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Comments on the absolute convergence of Fourier series

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Abstract. We give sufficient conditions for the convergence of the series having the following form

$$\sum_{k=1}^{\infty} k^{\delta}(\varphi(|a_{n_k}|)+\varphi(|b_{n_k}|)),$$

where a_k and b_k are Fourier coefficients, $\delta \geq 0$, $\varphi(u)$ $(u \geq 0)$ is an increasing concave function, and $\{n_k\}$ is a certain increasing sequence of natural numbers.

Key words: absolute convergence, lacunary series, modulus of continuity, best approximation.

1. Introduction

Let $f(x)$ be a 2π-periodic integrable function and let

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

be its Fourier series.

It is well known that O. Szász [6] proved that *if* $f \in L^2(-\pi,,\pi)$ *and*

$$\sum_{n=1}^{\infty} n^{-1/2}\omega^{(2)}\left(f;\frac{1}{n}\right) < \infty,$$

then the Fourier series of f *converges absolutely, where*

$$\omega^{(2)}(f;\delta) := \sup_{0<h\leq\delta}\left(\int_{-\pi}^{\pi}|f(x+h)-f(x-h)|^2 dx\right)^{1/2}.$$

S.B. Stečkin [5] showed that *if* $\{m_n\}$ *is an arbitrary increasing sequence*

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of natural numbers, then

$$\sum_{n=1}^{\infty} n^{-1/2}\omega^{(2)}\left(f;\frac{1}{m_n}\right) < \infty$$

implies that

$$\sum_{n=1}^{\infty}(|a_{m_n}| + |b_{m_n}|) < \infty,$$

assuming that the Fourier series of f has the following form

$$\sum_{n=1}^{\infty}(a_{m_n}\cos m_n x + b_{m_n}\sin m_n x).$$

J.R. Patadia and V.M. Shah [4] studied such a special Fourier series having the following form:

$$\sum_{k=1}^{\infty}(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x), \tag{1.1}$$

where $\{n_k\}$ $(k \in \mathbb{N})$ is a strictly increasing sequence of natural numbers with an infinity of gaps (n_k, n_{k+1}) and satisfying the so-called condition B_2.

A strictly increasing sequence $\{n_k\}$ of natural numbers is said to satisfy the condition B_2 if $\sup_n P_2(n)$ is finite, where $P_2(n)$ denotes the number of different representations of an integer n in the form

$$n = \varepsilon_1 n_{k_1} + \varepsilon_2 n_{k_2} \ (\varepsilon_i = \pm 1;\ n_{k_i} \in \{n_k\}).$$

Patadia and Shah, among others, verified that if the sequence $\{n_k\}$ satisfies the condition B_2 and (1.1) is the Fourier series of f, then

$$\sum_{k=1}^{\infty}(|a_{n_k}| + |b_{n_k}|) < \infty \tag{1.2}$$

holds, that is, the Fourier series (1.1) converges absolutely even when the hypothesis in the Stečkin's theorem is satisfied only in a set E of positive measure.

Before recalling their results more precisely let us give two definitions.

Let $E \subset [-\pi, \pi]$ be a set of positive measure and $|E|$ be its measure.

Denote

$$\omega^{(2)}(f,E;\delta) := \sup_{0<h\le\delta}\left(\int_E |f(x+h)-f(x-h)|^2 dx\right)^{1/2}$$

and

$$E_m^{(2)}(f,E) := \inf_{T_m}\left(\int_E |f(x)-T_m(x)|^2 dx\right)^{1/2},$$

where $T_m(x)$ is a trigonometric polynomial of order not higher than m.

Theorem A *If* $0<\beta\le 1$ *and*

$$\sum_{k=1}^{\infty} k^{-\beta/2}(\omega^{(2)}(f,E;1/n_k))^{\beta} < \infty,$$

or

$$\sum_{k=1}^{\infty} k^{-\beta/2}(E_{n_k}^{(2)}(f,E))^{\beta} < \infty,$$

furthermore if $\{n_k\}$ *satisfies the condition* B_2, *then for the Fourier series* (1.1) *of* f

$$\sum_{k=1}^{\infty}(|a_{n_k}|^{\beta}+|b_{n_k}|^{\beta}) < \infty.$$

Very recently N. Ogata [3] has generalized the Theorem A such a way that he replaced the function x^{β}, β is appearing in Theorem A as an exponent, by a more general increasing concave function.

In the paper we shall generalize her theorem further multiplying by a factor k^{δ} ($\delta\ge 0$) both the terms of the conditions and the terms in the statement.

Since our theorem in the special case $\delta=0$ reduces to that of N. Ogata, we leave out citing her result.

Applying the idea used by N. Ogata we extend one of Konjuškov's theorems from the classical function x^{β} ($0<\beta\le 1$) to any increasing concave function, too.

