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Mutual exclusiveness among spacelike, timelike, and lightlike leaves in totally geodesic foliations of lightlike complete Lorentzian two-dimensional tori

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Abstract. In this paper we prove an equation for totally geodesic foliations of pseudo-Riemannian manifolds which is originally established in Riemannian case. We prove that if $\mathcal{F}$ is a totally geodesic foliation of a lightlike complete Lorentzian 2-torus T^2, then $\mathcal{F}$ consists of only one kind of leaves among spacelike, timelike, and lightlike ones. As a corollary we prove that a totally geodesic foliation of a lightlike complete 2-torus has no Reeb components.

Key words: pseudo-Riemannian manifolds, totally geodesic foliations.

1. Introduction

In this section we explain what motivated us to consider totally geodesic foliations of pseudo-Riemannian manifolds.

There are a lot of results about codimension-one totally geodesic foliations of complete Riemannian manifolds as follows. Let $\mathcal{F}$ be a codimension-one totally geodesic foliation of a complete Riemannian manifold (M, g). The universal covering of M is a product $L \times \mathbf{R}$ and the lift of $\mathcal{F}$ is the product foliation, where L is the universal covering of the leaves of $\mathcal{F}$ (see [BH]). The foliation perpendicular to $\mathcal{F}$ is a Riemannian foliation ([CG]). G. Oshikiri proved that any Killing field with bounded length preserves $\mathcal{F}$ (see [Os1], [Os2]). In [Gh], E. Ghys classified totally geodesic foliations of a closed Riemannian manifold.

On the other hand, we do not know so many results on totally geodesic foliations of pseudo-Riemannian manifolds. Hence we consider those of pseudo-Riemannian manifolds, in particular, those of Lorentzian manifolds. We recall some results about them. In [Z], A. Zeghib constructed codimension-one, lightlike totally geodesic foliations. He made a foliation $\mathcal{F}$ lightlike totally geodesic in two cases:

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(1) The foliation is defined by a locally free action with codimension-one orbits of a Lie group with a one-dimensional normal subgroup;
(2) The foliation $\mathcal{F}$ is the suspension of a foliation $\mathcal{L}$ of a Riemannian manifold (M, g) by a diffeomorphism of M preserving $\mathcal{L}$ and $g|_{T\mathcal{L}}$.

He proved that the foliation $\mathcal{F}$ is lightlike geodesible by means of a construction of a Lorentzian metric. In fact, we do not know whether the metric is complete or not. So it seems significant to consider totally geodesic foliations of complete Lorentzian manifolds. In [CR], Y. Carrière and L. Rozoy proved that the canonical lightlike totally geodesic foliations of a lightlike complete 2-torus are C^0-linearizable. The above three persons considered lightlike totally geodesic foliations. The existence of a totally geodesic foliation having spacelike, timelike, and lightlike leaves seems unknown. So we ask the following.

Question 1 Does there exist a codimension-one totally geodesic foliation of a complete, closed Lorentzian manifold which contains spacelike, timelike, and lightlike leaves?

The condition of being codimension-one is necessary. There is an example of a codimension-two totally geodesic foliation having spacelike, timelike, and lightlike leaves of a complete, closed Lorentzian manifold (see Example A in Section 2).

We have a partial answer to Question 1 as follows.

Theorem 5.1 *Let (T^2, g) be a lightlike complete Lorentzian 2-torus. There exists no totally geodesic foliation which contains more than or equal to two kinds of leaves among spacelike, timelike, and lightlike ones.*

The assumption of lightlike completeness is necessary. There is an example of a totally geodesic foliation of a lightlike incomplete Lorentzian 2-torus that contains spacelike, timelike, and lightlike leaves (see Section 2).

As a corollary of Theorem 5.1, we have the following.

Corollary 5.5 *Let (T^2, g) be a lightlike complete Lorentzian 2-torus. Let $\mathcal{F}$ be a totally geodesic foliation of T^2. Then $\mathcal{F}$ has no Reeb components.*

The condition of lightlike completeness is necessary. There is an example of a totally geodesic foliation with Reeb components of a lightlike incomplete 2-torus (see Example D in Section 2).

