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Hilbert schemes and cyclic quotient surface singularities

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Abstract. Let G be a finite cyclic subgroup of $GL(2, \mathbf{C})$ of order n which contains no reflections. Let $\mathbf{A}^2$ be the complex affine plane. We consider a certain subscheme $\mathrm{Hilb}^G(\mathbf{A}^2)$ of $\mathrm{Hilb}^n(\mathbf{A}^2)$ consisting of G-invariant zero-dimensional subschemes of length n. We describe the structure of $\mathrm{Hilb}^G(\mathbf{A}^2)$ and prove this is the minimal resolution of the quotient surface singularity $\mathbf{A}^2/G$.

Key words: Hilbert scheme, cyclic, quotient singularities, resolution.

Introduction

Let $\mathbf{A}^2$ be the complex affine plane. Let $S^n(\mathbf{A}^2)$ be the nth symmetric product of $\mathbf{A}^2$, and $\mathrm{Hilb}^n(\mathbf{A}^2)$ the Hilbert scheme parametrizing all zero-dimensional subschemes of $\mathbf{A}^2$ of length n. By the natural morphism π : $\mathrm{Hilb}^n(\mathbf{A}^2) \to S^n(\mathbf{A}^2)$ called Hilbert-Chow morphism $\mathrm{Hilb}^n(\mathbf{A}^2)$ is a crepant resolution of $S^n(\mathbf{A}^2)$.

Let G be a small finite subgroup of $GL(2, \mathbf{C})$, that is, G is a finite subgroup of $GL(2, \mathbf{C})$ which contains no reflections. Then G acts on $\mathbf{A}^2$, hence it acts both $\mathrm{Hilb}^n(\mathbf{A}^2)$ and $S^n(\mathbf{A}^2)$ so that the Hilbert-Chow morphism is G-equivariant. Assume that n equals the order of G. Then the G-fixed point set of $S^n(\mathbf{A}^2)$ is isomorphic to the quotient space $\mathbf{A}^2/G$. Hence we see readily that there is a unique irreducible component of G-fixed point set in $\mathrm{Hilb}^n(\mathbf{A}^2)$ dominating $\mathbf{A}^2/G$, which we denote by $\mathrm{Hilb}^G(\mathbf{A}^2)$. For finite subgroups G of $SL(2, \mathbf{C})$, Ito and Nakamura proved in [IN96] [IN98] that $\mathrm{Hilb}^G(\mathbf{A}^2)$ is the minimal resolution of the simple singularity $\mathbf{A}^2/G$. Using this realization of the minimal resolution of $\mathbf{A}^2/G$, they also gave an explanation to the so-called McKay observation, though in part. Nakamura conjectured that $\mathrm{Hilb}^G(\mathbf{A}^3)$ is a crepant resolution of $\mathbf{A}^3/G$ for any finite subgroup G of $SL(3, \mathbf{C})$ and proved it for an Abelian group in [N]. Recentry the conjecture proved by Bridgeland, King and Reid in [BKR].

When G is a small finite cyclic subgroup of $GL(2, \mathbf{C})$ we call the germ

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of the quotient singularity $\mathbf{A}^2/G$ at the origin a cyclic quotient surface singularity. In the present article we prove that $\mathrm{Hilb}^G(\mathbf{A}^2)$ is the minimal resolution of the cyclic quotient surface singularity $\mathbf{A}^2/G$ and describe the structure of $\mathrm{Hilb}^G(\mathbf{A}^2)$ in detail.

In Section 1 we give some preparatory lemmas on continued fractions. In Section 2 we recall toric resolutions of cyclic quotient surface singularities. We recall some basic facts on $\mathrm{Hilb}^G(\mathbf{A}^2)$ in Section 3. We present our main theorem in Section 4 and 5.

1. Continued Fractions

Let n and ℓ are positive integers such that $1 \le \ell < n$ and $\gcd(n,\ell) = 1$. In this section we consider the modified continued fractions of $\frac{n}{\ell}$ and $\frac{n}{n-\ell}$. Let

$$\frac{n}{\ell} = [[b_1, b_2, b_3, \ldots, b_r]] := b_1 - \frac{1|}{|b_2} - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_r} \quad (b_\mu \ge 2)$$

$$\frac{n}{n-\ell} = [[a_1, a_2, a_3, \ldots, a_e]] \quad (a_\nu \ge 2) \tag{1.1}$$

be the Hirzebruch-Jung continued fractions. Then we define triples (i_μ, j_μ, k_μ) $(\mu = 0, 1, \ldots, r+1)$ and $(\alpha_\nu, \beta_\nu, \gamma_\nu)$ $(\nu = 0, 1, \ldots, e+1)$ of nonnegative integers as follows:

$$\begin{cases} (i_0, j_0, k_0) := (n, 0, 1), \quad (i_1, j_1, k_1) := (\ell, 1, 1), \\ (i_{\mu+1}, j_{\mu+1}, k_{\mu+1}) := b_\mu(i_\mu, j_\mu, k_\mu) - (i_{\mu-1}, j_{\mu-1}, k_{\mu-1}), \end{cases} \tag{1.2}$$

$$\begin{cases} (\alpha_0, \beta_0, \gamma_0) := (n, 0, 1), \quad (\alpha_1, \beta_1, \gamma_1) := (n-\ell, 1, 1), \\ (\alpha_{\nu+1}, \beta_{\nu+1}, \gamma_{\nu+1}) := a_\nu(\alpha_\nu, \beta_\nu, \gamma_\nu) - (\alpha_{\nu-1}, \beta_{\nu-1}, \gamma_{\nu-1}). \end{cases}$$

Then it is easy to see by $a_\nu, b_\mu \ge 2$ that

$$\begin{cases} i_0 > i_1 > \cdots > i_{r+1} = 0, \\ j_0 < j_1 < \cdots < j_{r+1} = n, \\ k_0 \le k_1 \le \cdots \le k_{r+1} = n - \ell, \end{cases} \qquad \begin{cases} \alpha_0 > \alpha_1 > \cdots > \alpha_{e+1} = 0, \\ \beta_0 < \beta_1 < \cdots < \beta_{e+1} = n, \\ \gamma_0 \le \gamma_1 \le \cdots \le \gamma_{e+1} = \ell. \end{cases}$$