2. Theorems

We prove the following theorems.

Theorem 1 *Let $\varphi(u)$ $(u \geq 0)$ be an increasing and concave function with $\varphi(0) = 0$ and let $E \subset [-\pi, \pi]$ be a set of positive measure. If a sequence $\{n_k\}$ of natural numbers satisfies the condition B_2, $\delta \geq 0$, and either*

$$\sum_{k=1}^{\infty} k^{\delta} \varphi(k^{-1/2} \omega^{(2)}(f, E; \pi/n_k) < \infty, \tag{2.1}$$

or

$$\sum_{k=1}^{\infty} k^{\delta} \varphi(k^{-1/2} E_{n_k}^{(2)}(f, E)) < \infty, \tag{2.2}$$

then for the Fourier series (1.1) of f

$$\sum_{k=1}^{\infty} k^{\delta} (\varphi(|a_{n_k}|) + \varphi(|b_{n_k}|)) < \infty \tag{2.3}$$

holds.

Theorem 1 in the special case $\delta = 0$ includes the mentioned theorem of N. Ogata. To be precise we note that she did not considere the condition (2.2), but her proof shows the sufficiency of (2.2) as well.

The next theorem will be a generalization of a well-known theorem of Konjuškov [1, Theorem 11].

Theorem 2 *Let $1 < p \leq 2$, $f \in L^p(0, 2\pi)$ and $\delta \geq 0$. If $\varphi(u)$ $(u \geq 0)$ is an increasing concave function with $\varphi(0) = 0$, and*

$$\sum_{n=1}^{\infty} n^{\delta} \varphi\left(n^{\frac{1}{p}-1} E_n(p)\right) < \infty, \tag{2.4}$$

then

$$\sum n^{\delta} (\varphi(|a_n|) + \varphi(|b_n|)) < \infty, \tag{2.5}$$

where $E_n(p)$ denotes the best approximation of f by trigonometric polynomials of order at most n in the space $L^p(0, 2\pi)$.

Theorem 2 in the special case $\varphi(x) = x^{\beta}$ $(0 < \beta \leq 1)$ reduces to the mentioned Konjuškov's theorem. The reader can find a different generalization of Konjuškov's theorem in [2, Satz V], where n^δ is replaced by an arbitrary nonnegative ω_n.

It seems to be an intricate problem to give a sharp sufficient condition for the convergence of the series

$$\sum_{n=1}^{\infty} \omega_n(\varphi(|a_n|) + \varphi(|b_n|)) < \infty,$$

where $\{\omega_n\}$ is an arbitrary sequence of nonnegative numbers.

3. Lemmas

The first two lemmas are certain generalizations used by N. Ogata in her work.

Lemma 1 *Let $\delta \geq 0$, $k \geq 1$ and $m \geq \delta + 1$, where k and m are natural numbers. Then the following inequalities*

$$k^\delta m 2^{1-m} \leq \sum_{j=k^m}^{(k+1)^m-1} j^{\frac{1+\delta}{m}-1} \leq k^\delta m 2^{m-1} \tag{3.1}$$

hold.

Proof of Lemma 1. By the mean value theorem, there exists c with $k < c < k+1$ such that the following two inequalities hold respectively;

$$\begin{aligned} \sigma(k,m;\delta) &:= \sum_{j=k^m}^{(k+1)^m-1} j^{\frac{1+\delta}{m}-1} \geq ((k+1)^m - k^m)(k+1)^{1+\delta-m} \\ &= mc^{m-1}(k+1)^{1+\delta-m} \geq m\left(\frac{k}{k+1}\right)^{m-1} k^\delta \end{aligned}$$

and

$$\begin{aligned} \sigma(k,m;\delta) &\leq ((k+1)^m - k^m)k^{1+\delta-m} = mc^{m-1}k^{1+\delta-m} \\ &\leq m\left(\frac{k+1}{k}\right)^{m-1} k^\delta. \end{aligned}$$

These two inequalities clearly imply (3.1) as stated.

In the sequel $[\alpha]$ will denote the integer part of α. □

Lemma 2 *Let $1 < p \leq 2$, $\delta \geq 0$ and m be an arbitrary natural number.*

Furthermore let $\{\alpha_n\}$ be a monotone nonincreasing sequence of nonnegative numbers and let $\varphi(u)$ $(u \ge 0,\ \varphi(0) = 0)$ be an increasing concave function. Then the conditions

$$\sigma(m,\delta) := \sum_{k=1}^{\infty} k^{\frac{1+\delta}{m}-1} \varphi\left(k^{\frac{1-p}{pm}} \alpha_{[k^{1/m}]}\right) < \infty \tag{3.2}$$

and

$$\sigma(\delta) := \sum_{k=1}^{\infty} k^{\delta} \varphi\left(k^{\frac{1-p}{p}} \alpha_k\right) < \infty \tag{3.3}$$

are equivalent.