2. Definitions and examples

This section contains definitions and examples. We prove an equation for totally geodesic foliations of pseudo-Riemannian manifolds which is originally established in Riemannian case. It is useful to calculate examples.

We assume that all the manifolds are smooth, connected, and orientable and that all the foliations are smooth and orientable.

Let M be a manifold.

Definition 2.1 A *pseudo-Riemannian metric* g on M is a nondegenerate, symmetric covariant 2-tensor. We call (M, g) a *pseudo-Riemannian manifold*. In particular, if the signature of g is $(+, \dots, +, -)$ we call it a *Lorentzian metric*.

Definition 2.2 Let g be a Lorentzian metric. A subspace $E \subset T_xM$ is called *spacelike* (resp. *timelike*, *lightlike*) if the signature of $g|_E$ is $(+, \dots, +)$ (resp. $(+, \dots, +, -)$, $(+, \dots, +, 0)$). A vector $v \in T_xM$ is called *spacelike* (resp. *timelike*, *lightlike*) if $g(v, v) > 0$ (resp. $g(v, v) < 0$, $g(v, v) = 0$).

For a pseudo-Riemannian manifold, it is well known that there exists the Levi-Civita connection, that is, a connection which is torsion free and compatible with the metric (see [ON]).

Definition 2.3 A pseudo-Riemannian metric g is called (*geodesically*) *complete* if an affine parameter of any geodesic can be defined on entire $\mathbf{R}$. Otherwise g is called (*geodesically*) *incomplete*. A Lorentzian metric g is called *lightlike* (*geodesically*) *complete* if an affine parameter of any geodesic with a lightlike initial vector can be defined on entire $\mathbf{R}$.

Definition 2.4 A foliation of a pseudo-Riemannian manifold (M, g) is called *totally geodesic* if each leaf L is a totally geodesic submanifold, that is, a submanifold such that any geodesic with any initial vector in TL is contained in L.

Definition 2.5 Let L be a submanifold in a Lorentzian manifold (M, g). We call L *spacelike* (resp. *timelike*, *lightlike*) if the tangent space T_xL is a spacelike (resp. timelike, lightlike) subspace of T_xM for each $x \in L$.

We can easily prove the following proposition.

Proposition 2.6 *Every leaf L of a totally geodesic foliation of a Lorentzian manifold is a spacelike, timelike, or lightlike submanifold.*

Now let (M, g) be a pseudo-Riemannian manifold and $\mathcal{F}$ a foliation. We define the *normal distribution* $\mathcal{H}$ by $\mathcal{H} = (T\mathcal{F})^{\perp}$, i.e., $\mathcal{H}$ is the distribution which consists of all vectors perpendicular to $T\mathcal{F}$.

The following proposition is fundamental to consider totally geodesic foliations of pseudo-Riemannian manifolds.

Proposition 2.7 *Let (M, g) be a pseudo-Riemannian manifold and $\mathcal{F}$ be a codimension k foliation of M. Then $\mathcal{F}$ is totally geodesic if and only if $(\mathcal{L}_X g)(Y, Z) = 0$ for all $X \in \Gamma(\mathcal{H})$ and for all Y, $Z \in \Gamma(T\mathcal{F})$.*

Furthermore if $\mathcal{F}$ is totally geodesic and $X \in \Gamma(\mathcal{H})$ is a foliation preserving[1] *local vector field which can define a local one-parameter group, then a local one-parameter group of local transformations generated by X preserves the metrics induced on plaques.*

Proof. Let ∇ be the Levi-Civita connection of g. We fix a splitting $TM = T\mathcal{F} \oplus \xi$ and define $\alpha_\xi(Y, Z)$ for $Y, Z \in \Gamma(T\mathcal{F})$ by ξ-component of $\nabla_Y Z$. Hence $\mathcal{F}$ is totally geodesic, if and only if $\nabla_Y Z \in \Gamma(T\mathcal{F})$ for all Y, $Z \in \Gamma(T\mathcal{F})$, if and only if $\alpha_\xi(Y, Z) = 0$ for all Y, $Z \in \Gamma(T\mathcal{F})$. We have

$$\begin{aligned}
(\mathcal{L}_X g)(Y, Z) &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \\
&= g(\nabla_X Y - [X, Y], Z) + g(Y, \nabla_X Z - [X, Z]) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\
&= g(X, -\nabla_Y Z - \nabla_Y Z - [Z, Y]) \\
&= -2g(X, \nabla_Y Z) \\
&= -2g(X, \alpha_\xi(Y, Z)).
\end{aligned}$$