By induction on μ and ν we get

$$\begin{cases} i_\mu + (n-\ell)j_\mu = nk_\mu, \\ i_{\mu-1}j_\mu - i_\mu j_{\mu-1} = n, \\ k_{\mu-1}j_\mu - k_\mu j_{\mu-1} = 1, \end{cases} \qquad \begin{cases} \alpha_\nu + \ell\beta_\nu = n\gamma_\nu, \\ \alpha_{\nu-1}\beta_\nu - \alpha_\nu\beta_{\nu-1} = n, \\ \gamma_{\nu-1}\beta_\nu - \gamma_\nu\beta_{\nu-1} = 1. \end{cases} \tag{1.3}$$

Next we investigate the relations between (i_μ, j_μ, k_μ) and $(\alpha_\nu, \beta_\nu, \gamma_\nu)$. First we review a lemma from Riemenschneider [R74, Lemma 3].

Lemma 1.1 *Let $\frac{n}{\ell} = [[b_1, b_2, \dots, b_r]]$ and $\frac{n_1}{\ell_1} := [[b_2, b_3, \dots, b_r]]$. Suppose $\frac{n_1}{n_1-\ell_1} = [[a_2, a_3, \dots, a_e]]$. Then we have*

$$\frac{n}{n-\ell} = [[\underbrace{2, \dots, 2}_{b_1-2}, a_2+1, a_3, \dots, a_e]].$$

Proof. We prove this by induction on the first term b_1 of n/ℓ. Assume $b_1 = 2$. Then we have $\frac{n}{\ell} = 2 - \frac{\ell_1}{n_1} = \frac{2n_1-\ell_1}{n_1}$. On the other hand, $\frac{n}{n-\ell} - 1 = \frac{n_1}{n_1-\ell_1}$. It follows that $\frac{n}{n-\ell} = [[a_2+1, a_3, \dots, a_e]]$. This prove the lemma in this case.

Next we consider the case $b_1 \geq 3$. Let $\frac{n}{\ell} = [[b_1, b_2, \dots, b_r]]$ and $n_1/\ell_1 = [[b_2, b_3, \dots, b_r]]$. Let $\frac{n'}{\ell'} := \frac{n-\ell}{\ell}$. Then we have $\frac{n'}{\ell'} = [[b_1 - 1, b_2, \dots, b_r]]$. We put $\frac{n'_1}{\ell'_1} := [[b_2, b_3, \dots, b_r]]$ and suppose $\frac{n'_1}{n'_1-\ell'_1} = [[a_2, a_3, \dots, a_e]]$. Then by the induction hypothesis we have

$$\frac{n'}{n'-\ell'} = [[\underbrace{2, \dots, 2}_{b_1-3}, a_2+1, a_3, \dots, a_e]].$$

It follows from $\frac{n}{n-\ell} = \frac{n'+\ell'}{n'} = 2 - \frac{n'-\ell'}{n'}$ that

$$\frac{n}{n-\ell} = [[\underbrace{2, \dots, 2}_{b_1-2}, a_2+1, a_3, \dots, a_e]]$$

where a_i's are the terms of $\frac{n'_1}{n'_1-\ell'_1}$. Since $\frac{n'_1}{\ell'_1} = \frac{n}{\ell}$ and $\frac{n'_1}{n'_1-\ell'_1} = \frac{n_1}{n_1-\ell_1}$, the lemma holds for $\frac{n}{\ell}$ and $\frac{n_1}{n_1-\ell_1}$. □

Proposition 1.2 *There is a duality between the continued fraction expansions of $\frac{n}{\ell}$ and $\frac{n}{n-\ell}$. To be more precise there are positive integers c_i and*

d_i such that

$$\frac{n}{\ell}=[[d_1+1,\underbrace{2,\dots,2}_{c_1-1},d_2+2,\dots,d_{m-1}+2,\underbrace{2,\dots,2}_{c_{m-1}-1},d_m+2,\underbrace{2,\dots,2}_{c_m-1}]],$$

$$\frac{n}{n-\ell}=[[\underbrace{2,\dots,2}_{d_1-1},c_1+2,\underbrace{2,\dots,2}_{d_2-1},c_2+2,\dots,c_{m-1}+2,\underbrace{2,\dots,2}_{d_m-1},c_m+1]].$$

Proof. We prove the lemma by induction on the length of the continued fraction $\frac{n}{\ell}$. If the length of the continued fraction equals one, we see $m=1$, $c_1=1$ and $\frac{n}{\ell}=d_1+1$. Then $\frac{n}{n-\ell}=\frac{d_1+1}{d_1}=[[\underbrace{2,\dots,2}_{d_1}]]$.

Now we consider the general case. For $\frac{n}{\ell}$ as in proposition we define

$$\frac{n_1}{\ell_1}:=[[\underbrace{2,\dots,2}_{c_1-1},d_2+2,\dots,d_{m-1}+2,\underbrace{2,\dots,2}_{c_{m-1}-1},d_m+2,\underbrace{2,\dots,2}_{c_m-1}]].$$

By the induction hypothesis, we have

$$\frac{n_1}{n_1-\ell_1}=[[c_1+1,\underbrace{2,\dots,2}_{d_2-1},c_2+2,\dots,c_{m-1}+2,\underbrace{2,\dots,2}_{d_m-1},c_m+1]].$$

Then by Lemma 1.1

$$\frac{n}{n-\ell}=[[\underbrace{2,\dots,2}_{d_1-1},c_1+2,\underbrace{2,\dots,2}_{d_2-1},c_2+2,\dots,c_{m-1}+2,\underbrace{2,\dots,2}_{d_m-1},c_m+1]].$$

□

Notation Let the modified continued fraction of $\frac{n}{\ell}$ be

$$\frac{n}{\ell}=[[d_1+1,\underbrace{2,\dots,2}_{c_1-1},d_2+2,\dots,d_{m-1}+2,\underbrace{2,\dots,2}_{c_{m-1}-1},d_m+2,\underbrace{2,\dots,2}_{c_m-1}]].$$