Proof of Lemma 2. First we show that (3.2) ⇒ (3.3). Taking into account the first inequality in (3.1), the monotonicity of φ and α_n, an easy calculation shows that

$$\begin{aligned} \sigma(m,\delta) &= \sum_{k=1}^{\infty} \sum_{j=k^m}^{(k+1)^m-1} j^{\frac{1+\delta}{m}-1} \varphi\left(j^{\frac{1-p}{pm}} \alpha_{[j^{1/m}]}\right) \\ &\ge \sum_{k=1}^{\infty} \varphi\left((k+1)^{\frac{1-p}{p}} \alpha_{k+1}\right) \sum_{j=k^m}^{(k+1)^m-1} j^{\frac{1+\delta}{m}-1} \\ &\ge 2^{-m} \sum_{k=1}^{\infty} k^{\delta} \varphi\left((k+1)^{\frac{1-p}{p}} \alpha_{k+1}\right) \\ &\ge 2^{-m-\delta} \sum_{k=2}^{\infty} k^{\delta} \varphi\left(k^{\frac{1-p}{p}} \alpha_k\right). \end{aligned} \tag{3.4}$$

Hence the implication (3.2) ⇒ (3.3) follows obviously.

The proof of (3.3) ⇒ (3.2) runs similarly. Using the first equality in (3.4), the monotonicity assumptions and the second inequality in (3.1), we get immediately that

$$\sigma(m,\delta) \le m2^m \sum_{k=1}^{\infty} \varphi\left(k^{\frac{1-p}{p}} \alpha_k\right) k^{\delta} \equiv m2^m \sigma(\delta).$$

The proof of Lemma 2 is complete. □

In the sequel K, K_i will denote some positive constants, not necessarily the same one in every occurrence, furthermore $K(\cdot)$ will denote a constant depending only on those parameters as indicated.

Lemma 3 [4; Lemma 2] *Let $f \in L^2$ with the special Fourier series* (1.1), *and let $E \subset [-\pi, \pi]$ be a set of positive measure. Let the sequence $\{n_k\}$ of natural numbers satisfy the condition B_2. Then, if ℓ is large enough, the following inequalities*

$$\sum_{k=\ell}^{\infty}(a_{n_k}^2 + b_{n_k}^2) \le \frac{K}{|E|}\omega^{(2)}\left(f, E; \frac{\pi}{n_\ell}\right)^2 \tag{3.5}$$

and

$$\sum_{k=\ell}^{\infty}(a_{n_k}^2 + b_{n_k}^2) \le \frac{K}{|E|}(E_{n_\ell}^{(2)}(f, E))^2 \tag{3.6}$$

hold.

Lemma 4 (Jensen's inequality) *Let $\varphi(u)$ $(u \ge 0,\ \varphi(0) = 0)$ be an increasing concave function. Then, for any infinite sequence of nonnegative numbers $x_1, x_2, \dots, x_n, \dots$ and any infinite sequence of positive numbers $p_1, p_2, \dots, p_n, \dots$, the following inequality*

$$\frac{\sum_{k=1}^{\infty} p_k \varphi(x_k)}{\sum_{k=1}^{\infty} p_k} \le \varphi\left(\frac{\sum_{k=1}^{\infty} p_k x_k}{\sum_{k=1}^{\infty} p_k}\right)$$

holds assuming that each series in the above inequality converges.

Lemma 5 [7] *Let $1 < p \le 2$ and $p' := \frac{p}{p-1}$. Then the following inequality*

$$\sum_{k=n}^{\infty}(a_k^2 + b_k^2)^{p'/2} \le K(E_n(p))^{p'}$$

holds for all n, where K is a constant independent of n.

4. Proof of the theorems

Proof of Theorem 1. In order to simplify writing, in the sequel, we shall write only $k^{1/m}$ instead of $[k^{1/m}]$. Denote

$$\rho_n := (a_n^2 + b_n^2)^{1/2},$$

and let $m > \delta + 1$. Then

$$\sum_{j=1}^{\infty} j^{\delta}\varphi(\rho_{n_j}) = \sum_{j=1}^{\infty}\sum_{k=1}^{j^m} j^{\delta-m}\varphi(\rho_{n_j})$$
$$\le \sum_{k=1}^{\infty}\sum_{j=k^{1/m}}^{\infty} j^{\delta-m}\varphi(\rho_{n_j}) =: S_1.$$