Notice that $(\mathcal{H})^{\perp} = T\mathcal{F}$. Thus given a non-zero vector $v \in \xi$ there exists $w \in \mathcal{H}$ such that $g(v, w) \neq 0$. Therefore the condition $\alpha_\xi \equiv 0$ is equivalent to the following condition.

$$g(X, \alpha_\xi(Y, Z)) = 0 \text{ for all } X \in \Gamma(\mathcal{H}) \text{ and for all } Y, Z \in \Gamma(T\mathcal{F}).$$

This proves the first part of Proposition 2.7.

We will prove the second part of Proposition 2.7. First we assume that $\mathcal{F}$ is totally geodesic and X is a foliation preserving local vector field tangent to $\mathcal{H}$. Next we decompose g into $g\,|_{T\mathcal{F}} + (g - g\,|_{T\mathcal{F}})$. By straightforward

[1] We call X foliation preserving if $[X, Y] \in \Gamma(T\mathcal{F})$ for all $Y \in \Gamma(T\mathcal{F})$.

computation, we have

$$(\mathcal{L}_X(g - g\,|_{T\mathcal{F}}))(Y, Z) = 0.$$

Since $0 = (\mathcal{L}_X(g\,|_{T\mathcal{F}}))(Y, Z) + (\mathcal{L}_X(g - g\,|_{T\mathcal{F}}))(Y, Z)$, we get

$$(\mathcal{L}_X(g\,|_{T\mathcal{F}}))(Y, Z) = 0.$$

This condition means that a local one-parameter group of local transformations generated by X preserves the metrics induced on plaques. More precisely, let ψ_t be a local one-parameter group of local transformations generated by X. Let P be a plaque of $\mathcal{F}$ which is contained in domains of ψ_t for all $t \in (-\epsilon, \epsilon)$. Because $\psi_t(P)$ is a submanifold and its tangent vectors lie in $T\mathcal{F}$ by the definition of foliation preserving, $\psi_t(P)$ is also a plaque of $\mathcal{F}$. If $p \in P$ and $v \in T_p\mathcal{F}$ then $\psi_{t*}v \in T_{\psi_t(p)}\mathcal{F}$. We have

$$\lim_{t \to 0} \frac{1}{t}\{(g\,|_{T\mathcal{F}})(\psi_{t*}v, \psi_{t*}v) - (g\,|_{T\mathcal{F}})(v, v)\} = 0,$$

$$\lim_{t \to 0} \frac{1}{t}\{(g\,|_{T\mathcal{F}})(\psi_{t+s*}v, \psi_{t+s*}v) - (g\,|_{T\mathcal{F}})(\psi_{s*}v, \psi_{s*}v)\} = 0.$$

Hence the differential of the map $s \mapsto (g\,|_{T\mathcal{F}})(\psi_{s*}v, \psi_{s*}v)$ is zero. Therefore

$$(g\,|_{T\mathcal{F}})(\psi_{s*}v, \psi_{s*}v) = (g\,|_{T\mathcal{F}})(v, v).$$

This proves the second part of Proposition 2.7. □

We introduce the concept of an element of holonomy.

Definition 2.8 A piecewise smooth curve $\sigma : [0, t_0] \to M$ is called an *$\mathcal{H}$-curve* if its tangent vectors lie in $\mathcal{H}$. An *element of holonomy* along the $\mathcal{H}$-curve σ is a family of maps $\{\psi_t : V_{\sigma(0)} \to V_{\sigma(t)}\}_{t \in [0,t_0]}$ which satisfies the following conditions:

(1) The set $V_{\sigma(t)}$ is a plaque of the leaf containing the point $\sigma(t)$ for each $t \in [0, t_0]$;
(2) The map ψ_t is an isometry from $(V_{\sigma(0)}, g\,|_{V_{\sigma(0)}})$ to $(V_{\sigma(t)}, g\,|_{V_{\sigma(t)}})$ for each $t \in [0, t_0]$;
(3) The curve $\psi_t(x)$ with parameter $t \in [0, t_0]$ is an $\mathcal{H}$-curve for each $x \in V_{\sigma(0)}$ and $\psi_t(\sigma(0)) = \sigma(t)$;
(4) The map ψ_0 is the identity map of $V_{\sigma(0)}$.