We define

$$\mu(\lambda):=1+\sum_{j=0}^{\lambda}c_j,\quad \nu(\lambda):=\sum_{j=0}^{\lambda}d_j,\quad (\lambda=0,1,\dots,m),$$

where $c_0:=0$, $d_0:=0$. And we define positive integers by

$$\begin{aligned} &i(\mu):=i_{\mu-1}-i_\mu, && j(\mu):=j_\mu-j_{\mu-1}, && (1\le\mu\le\mu(m)),\\ &\alpha(\nu):=\alpha_{\nu-1}-\alpha_\nu, && \beta(\nu):=\beta_\nu-\beta_{\nu-1}, && (1\le\nu\le\nu(m)+1). \end{aligned} \tag{1.4}$$

Proposition 1.3
(i) $i(1) = \alpha_1$, $j(1) = \beta_1$,
(ii) $i(\mu) = \alpha_{\nu(\lambda+1)}$, $j(\mu) = \beta_{\nu(\lambda+1)}$ *for* $0 \le \lambda \le m-1$ *and* $\mu(\lambda)+1 \le \mu \le \mu(\lambda+1)$,
(iii) $\alpha(\nu(m)+1) = i_{\mu(m)-1}$, $\beta(\nu(m)+1) = j_{\mu(m)-1}$,
(iv) $\alpha(\nu) = i_{\mu(\lambda)}$, $\beta(\nu) = j_{\mu(\lambda)}$ *for* $0 \le \lambda \le m-1$ *and* $\nu(\lambda)+1 \le \nu \le \nu(\lambda+1)$.

Proof. We put $r := \mu(m)$, $e := \nu(m)+1$. We write $\frac{n}{\ell} = [[b_1, b_2, \dots, b_r]]$ and $\frac{n}{n-\ell} = [[a_1, a_2, \dots, a_e]]$ for simplicity. Then we have

$$\begin{aligned} i(1) &= i_0 - i_1 = n - \ell = \alpha_1, \\ i(\mu+1) &= i_\mu - (b_\mu i_\mu - i_{\mu-1}) \\ &= i_{\mu-1} - i_\mu - (b_\mu - 2) i_\mu \\ &= i(\mu) - (b_\mu - 2) i_\mu \quad \text{for} \quad \mu \ge 1. \end{aligned}$$

In the same way we see $\alpha(1) = i_1$ and $\alpha(\nu+1) = \alpha(\nu) - (a_\nu - 2)\alpha_\nu$ $(\nu \ge 1)$.

Similarly we have $j(1) = 1 = \beta_1$, $j(\mu+1) = j(\mu) + (b_\mu - 2) j_\mu$, $\beta(1) = j_1$ and $\beta(\nu+1) = \beta(\nu) + (a_\nu - 2)\beta_\nu$. Therefore by Proposition 1.2

$$\begin{aligned} &i(\mu) = i(\mu(\lambda)+1), \quad j(\mu) = j(\mu(\lambda)+1) \\ &\qquad\qquad \text{for} \quad \mu(\lambda)+1 \le \mu \le \mu(\lambda+1), \\ &\alpha(\nu) = \alpha(\nu(\lambda)+1), \quad \beta(\nu) = \beta(\nu(\lambda)+1) \\ &\qquad\qquad \text{for} \quad \nu(\lambda)+1 \le \nu \le \nu(\lambda+1). \end{aligned}$$

By definition $\alpha(\nu(0)+1) = \alpha(1) = i_1$, $\beta(\nu(0)+1) = \beta(1) = j_1$. Then

$$\begin{aligned} i(\mu(0)+1) &= i(2) = i(1) - (d_1 - 1) i_1 \\ &= \alpha_1 - \{\alpha(2) + \alpha(3) + \cdots + \alpha(\nu(1))\} = \alpha_{\nu(1)}, \\ j(\mu(0)+1) &= j(2) = j(1) + (d_1 - 1) j_1 \\ &= \beta_1 + \{\beta(2) + \beta(3) + \cdots + \beta(\nu(1))\} = \beta_{\nu(1)}. \end{aligned}$$

We suppose that (i)–(iv) hold for $\mu \le \mu(\lambda)$ and $\nu \le \nu(\lambda)$. Assume first $\lambda < m$. Then we have

$$\begin{aligned} \alpha(\nu(\lambda)+1) &= \alpha(\nu(\lambda)) - c_\lambda \alpha_{\nu(\lambda)} \\ &= i_{\mu(\lambda-1)} - \{i(\mu(\lambda-1)+1) + \cdots + i(\mu(\lambda)-1) + i(\mu(\lambda))\} \\ &= i_{\mu(\lambda)}. \end{aligned}$$

Assume next $\lambda = m$. Then we have

$$\begin{aligned}\alpha(\nu(m)+1) &= \alpha(\nu(m)) - (c_m - 1)\alpha_{\nu(m)} \\ &= i_{\mu(m-1)} - \{i(\mu(m-1)+1) + \cdots + i(\mu(m)-1)\} \\ &= i_{\mu(m)-1}.\end{aligned}$$

Similarly we see

$$\begin{aligned}i(\mu(\lambda)+1) &= i(\mu(\lambda)) - d_{\lambda+1} i_{\mu(\lambda)} \\ &= \alpha_{\nu(\lambda)} - \{\alpha(\nu(\lambda)+1) + \alpha(\nu(\lambda)+2) + \cdots + \alpha(\nu(\lambda+1))\} \\ &= \alpha_{\nu(\lambda+1)}.\end{aligned}$$

Similarly we can also prove the assertions for $\beta(\nu)$ and $j(\mu)$. □

2. Cyclic Quotient Singularities

The isomorphism classes of cyclic quotient surface singularities are in one-to-one correspondence to the conjugacy classes of small finite cyclic subgroups of $GL(2, \mathbf{C})$. Up to conjugacy we may assume that any small abelian subgroup of $GL(2, \mathbf{C})$ is generated by $\sigma := \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^\ell \end{pmatrix}$ where ζ is a primitive n-th root of unity and ℓ is a positive integer such that $1 \le \ell < n$ and $\gcd(n, \ell) = 1$. We denote the group by $C_{n,\ell}$. Let (x, y) be a coordinate system of the complex affine space $\mathbf{A}^2$. Then $C_{n,\ell}$ operates upon $\mathbf{A}^2$ from the right by $(x, y) \to (x, y)g$ $(g \in C_{n,\ell})$. We denote the quotient space $\mathbf{A}^2/C_{n,\ell}$ by $A_{n,\ell}$. We remark that two germs $(A_{n,\ell}, 0)$ and $(A_{n',\ell'}, 0)$ are equivalent if and only if $n = n'$ and $\ell = \ell'$ or $\ell\ell' \equiv 1 \pmod{n}$ ([B]), if and only if $A_{n,\ell} \simeq A_{n',\ell'}$.