Next using Lemma 4 and the Cauchy inequality we get that

$$S_1 \le \sum_{k=1}^{\infty}\Bigg(\sum_{j=k^{1/m}}^{\infty} j^{\delta-m}\Bigg)\varphi\Bigg(\Bigg\{\sum_{j=k^{1/m}}^{\infty} j^{\delta-m}\Bigg\}^{-1}\sum_{j=k^{1/m}}^{\infty} j^{\delta-m}\rho_{n_j}\Bigg)$$
$$\le K(\delta,m)\sum_{k=1}^{\infty} k^{\frac{1+\delta-m}{m}}\varphi\Bigg(K(\delta,m)k^{\frac{m-\delta-1}{m}}$$
$$\times\Bigg\{\sum_{j=k^{1/m}}^{\infty} j^{2(\delta-m)}\Bigg\}^{1/2}\Bigg\{\sum_{j=k^{1/m}}^{\infty}\rho_{n_j}^2\Bigg\}^{1/2}\Bigg) =: S_2.$$

To estimate S_2 we utilize Lemma 3 with (3.5) and we obtain that

$$S_2 \le K(\delta,m,|E|,\varphi)\sum_{k=1}^{\infty} k^{\frac{1+\delta}{m}-1}\varphi\left(k^{-\frac{1}{2m}}\omega^{(2)}\left(f,E;\frac{\pi}{n_{k^{1/m}}}\right)\right).$$

Finally the last sum, by Lemma 2 with $p = 2$ and $\alpha_i = \omega^{(2)}\left(f,E,\frac{\pi}{n_i}\right)$, is finite if and only if

$$\sum_{k=1}^{\infty} k^{\delta}\varphi\left(k^{-\frac{1}{2}}\omega^{(2)}\left(f,E;\frac{\pi}{n_k}\right)\right) < \infty.$$

Collecting our fractional estimates we obtain that (2.1) implies that

$$\sum_{j=1}^{\infty} j^{\delta}\varphi(\rho_{n_j}) < \infty, \tag{4.1}$$

and the inequality (4.1) is clearly equivalent to (2.3).

If we use Lemma 3 with (3.6) in order to estimate the sum S_2, and follow the same way as before we can verify that the condition (2.2) also implies the statement (2.3).

The proof of Theorem 1 is ended. □

Proof of Theorem 2. First we show that already the condition

$$\sum_{n=1}^{\infty} n^{\delta}\varphi\left(\left\{\frac{1}{n}\sum_{k=n}^{\infty}\rho_k^{p'}\right\}^{1/p'}\right) < \infty \qquad \left(p' = \frac{p}{p-1}\right) \tag{4.2}$$

implies that

$$S_3 := \sum_{n=1}^{\infty} n^{\delta}\varphi(\rho_n) < \infty, \tag{4.3}$$

whence (2.5) clearly follows.

Let $m > \delta + 1$. A similar consideration as in the proof of Theorem 1, first an Abel rearrangement, followed by a Jensen and Hölder inequality, gives that

$$\begin{aligned} S_3 &:= \sum_{n=1}^{\infty}\sum_{k=1}^{n^m} n^{\delta-m}\varphi(\rho_n) \\ &\le \sum_{k=1}^{\infty}\sum_{n=k^{1/m}}^{\infty} n^{\delta-m}\varphi(\rho_n) \\ &\le \sum_{k=1}^{\infty}\Bigg(\sum_{n=k^{1/m}}^{\infty} n^{\delta-m}\Bigg)\varphi\Bigg(\Bigg\{\sum_{n=k^{1/m}}^{\infty} n^{\delta-m}\Bigg\}^{-1}\sum_{n=k^{1/m}}^{\infty} n^{\delta-m}\rho_n\Bigg) \\ &\le K(\delta,m,\varphi)\sum_{k=1}^{\infty} k^{\frac{1+\delta}{m}-1} \\ &\qquad\qquad \varphi\Bigg(k^{1-\frac{1+\delta}{m}}\Bigg(\sum_{n=k^{1/m}}^{\infty}\rho_n^{p'}\Bigg)^{1/p'}\Bigg(\sum_{n=k^{1/m}}^{\infty} n^{(\delta-m)p}\Bigg)^{1/p}\Bigg) \\ &\le K_1(\delta,m,\varphi)\sum_{k=1}^{\infty} k^{\frac{1+\delta}{m}-1}\varphi\Bigg(k^{\frac{1-p}{pm}}\Bigg(\sum_{n=k^{1/m}}^{\infty}\rho_k^{p'}\Bigg)^{1/p'}\Bigg) =: S_4. \end{aligned}$$

To estimate the sum S_4 we use Lemma 2 with $\alpha_k = \left\{\sum_{n=k}^{\infty}\rho_n^{p'}\right\}^{1/p'}$, whence we get that $S_4 < \infty$ if and only if the inequality (4.2) holds, namely $\frac{1-p}{p} = -\frac{1}{p'}$.

Herewith we have verified that (4.2) implies (4.3), and as stated, thus the implication (4.2) $\Rightarrow$ (2.5) also holds.

Finally, on account of the hitherto obtained result, Lemma 5 conveys the statement of Theorem 2. □

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