By using this term, we can interpret Proposition 2.7: If we can take a foliation preserving vector field $X \in \Gamma(\mathcal{H})$ for an $\mathcal{H}$-curve σ such that $X_{\sigma(t)} = \dot{\sigma}(t)$, then we can construct an element of holonomy along the $\mathcal{H}$-curve σ.

If $\mathcal{F}$ is a totally geodesic foliation of a complete Riemannian manifold, then there exists an element of holonomy for each $\mathcal{H}$-curve, see [BH] for more details. We will consider sufficient conditions for the existence of an element of holonomy for a totally geodesic foliation of a Lorentzian manifold in Section 3.

Now we consider some examples related to Question 1.

There is an example of a totally geodesic foliation of a complete Lorentzian manifold which contains spacelike, timelike, and lightlike leaves. This foliation is codimension-two and its normal distribution is non-integrable.

Example A We consider 4-dimensional torus $T^4 = \mathbf{R}^4/2\pi\mathbf{Z}^4$ and denote the canonical coordinates of $\mathbf{R}^4$ by (x, y, z, w). Put

$$\tilde{g} = dx \otimes dx + dy \otimes dy + dz \otimes dz - dw \otimes dw.$$

It is invariant under the $\mathbf{Z}^4$-action. Hence it induces a metric g on T^4. The metric g on T^4 is complete because $\tilde{g}$ on $\mathbf{R}^4$ is complete. Let $T\mathcal{F}$ denote the distribution generated by the vector fields

$$\frac{\partial}{\partial x} \quad \text{and} \quad \cos y \frac{\partial}{\partial z} + \sin y \frac{\partial}{\partial w}.$$

Because strait lines in $\mathbf{R}^4$ are geodesics, this foliation $\mathcal{F}$ is totally geodesic. We can classify the leaves passing through $(0, y, 0, 0)$ as follows: The leaves passing through $y = \frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$ are lightlike; The leaves passing through $\frac{\pi}{4} < y < \frac{3}{4}\pi$ and $\frac{5}{4}\pi < y < \frac{7}{4}\pi$ are timelike; The other leaves are spacelike. Hence $\mathcal{F}$ contains spacelike, timelike, and lightlike leaves.

In the case of codimension one, we have the following examples without Reeb components: One has three kinds of leaves; Another has two kinds of leaves.

Example B We consider 2-dimensional torus $T^2 = \mathbf{R}^2/2\pi\mathbf{Z}^2$. Denote the canonical coordinates of $\mathbf{R}^2$ by (x, y). Define the Lorentzian metric g

by

$$g = \frac{1}{2}dx^2 + (dx \otimes dy + dy \otimes dx) - \cos x \, dy^2.$$

Define the foliation $\mathcal{F}$ by

$$T\mathcal{F} = \left\{ \frac{\partial}{\partial y} + \cos x \frac{\partial}{\partial x} \right\}.$$

Since the vector field $\frac{\partial}{\partial y}$ is perpendicular to $\mathcal{F}$ and it is a Killing field, $\mathcal{F}$ is totally geodesic. The leaves passing through $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3}{2}\pi, 0\right)$ are lightlike. The leaves passing through $(x, 0)$ where $\frac{\pi}{2} < x < \frac{3}{2}\pi$ are timelike. The other leaves are spacelike. Hence $\mathcal{F}$ contains spacelike, timelike, and lightlike leaves. This Lorentzian metric is incomplete by Theorem 5.1.