In what follows we put $G := C_{n,\ell}$ for simplicity. The quotient space $A_{n,\ell}$ and its minimal resolution are in fact torus embeddings as we see below.

Proposition 2.1 *Let $N \simeq \mathbf{Z}^2$ be a free abelian group of rank 2 with a basis e_1 and e_2 and $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. Let $\tau := \langle ne_1 + (n-\ell)e_2, e_2\rangle \subset N \otimes_{\mathbf{Z}} \mathbf{R}$ be the cone in $N \otimes \mathbf{R}$ generated by $ne_1 + (n-\ell)e_2$ and e_2. Then*

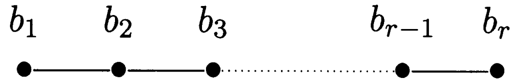
$$A_{n,\ell} \simeq X_\tau := \mathrm{Spec}\, \mathbf{C}[\check{\tau} \cap M] = \mathrm{Spec}\, \mathbf{C}[x, y]^G.$$

Proof. Let $\{f_1, f_2\}$ be the dual basis of M such that $\langle e_i, f_j\rangle = \delta_{ij}$. We put $N^* := (n\mathbf{Z})e_1 \oplus \mathbf{Z}e_2$, $M^* := (\frac{1}{n}\mathbf{Z})f_1 \oplus \mathbf{Z}f_2 = \mathrm{Hom}_{\mathbf{Z}}(N^*, \mathbf{Z})$. Since $\check{\tau} = \langle f_1, f_2 - \frac{n-\ell}{n} f_1\rangle$, $\mathrm{Spec}\, \mathbf{C}[\check{\tau} \cap M^*] = \mathrm{Spec}\, \mathbf{C}[x, y] \simeq \mathbf{A}^2$ where $x := \mathbf{e}(\frac{1}{n} f_1)$

$y := \mathbf{e}(f_2 - \frac{n-\ell}{n} f_1)$ and $\mathbf{e}(*) := \exp(2\pi\sqrt{-1}\ *)$. We define a symmetric pairing $f : M^*/M \times N/N^* \to \mu_n$ by $f(\bar{a}, \bar{b}) := \zeta^{n\langle a,b\rangle}$ where μ_n is a cyclic group generated by ζ and $\bar{a}$ (resp. $\bar{b}$) is represented by $a \in M^*$ (resp. $b \in N$). The action of $N/N^* \simeq \mu_n$ on $\operatorname{Spec} \mathbf{C}[\check{\tau} \cap M^*]$ is defined by $\bar{b} \cdot \mathbf{e}(a) := f(\bar{a}, \bar{b})\mathbf{e}(a)$ $(a \in \check{\tau} \cap M^*, b \in N)$. Then $\bar{e_1} \cdot x = \zeta x$ and $\bar{e_1} \cdot y = \zeta^\ell y$. Because f is non-singular we see $\operatorname{Spec} \mathbf{C}[\check{\tau} \cap M] \simeq \operatorname{Spec} \mathbf{C}[x, y]^{N/N^*} \simeq \mathbf{A}^2/C_{n,\ell}$. □

The minimal resolution S of X_τ is constructed by using the continued fraction $\frac{n}{\ell} = [[b_1, b_2, \ldots, b_r]]$ as follows.

Let $v_\mu := j_\mu e_1 + k_\mu e_2$ and we subdivide τ into $\tau_\mu := \langle v_{\mu-1}, v_\mu \rangle$ $(\mu = 1, \ldots, r+1)$. Let Δ be the fan consisting of all of τ_μ and its faces. (1.3) shows that affine charts $U_\mu := \operatorname{Spec} \mathbf{C}[\check{\tau_\mu} \cup M]$ $(\mu = 1, 2, \ldots, r+1)$ are smooth. Since $v_{\mu+1} + v_{\mu-1} = b_\mu v_\mu$ and $b_\mu \geq 2$, $\tilde{S} := T_N\mathrm{emb}(\Delta)$ is the minimal resolution of X_τ. And the dual graph of the exceptional set of this minimal resolution is:



By the proof of Proposition 2.1 we see $\mathbf{A}^2 \simeq \operatorname{Spec} \mathbf{C}[x, y]$ and $\mathbf{A}^2/G \simeq X_\tau$ where $x = \mathbf{e}(\frac{1}{n} f_1)$, $y = \mathbf{e}(f_2 - \frac{n-\ell}{n} f_1)$. By definition

$$U_\mu = \operatorname{Spec} \mathbf{C}[\mathbf{e}(k_{\mu-1} f_1 - j_{\mu-1} f_2), \mathbf{e}(-k_\mu f_1 + j_\mu f_2)].$$

Hence by (1.3) we see

$$-k_\mu f_1 + j_\mu f_2 = -i_\mu \Big(\frac{1}{n} f_1\Big) + j_\mu \Big(f_2 - \frac{n-\ell}{n} f_1\Big),$$
$$U_\mu = \operatorname{Spec} \mathbf{C}[s_\mu, t_\mu], \quad s_\mu = x^{i_{\mu-1}}/y^{j_{\mu-1}}, \quad t_\mu = y^{j_\mu}/x^{i_\mu}. \qquad (2.1)$$

3. Hilbert Schemes and Symmetric Products

Let $S^n(\mathbf{A}^2)$ be the nth symmetric product of $\mathbf{A}^2$. This is by definition the quotient of the product of n copies of $\mathbf{A}^2$ by the natural permutation action of the symmetry group of n letters.

