_1, R_2 \subset \mathbb{R}_0^+$ with $R_1 \cup R_2 = \mathbb{R}_0^+$, such that:

(a) $\varphi(x, u)$ is a nonincreasing function of $x \in [c, b[$, for every $u \in R_1$.

(b) $\varphi(x,u)$ is a nondecreasing function of $x \in [c,b[$, for every $u \in R_2$. Finally, we will assume that the family $\{\varphi(x,u)\}_{x\in[a,b[}$ is equicontinuous at $u=0$.

Then (see [1]) the modular $\mathcal{A}_\varphi$ is absolutely continuous and absolutely finite. Thus Theorem 2 can be applied to $\mathcal{A}_\varphi$ under the above assumptions.

In [1] there are described other interesting particular cases of $\mathcal{A}_\varphi$.

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