Example C Let T^2, x, y be as in Example B. Define the Lorentzian metric g by

$$g = dx^2 + (dx \otimes dy + dy \otimes dx) - \cos^2 x \, dy^2.$$

Define the foliation $\mathcal{F}$ by

$$T\mathcal{F} = \left\{ \frac{\partial}{\partial y} + \cos^2 x \frac{\partial}{\partial x} \right\}.$$

Since $\mathcal{H} = \left\{ \frac{\partial}{\partial y} \right\}$, the foliation $\mathcal{F}$ is totally geodesic. It contains spacelike and lightlike leaves. This metric is incomplete by Theorem 5.1.

There is an example of a totally geodesic foliation with Reeb components of an incomplete Lorentzian 2-torus as follows.

Example D (torus of Clifton-Pohl [CR]) Consider

$$\left(\mathbf{R}^2 - \{0\}, \ \frac{1}{x^2 + y^2}(dx \otimes dy + dy \otimes dx)\right).$$

This metric is invariant under homotheties. Hence we get a Lorentzian torus (T^2, g). Each of the canonical lightlike totally geodesic foliations has Reeb components. The Lorentzian metric g on T^2 is incomplete by Corollary 5.5.

3. Some properties of totally geodesic foliations of Lorentzian manifolds

In this section we consider general properties of totally geodesic foliations of Lorentzian manifolds. We will prove that the union of all spacelike leaves and that of all timelike ones are open, and that the union of all lightlike leaves is closed. We will give sufficient conditions for the existence of an element of holonomy: When an $\mathcal{H}$-curve σ intersects only spacelike (or timelike) leaves or is contained in a lightlike leaf, we construct an element of holonomy along σ.

Let (M, g) be a Lorentzian manifold and $\mathcal{F}$ be a totally geodesic foliation of codimension k of M. Geodesically completeness of g is not assumed.

We denote the union of all spacelike leaves, timelike ones and lightlike ones of $\mathcal{F}$ by $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{L}$ respectively. Hence we get a decomposition $M = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ (disjoint union). We call this the *STL-decomposition* of M by $\mathcal{F}$.

We will inquire into several properties of the STL-decomposition.

Proposition 3.1 *Let L be a spacelike (resp. timelike) leaf. Then there exists a saturated neighborhood U of L such that all leaves in U are spacelike (resp. timelike).*

Proof. This proposition follows from the following lemma. □

Lemma 3.2 *Let L be a spacelike (resp. timelike) leaf. Then for all $p \in L$ there exists a foliation chart U around p such that all plaques in U are spacelike (resp. timelike).*

Proof. For a neighborhood V around p we take a frame of $T\mathcal{F}$ and express $g|_{T\mathcal{F}}$ by a matrix-valued function G. If a leaf L is spacelike (resp. timelike), then $\det G|_{L\cap V} \neq 0$. Since the condition $\det G \neq 0$ is an open condition, the fact $\det G|_{L\cap V} \neq 0$ implies that there exists a foliation chart U around p such that all plaques in U are spacelike (resp. timelike). □

Corollary 3.3 *The sets $\mathbf{S}$ and $\mathbf{T}$ are open in M, and $\mathbf{L}$ is closed in M.*

Proposition 3.4 *If $\mathcal{F}$ is codimension one and all the leaves of $\mathcal{F}$ are dense in M, then one of the following occurs:*

(1) *$M = \mathbf{S}$, that is, all the leaves are spacelike;*
(2) *$M = \mathbf{T}$, that is, all the leaves are timelike;*
(3) *$M = \mathbf{L}$, that is, all the leaves are lightlike.*

Proof. Assume that $\mathcal{F}$ has a spacelike leaf L. By assumption

$$M = \bar{L} \subset \bar{\mathbf{S}} \subset M.$$

Hence $\bar{\mathbf{S}} = M$. If $\bar{\mathbf{S}} - \mathbf{S} \neq \phi$ then there is a leaf $L' \subset \bar{\mathbf{S}} - \mathbf{S}$, which implies $\bar{L'} \subsetneqq M$. This contradicts that all the leaves are dense in M. Therefore $\mathbf{S} = M$.

Assume that $\mathcal{F}$ has a timelike leaf $L \in \mathcal{F}$. By the same argument as above, we see that $\mathbf{T} = M$.