Lemma 3.1 *Let $S^n(\mathbf{A}^2)^G$ be the subset of $S^n(\mathbf{A}^2)$ consisting of all the points of $S^n(\mathbf{A}^2)$ fixed by any element of G. Then $S^n(\mathbf{A}^2)^G$ has a unique natural normal surface structure isomorphic to $\mathbf{A}^2/G$.*

Proof. Let $\mathfrak{q}(\neq 0) \in \mathbf{A}^2$. The point $\mathfrak{q}$ is fixed by no element of G except the identity. Therefore the set $G \cdot \mathfrak{q} := \{g(\mathfrak{q}); g \in G\}$ determines a point in $S^n(\mathbf{A}^2)^G$. Conversely any point of $S^n(\mathbf{A}^2)^G$ is an unordered set Σ of n points in $\mathbf{A}^2$. If Σ contains a point $\mathfrak{q}$ different from the origin, the above argument shows that Σ contains the set $G \cdot \mathfrak{q}$. Since $|\Sigma| = |n| = |G|$, we have $\Sigma = G \cdot \mathfrak{q}$.

We see that $G \cdot \mathfrak{q} = G \cdot \mathfrak{q}'$ for a pair of points $\mathfrak{q}(\neq 0)$ and $\mathfrak{q}'(\neq 0)$ if and only if $\mathfrak{q}' \in G \cdot \mathfrak{q}$. Therefore we have the isomorphism $S^n(\mathbf{A}^2 \setminus \{0\})^G \simeq (\mathbf{A}^2 \setminus \{0\})/G$, which extends to a natural bijection j between $S^n(\mathbf{A}^2)^G$ and $\mathbf{A}^2/G$. Since $\mathbf{A}^2/G$ is normal, $S^n(\mathbf{A}^2)^G$ has a unique structure of normal complex space via the bijection j. Hence j gives the isomorphism $S^n(\mathbf{A}^2)^G \simeq \mathbf{A}^2/G$. □

Definition 3.2 Let $\mathrm{Hilb}^n(\mathbf{A}^2)$ be the *Hilbert scheme of n points* on $\mathbf{A}^2$. By definition any $Z \in \mathrm{Hilb}^n(\mathbf{A}^2)$ is a zero dimensional subscheme with $h^0(Z, \mathcal{O}_Z) = \dim(\mathcal{O}_Z) = n$.

Remark We identify a subscheme Z and the defining ideal I_Z of Z, so that we consider $I_Z \in \mathrm{Hilb}^n(\mathbf{A}^2)$ since no confusion is possible.

The group G acts on $\mathbf{A}^2$ so that it acts on $\mathrm{Hilb}^n(\mathbf{A}^2)$ canonically. Let $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ be the subset of $\mathrm{Hilb}^n(\mathbf{A}^2)$ consisting of all the points fixed by any element of G. The Hilbert scheme $\mathrm{Hilb}^n(\mathbf{A}^2)$ is nonsingular ([F]) and the action of G on $\mathrm{Hilb}^n(\mathbf{A}^2)$ at any point of $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ is linearlized, therefore $\mathrm{Hilb}^n(\mathbf{A}^2)^G$ is also nonsingular ([IN98, Lemma 9.1]).

Definition 3.3 Let $\mathrm{Hilb}^G(\mathbf{A}^2)$ be a unique irreducible component of $\mathrm{Hilb}^n(\mathbf{A}^2)$ dominating $S^n(\mathbf{A}^2)^G$.

We have a natural morphism $\pi : \mathrm{Hilb}^G(\mathbf{A}^2) \to S^n(\mathbf{A}^2)^G$ defined by $\pi(Z) = \sum_{p \in \mathbf{A}^2} (\dim \mathcal{O}_{Z,p})p$ for $Z \in \mathrm{Hilb}^G(\mathbf{A}^2)$. Any point of $S^n(\mathbf{A}^2)^G \setminus \{0\}$ is a G-orbit of a point $\mathfrak{q}(\neq (0,0)) \in \mathbf{A}^2$. It determines a G-invariant reduced zero dimensional subscheme. This gives the inverse map of π over $(\mathbf{A}^2 \setminus \{0\})/G$. It follows that $\mathrm{Hilb}^G(\mathbf{A}^2)$ is birationally equivalent to $S^n(\mathbf{A}^2)^G$. In fact, we prove that $\mathrm{Hilb}^G(\mathbf{A}^2)$ is the minimal resolution of $\mathbf{A}^2/G$ in Section 5.

Lemma 3.4 *Let I_Z be the defining ideal of $Z \in \mathrm{Hilb}^G(\mathbf{A}^2)$. Any G-invariant function vanishing on* $\mathrm{supp}(Z)$ *is contained in I_Z.*

Especially if $\mathrm{supp}(Z) = \{0\}$ *then I_Z contains all of $x^{\alpha_\nu} y^{\beta_\nu}$ and $x^{i(\mu)} y^{j(\mu)}$.*

Proof. First we check $\mathcal{O}_{\mathbf{A}^2}/I_Z \simeq \mathbf{C}[G]$ as G-modules. Let $H := \mathrm{Hilb}^G(\mathbf{A}^2)$ and $G := \{g_1 = \mathrm{id}, g_2, \ldots, g_n\}$. If Z is a G-orbit of a point $p \neq (0,0)$ then $\mathcal{O}_{\mathbf{A}^2}/I_Z = \bigoplus_{i=1}^n \mathbf{C}\delta_{g_i}$ where $\delta_{g_i}(g_j p) := \delta_{ij}$. G acts on $\oplus \mathbf{C}\delta_{g_i}$ by $(g_j \circ \delta_{g_i})(p) := \delta_{g_i}(g_j^{-1}p) = \delta_{g_j g_i}(p)$ and it gives $\mathbf{C}[G] \simeq \oplus \mathbf{C}\delta_{g_i}$ $(g_i \mapsto \delta_{g_i})$ as G-modules.