Assume that $\mathcal{F}$ has neither spacelike leaves nor timelike leaves. Hence all the leaves of $\mathcal{F}$ are lightlike. This proves the proposition. □

Now we consider elements of holonomy. The following propositions give sufficient conditions for the existence of an element of holonomy.

Proposition 3.5 *If an $\mathcal{H}$-curve $\sigma : [0, t_0] \to M$ intersects only spacelike or timelike leaves then there exists an element of holonomy along σ.*

Proof. We may suppose that σ is contained in a foliation chart U by decomposing the domain $[0, t_0]$. We may also assume that U intersects only spacelike leaves, because the same argument works for the case of timelike leaves.

We denote the submersion defining the foliation $\mathcal{F}|_U$ by $f : U \to \mathbf{R}^k$. We can assume that $f \circ \sigma$ is a simple curve in $\mathbf{R}^k$. Define the vector field V by

$$V_{f\circ\sigma(t)} = \frac{d}{dt}(f \circ \sigma)\Big|_t$$

and put

$$U' = \bigcup \{P \in \mathcal{F}|_U \mid f(P) \in f \circ \sigma([0, t_0])\}.$$

Define the local vector field X on U' tangent to $\mathcal{H}$ by $f_* X = V$. By the condition that X is tangent to $\mathcal{H}$, this vector field X is well-defined. And by the construction, X is foliation preserving. Therefore after shaving U' we get an element of holonomy along σ. This proves the proposition. □

The following proposition was given in different terms in Zeghib [Z].

Proposition 3.6 (Zeghib [Z]) *If an $\mathcal{H}$-curve $\sigma : [0, t_0] \to M$ is contained in a lightlike leaf then there exists an element of holonomy along σ.*

Proof. Assume that σ is contained in a lightlike leaf L. By definition, there exists a distribution $\mathcal{N} \subset TL$ which consists of all lightlike vectors tangent to L. Since the velocity vectors of σ are contained in $\mathcal{H}$, those must be contained in $\mathcal{N}$. There is a vector field $X \in \Gamma(\mathcal{N})$ which is defined on an open set in L and satisfies $X_{\sigma(t)} = \dot{\sigma}(t)$. This vector field X is foliation preserving. Thus we can construct an element of holonomy along σ. $\square$

4. The key proposition

In this section we prove a proposition, which will be used to prove the main theorem in the next section.

We denote the coordinates of $S^1 \times [0,1] = \mathbf{R}/2\pi\mathbf{Z} \times [0,1]$ by (x, y). Let

$$g = f(dx \otimes dy + dy \otimes dx)$$

be a Lorentzian metric on $S^1 \times [0,1]$, where f is a function defined on $S^1 \times [0,1]$. Hence f is the function whose range is bounded away from 0. Note that the vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are lightlike.

Proposition 4.1 *Let $(S^1 \times [0,1], g)$ be as above. Then there exists no totally geodesic foliation $\mathcal{F}$ which satisfies the following conditions:*

(1) *$S^1 \times \{0\}$ is a leaf of $\mathcal{F}$;*

(2) *All the leaves of $\mathcal{F}|_{S^1\times(0,1]}$ are spacelike (or timelike).*

We will prove the proposition by contradiction. We assume the existence of a totally geodesic foliation $\mathcal{F}$ satisfying (1) and (2) in Proposition 4.1. We are going to prove that the existence of such a foliation leads to contradiction.

We may assume that $f > 0$ and all the leaves of $\mathcal{F}|_{S^1\times(0,1]}$ are spacelike by considering the pull back of $\mathcal{F}$ and g by the diffeomorphism $(x, y) \to (-x, y)$ and considering $-g$, if necessary. We take the frame $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ of $T(S^1 \times [0,1])$. Hence the tangent vectors of a leaf of $\mathcal{F}|_{S^1\times(0,1]}$ lie in the first and the third quadrants (Figure 1).

Lemma 4.2 *All the leaves of $\mathcal{F}|_{S^1\times(0,1]}$ are non-closed (Figure 2).*

Proof. Assume there exists a compact leaf $L \in \mathcal{F}|_{S^1\times(0,1]}$. Taking a finite covering, we can assume $[L] = \pm 1 \in H_1(S^1 \times [0,1])$, because the vector field $\frac{\partial}{\partial y}$ cannot be tangent to L. So we can regard L as a "graph" of S^1. Therefore there is a point $p \in S^1$ such that $T_pL = \mathbf{R}\frac{\partial}{\partial x}$. This contradicts

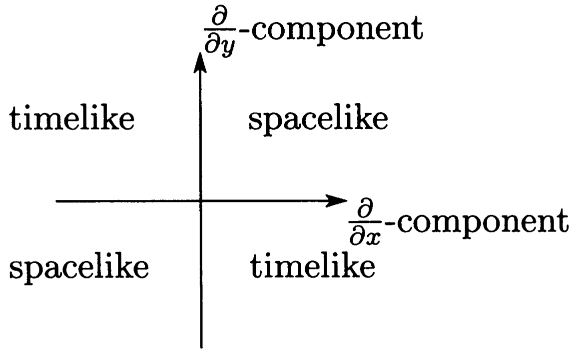

Fig. 1. $T_p(S^1 \times [0,1])$

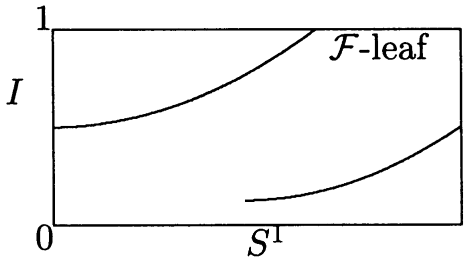

Fig. 2. a leaf $\in \mathcal{F}|_{S^1\times(0,1]}$

that the tangent vectors of L are spacelike. This proves Lemma 4.2. □

Define the Riemannian metric g_R on $S^1 \times [0,1]$ by $g_R = dx^2 + dy^2$. Let d be the distance function induced by g_R. For a submanifold $N \subset S^1 \times [0,1]$, define $\operatorname{diam}_N N$ by

$$\operatorname{diam}_N N = \sup_{p,q\in N} \mathrm{d}_N(p,q), \tag{1}$$

where d_N denotes the distance function induced by $g_R|_N$. Recall that all the leaves of $\mathcal{F}|_{S^1\times(0,1]}$ are spacelike.

Lemma 4.3 *Let $U = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$ ($\beta > 0$) be a vector field tangent to $\mathcal{F}|_{S^1\times(0,1]}$ such that $g(U_p, U_p) = 1$ for all $p \in S^1 \times (0,1]$. Then for each $A > 0$ there exists a $\delta > 0$ such that $g_R(U_p, U_p) > A$ for all $p \in S^1 \times (0,\delta)$.*

Proof. All the leaves of $\mathcal{F}|_{S^1\times(0,1]}$ approach $S^1 \times \{0\}$. Let $\{p_n\}_{n=1}^{\infty}$ be an arbitrary sequence which converges to some point in $S^1 \times \{0\}$. So

$$\lim_{n\to\infty} \frac{\beta(p_n)}{\alpha(p_n)} = 0.$$

By the assumption of U, we have $2\alpha\beta f = 1$. Hence

$$\lim_{n\to\infty} 2\beta(p_n)^2 f(p_n) = \lim_{n\to\infty} \frac{\beta(p_n)}{\alpha(p_n)} = 0.$$

We have $\lim_{n\to\infty} \beta(p_n)^2 = 0$, because the range of f is bounded away from 0. Hence $\lim_{n\to\infty} \alpha(p_n) = +\infty$. By setting $\beta(S^1 \times \{0\}) = 0$, we can extend β to a C^0 function on $S^1 \times [0,1]$. Because $\beta|_{S^1\times(0,1]} > 0$, the set $\beta^{-1}([0,\nu))$ for each $\nu > 0$ is open in $S^1 \times [0,1]$ and contains $S^1 \times \{0\}$. Hence for each $\nu > 0$ there exists a $\delta > 0$ such that $\beta(S^1 \times [0,\delta]) \subset [0,\nu)$.