Because $\dim \mathcal{O}_{\mathbf{A}^2}/I_Z = n$ for any $Z \in H$, $\mathcal{O}_{\mathbf{A}^2 \times H}$ is a locally free $\mathcal{O}$-module of rank n. G operates upon the vector space $\mathcal{O}_{\mathbf{A}^2 \times \{Z\}}/I_Z$ and the coefficients of the action of g_i are regular functions on H. By $g_i^n = 1$ $(\forall i)$ we see that all the eigenvalues are roots of unity. In particular its trace is independent of $Z \in H$. Since any representation of a finite group is uniquely determined by its character, the representation of G in $\mathcal{O}_{\mathbf{A}^2 \times \{Z\}}/I_Z$ is independent of $Z \in H$. Therefore $\mathcal{O}_{\mathbf{A}^2}/I_Z \simeq \mathbf{C}[G]$ for any $Z \in H$.

$\mathbf{C}[G]$ has a unique trivial G-submodule $\mathbf{C}(\sum_{g \in G} g)$ by the complete reducibility of G-module. Therefore a G-submodule spanned by G-invariant functions in $\mathcal{O}_{\mathbf{A}^2}/I_Z$ is isomorphic to $\mathbf{C}$ as G-modules. It follows that any G-invariant function vanishing on Z is contained in I_Z.

By (1.4) $x^{\alpha_\nu} y^{\beta_\nu}$ are G-invariant functions. Combining with Proposition 1.3 if $\mathrm{supp}(Z) = \{0\}$ then $x^{\alpha_\nu} y^{\beta_\nu}, x^{i(\mu)} y^{j(\mu)} \in I_Z$. □

4. $\mathbf{Hilb}^G(\mathbf{A}^2)$

Theorem 4.1 *Let $G = C_{n,\ell}$. Then $\mathrm{Hilb}^G(\mathbf{A}^2)$ set-theoretically consists of the following G-invariant ideals of colength $n = |G|$:*

$$I_\mu(p_\mu, q_\mu) := (x^{i_{\mu-1}} - p_\mu y^{j_{\mu-1}}, y^{j_\mu} - q_\mu x^{i_\mu}, x^{i(\mu)} y^{j(\mu)} - p_\mu q_\mu)$$

where $1 \leq \mu \leq r+1$ and $(p_\mu, q_\mu) \in \mathbf{A}^2$.

Remark

(i) r, i_μ, j_μ, $i(\mu)$ and $j(\mu)$ are nonnegative integers defined in (1.1), (1.2) and (1.4).

(ii) $I_{r+1}(p_{r+1}, 0) = (x, y^n)$ because $i_r = 1$, $i_{r+1} = 0$.

(iii) $\{x^{i_\mu}; 1 \leq \mu \leq r\}$ is a set of *special* representations which are associated to the irreducible components of the exceptional set in the minimal resolution of $\mathbf{A}^2/G$ by the work of Riemenschneider [R98] and Wunram [W].

Proof. Let $\mathfrak{m}_\mathfrak{p}$ (resp. $\mathfrak{m}_{A_{n,\ell}}$) be the maximal ideal of $\mathfrak{p} \in \mathbf{A}^2$ (resp. of the origin of $A_{n,\ell}$) and $\mathfrak{n} = \mathfrak{m}_{A_{n,\ell}} \mathcal{O}_{\mathbf{A}^2}$. We put $\mathfrak{m} := \mathfrak{m}_{(0,0)}$.

We note $I_\mu(p_\mu, q_\mu)$ is a G-invariant ideal. In fact, $i_\mu \equiv \ell j_\mu \pmod n$ by (1.3) and $x^{i(\mu)}y^{j(\mu)}$ is a G-invariant function by Proposition 1.3 and (1.3).

First we consider the case where $\mathrm{supp}(Z) \neq \{0\}$. We recall that the subset $\{Z \in \mathrm{Hilb}^G(\mathbf{A}^2); \mathrm{supp}(Z) \neq \{0\}\}$ is bijective to $(\mathbf{A}^2 \setminus \{0\})/G$. Hence for any $Z \in \mathrm{Hilb}^G(\mathbf{A}^2)$ with $\mathrm{supp}(Z) \neq \{0\}$, there exists a point $\mathfrak{p} \in \mathrm{supp}(Z)$ such that $I_Z = \prod_{\mathfrak{q} \in G\mathfrak{p}} \mathfrak{m}_\mathfrak{q}$. Next we prove that I_Z coincides with one of the I_μ for a suitable pair (p_μ, q_μ). If $\mathfrak{p} = (u, v) \in \mathbf{A}^2$ with $uv \neq 0$, we put $p_\mu := u^{i_{\mu-1}}/v^{j_{\mu-1}}$ and $q_\mu := v^{j_\mu}/u^{i_\mu}$ for any $1 \leq \mu \leq r$. Then $I_\mu(p_\mu, q_\mu) \subset \mathfrak{m}_\mathfrak{p}$. If $v = 0$ and $u \neq 0$ (resp. if $u = 0$ and $v \neq 0$), then we see $I_1(u^n, 0) \subset \mathfrak{m}_\mathfrak{p}$ (resp. $I_{r+1}(0, v^n) \subset \mathfrak{m}_\mathfrak{p}$).

Since $I_\mu(p_\mu, q_\mu)$ is G-invariant, we infer $I_\mu(p_\mu, q_\mu) \subset \prod_{\mathfrak{q} \in G\mathfrak{p}} \mathfrak{m}_\mathfrak{q}$. On the other hand $\dim \mathcal{O}_{\mathbf{A}^2}/I_\mu(p_\mu, q_\mu) \leq n$. In fact $\mathcal{O}_{\mathbf{A}^2}/I_\mu(p_\mu, q_\mu)$ is spanned by monomials $x^{\lambda_1}y^{\lambda_2}$ where

$$(\lambda_1, \lambda_2) \in \Lambda := \{(\lambda_1, \lambda_2); 0 \leq \lambda_1 < i_{\mu-1} \text{ and } 0 \leq \lambda_2 < j_\mu - j_{\mu-1}, \\ \text{or } 0 \leq \lambda_1 < i_{\mu-1} - i_\mu \text{ and } j_\mu - j_{\mu-1} \leq \lambda_2 < j_\mu\}.$$

And by (1.3) $i_{\mu-1}(j_\mu - j_{\mu-1}) + \{j_\mu - (j_\mu - j_{\mu-1})\}(i_{\mu-1} - i_\mu) = n$. It follows that $I_\mu(p_\mu, q_\mu) = \prod_{\mathfrak{q} \in G\mathfrak{p}} \mathfrak{m}_\mathfrak{q}$.

As we remarked after Definition 3.3, $\pi : \mathrm{Hilb}^G(\mathbf{A}^2) \to S^n(\mathbf{A}^2)^G \simeq \mathbf{A}^2/G$ is a resolution, which is an isomorphism over $(\mathbf{A}^2 \setminus \{0\})/G$. Now we study the exceptional set $\pi^{-1}(0) = \{Z \in \mathrm{Hilb}^G(\mathbf{A}^2); \mathrm{supp}(Z) = \{0\}\}$. We prove that it is the union of $I_\mu(p_\mu, q_\mu)$ with $p_\mu q_\mu = 0$ $(1 \leq \mu \leq r-1)$ and $I_1(0, q_1)$, $I_{r+1}(p_{r+1}, 0)$. In fact, since $\mathbf{A}^2/G$ is a normal surface, it follows from Zariski's connectedness theorem ([EGA], III 4.3) that $\pi^{-1}(0)$ is connected. Hence we can determine $\pi^{-1}(0)$ by using deformations.

We remark first that by definition of I_μ, $I_\mu(p_\mu, q_\mu) \subset \mathfrak{m}$ if and only if $p_\mu q_\mu = 0$ for $2 \leq \mu \leq r$ or $p_1 = 0$ or $q_{r+1} = 0$. Moreover we check that these ideals belong to $\mathrm{Hilb}^G(\mathbf{A}^2)$. In fact, if $p_\mu q_\mu = 0$ the monomials $\{x^{\lambda_1}y^{\lambda_2}; (\lambda_1, \lambda_2) \in \Lambda\}$ is a basis of $\mathcal{O}_{\mathbf{A}^2}/I_\mu(p_\mu, q_\mu)$. Therefore $I_\mu(p_\mu, q_\mu) \in \mathrm{Hilb}^G(\mathbf{A}^2)$.

Now let Z be the subscheme defined by one of the ideals $I_\mu(p_\mu, q_\mu)$. We consider G-equivariant versal deformations of Z. The tangent space of $\mathrm{Hilb}^G(\mathbf{A}^2)$ at a point I_Z is isomorphic to $\mathrm{Hom}_{\mathcal{O}_{\mathbf{A}^2}}(I_Z, \mathcal{O}_{\mathbf{A}^2}/I_Z)^G$. Now we prove a lemma to determine deformations of Z inside $\pi^{-1}(0)$.

Lemma 4.2 *There is a basis* $\{\phi_-, \phi_+\}$ *of* $T := \mathrm{Hom}_{\mathcal{O}_{\mathbf{A}^2}}(I_\mu(0,0), \mathcal{O}_{\mathbf{A}^2}/$

$I_\mu(0,0))^G$ *defined by*

$$\phi_-(x^{i_{\mu-1}}) = y^{j_{\mu-1}}, \quad \phi_-(y^{j_\mu}) = 0,$$
$$\phi_+(x^{i_{\mu-1}}) = 0, \qquad \phi_+(y^{j_\mu}) = x^{i_\mu}.$$

Proof. We put $I := I_\mu(0,0) = (x^{i_{\mu-1}}, y^{j_\mu}, x^{i(\mu)}y^{j(\mu)})$. It follows from $\mathcal{O}_{\mathbf{A}^2}/I \simeq \mathbf{C}[G]$ that G-invariant $\mathcal{O}_{\mathbf{A}^2}$-homomorphism $\phi \in T$ does not change the characters of elements of I. Since $\mathcal{O}_{\mathbf{A}^2}/I$ has a basis $\{x^{\lambda_1}y^{\lambda_2}; (\lambda_1,\lambda_2) \in \Lambda\}$ and $i_\mu \equiv \ell j_\mu \pmod n$, ϕ is defined by

$$\phi(x^{i_{\mu-1}}) = c_1 y^{j_{\mu-1}}, \quad \phi(y^{j_\mu}) = c_2 x^{i_\mu}, \quad \phi(x^{i(\mu)}y^{j(\mu)}) = c_3 \quad (c_i \in \mathbf{C}).$$

Applying ϕ to $x^{i_{\mu-1}}y^{j(\mu)} \in I$, we see

$$\phi(x^{i_{\mu-1}}y^{j(\mu)}) = y^{j(\mu)}\phi(x^{i_{\mu-1}}) = c_1 y^{j_\mu} = 0 \quad \text{in } \mathcal{O}_{\mathbf{A}^2}/I.$$

Since $\phi(x^{i_{\mu-1}}y^{j(\mu)}) = \phi(x^{i_\mu}x^{i(\mu)}y^{j(\mu)}) = c_3 x^{i_\mu}$ and $x^{i_\mu} \neq 0$ in $\mathcal{O}_{\mathbf{A}^2}$, we infer $c_3 = 0$. Thus the lemma follows from $\dim T = \dim \mathrm{Hilb}^G(\mathbf{A}^2) = 2$. □

By Lemma 4.2 $\{I_\mu(p_\mu,q_\mu); (p_\mu,q_\mu) \in \mathbf{A}^2\}$ is a G-equivariant versal deformation of $I_\mu(0,0)$. On the other hand, we have

$$x^{i_{\mu-1}}y^{j_\mu} = x^{i_\mu}y^{j_{\mu-1}}x^{i(\mu)}y^{j(\mu)},$$
$$y^{j_{\mu+1}} = (y^{j_\mu} - q_\mu x^{i_\mu})y^{j(\mu+1)} + q_\mu x^{i_{\mu+1}}x^{i(\mu+1)}y^{j(\mu+1)} \in I_\mu(0,q_\mu).$$

because $x^{i(\mu+1)}y^{j(\mu+1)} \in I_\mu(0,q_\mu)$ by Lemma 3.4. We see $I_\mu(0,q_\mu) = I_{\mu+1}(q_\mu^{-1},0)$ for $q_\mu \neq 0$. Hence $\lim_{q_\mu\to\infty} I_\mu(0,q_\mu) = I_{\mu+1}(0,0)$ for $\mu \le r$. To be more precise in $\mathrm{Grass}(\mathfrak{m}/\mathfrak{n}+\mathfrak{m}^n, n-1)$ we get $\lim_{q_\mu\to\infty} I_\mu(0,q_\mu)/\mathfrak{n}+\mathfrak{m}^n = I_{\mu+1}(0,0)/\mathfrak{n}+\mathfrak{m}^n$. Similarly we infer $\lim_{p_\mu\to\infty} I_\mu(p_\mu,0) = I_{\mu-1}(0,0)$ for $\mu \ge 2$.

Since $\pi^{-1}(0)$ is connected. We have

$$\pi^{-1}(0) = \{I_1(0,q_1)\} \cup \{I_{r+1}(p_{r+1},0)\}$$
$$\cup \{I_\mu(p_\mu,q_\mu); p_\mu q_\mu = 0, 2 \le \mu \le r\}.$$

Thus Theorem 4.1 is proved. □

5. The isomorphism $\mathrm{Hilb}^G(\mathbf{A}^2) \simeq S$

Theorem 5.1 *Let S be the toric minimal resolution of the cyclic singularity $A_{n,\ell} = \mathbf{A}^2/G$. Then $S \simeq \mathrm{Hilb}^G(\mathbf{A}^2)$. In fact, let $U_\mu = \mathrm{Spec}\,\mathbf{C}[s_\mu, t_\mu]$*

the affine charts of S $(1 \le \mu \le r+1)$ given in Section 2. Then the isomorphism of S with $\mathrm{Hilb}^G(\mathbf{A}^2)$ is given by the morphism defined by the universal property of $\mathrm{Hilb}^n(\mathbf{A}^2)$ from the S-flat family of zero dimensional subschemes defined by the G-invariant ideals of $\mathcal{O}_{\mathbf{A}^2}$;

$$I_\mu(s_\mu, t_\mu) := (x^{i_{\mu-1}} - s_\mu y^{j_{\mu-1}}, y^{j_\mu} - t_\mu x^{i_\mu}, x^{i(\mu)} y^{j(\mu)} - s_\mu t_\mu).$$

Proof. First we check that $I_\mu(s_\mu, t_\mu) = I_{\mu+1}(s_{\mu+1}, t_{\mu+1})$ if two points $(s_\mu, t_\mu) \in U_\mu$ and $(s_{\mu+1}, t_{\mu+1}) \in U_{\mu+1}$ are coincident in S. In fact, if both the points represent the same point in S, then it follows from (2.1) that $s_{\mu+1} t_\mu = 1$ and $t_{\mu+1} = t_\mu^{b_\mu} s_\mu$. Then $x^{i_\mu} - s_{\mu+1} y^{j_\mu} = s_{\mu+1}(t_\mu x^{i_\mu} - y^{j_\mu}) \in I_\mu(s_\mu, t_\mu)$. We check

$$\begin{aligned} h(b_\mu) &:= x^{i(\mu+1)} y^{j(\mu+1)} - s_{\mu+1} t_{\mu+1} \\ &= x^{i(\mu)-(b_\mu-2)i_\mu} y^{j(\mu)+(b_\mu-2)j_\mu} - t_\mu^{b_\mu-1} s_\mu \end{aligned}$$

is contained in $I_\mu(s_\mu, t_\mu)$ by induction on b_μ. If $b_\mu = 2$ then $h(b_\mu) = x^{i(\mu)} y^{j(\mu)} - s_\mu t_\mu \in I_\mu(s_\mu, t_\mu)$. If $b_\mu > 2$ then $j(\mu+1) > j_\mu$ and

$$h(b_\mu) = x^{i(\mu+1)} y^{j(\mu+1)-j_\mu} (y^{j_\mu} - t_\mu x^{i_\mu}) + t_\mu h(b_\mu - 1).$$

By the induction hypothesis $h(b_\mu - 1) \in I_\mu(s_\mu, t_\mu)$ and we get $h(b_\mu) \in I_\mu(s_\mu, t_\mu)$. Since $y^{j_{\mu+1}} - t_{\mu+1} x^{i_{\mu+1}} = y^{j(\mu+1)}(y^{j_\mu} - t_\mu x^{i_\mu}) + t_\mu x^{i_{\mu+1}} h(b_\mu) \in I_\mu(s_\mu, t_\mu)$ we see $I_{\mu+1}(s_{\mu+1}, t_{\mu+1}) \subset I_\mu(s_\mu, t_\mu)$. Both the ideals have the same colength n. Hence $I_{\mu+1}(s_{\mu+1}, t_{\mu+1}) = I_\mu(s_\mu, t_\mu)$.

Therefore the family of the zero dimensional subschemes defined by I_μ is well-defined on S. Since $\dim \mathcal{O}_{\mathbf{A}^2}/I_\mu(s_\mu, t_\mu)$ is constant, this family is S-flat. By the universality of $\mathrm{Hilb}^n(\mathbf{A}^2)$ we have a natural morphism $f : S \to \mathrm{Hilb}^n(\mathbf{A}^2)$, which factors through $\mathrm{Hilb}^G(\mathbf{A}^2)$ by the G-invariance of the ideals I_μ. By the proof of Theorem 4.1, in fact because no irreducible component of $\pi^{-1}(0)$ is contracted by f by Lemma 4.2, we have a finite birational morphism of S onto $\mathrm{Hilb}^G(\mathbf{A}^2)$. Since $\mathrm{Hilb}^G(\mathbf{A}^2)$ is nonsingular and S is minimal we infer $S \simeq \mathrm{Hilb}^G(\mathbf{A}^2)$. □

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