Now assume that an arbitrary $A > 0$ are given. Take a $\nu > 0$ such that

$$0 < \nu < \frac{1}{2A(\max f)}.$$

Thus there exists a $\delta > 0$ such that $\beta(S^1 \times (0,\delta)) \subset (0,\nu)$. Hence for $p \in S^1 \times (0,\delta)$,

$$\alpha(p) = \frac{1}{2\beta(p)f(p)} > \frac{1}{2\nu f(p)} > \frac{2A(\max f)}{2f(p)} \geq A.$$

This means $g_R(U_p, U_p) > A$. This proves Lemma 4.3. □

Fix $L \in \mathcal{F}|_{S^1\times(0,1]}$ and $p \in L$. Let $\mathcal{H}$ be the foliation defined by the normal distribution of $\mathcal{F}$. We denote by $V_\epsilon(p)$ the ϵ-neighborhood of p in L about the distance induced from the Riemannian metric $g|_L$. Orient $\mathcal{F}$ and $\mathcal{H}$ so that ω limit sets of leaves in $S^1 \times (0,1]$ are $S^1 \times \{0\}$. (Notice that orientations of $S^1 \times \{0\}$ from $\mathcal{F}$ and $\mathcal{H}$ are opposite.) Define an $\mathcal{H}$-curve $h : [0,\infty) \to S^1 \times [0,1]$ such that $h(0) = p$ and $h(t)$ approaches $S^1 \times \{0\}$ as $t \to \infty$ (see Figure 3).

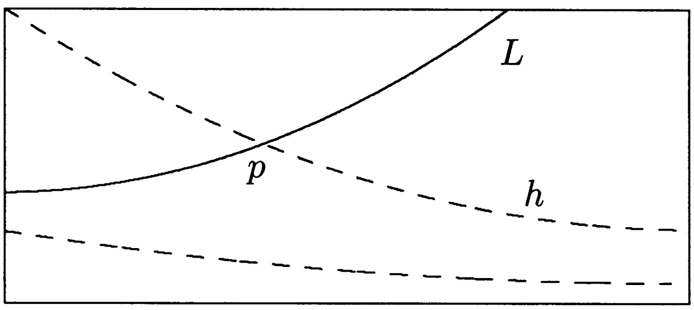

Fig. 3.

Lemma 4.4 *There is an $\epsilon > 0$ such that any $\mathcal{H}$-curve passing through $V_\epsilon(p)$ intersects $V_\epsilon(p)$ only one time.*

Proof. By the same argument as in Lemma 4.2, all the leaves of $\mathcal{H}|_{S^1\times(0,1]}$ are non-closed. We assume that for arbitrary $n \in \mathbf{N}$ there exist y_n and $y'_n \in V_{\frac{1}{n}}(p)$ such that the $\mathcal{H}$-curve from y_n with the positive direction intersects $V_{\frac{1}{n}}(p)$ at y'_n for the first time. Notice that

$$\lim_{n\to\infty} y_n = p \quad \text{and} \quad \lim_{n\to\infty} y'_n = p.$$

Let W be the union of $\{p\}$ and a connected component of $V_{\frac{1}{2}}(p) - \{p\}$ containing infinitely many y_n. If $y_n \in W$ then $y'_n \in W$. Take a subsequence $\{z_m\} \subset W$ of $\{y_n\}$ such that $\{z_m\}$ monotonely converges to p. Define the subsequence $\{z'_m\}$ of $\{y'_n\}$ corresponding to $\{z_m\}$. Taking subsequences of $\{z_m\}$ and $\{z'_m\}$, we can assume

$$z_1 < z'_1 < z_2 < z'_2 < \cdots < p \quad \text{or} \quad p < \cdots < z_2 < z'_2 < z_1 < z'_1,$$

where "$a < b$" denotes the existence of $\mathcal{F}$-curve from a to b with the positive direction.

Case 1: $z_1 < z'_1 < z_2 < z'_2 < \cdots < p$.

We take a C^∞ immersion

$$C : [0,1] \times [0,1] \to S^1 \times (0,1)$$

such that

$C([0,1] \times \{*\})$ is contained in an $\mathcal{F}$-leaf,
$C(\{*\} \times [0,1])$ is contained in an $\mathcal{H}$-leaf,
$C(0,0) = z_1$,
$C(1,0) = p$, and
$C([0,1] \times \{0\})$ and $C([0,1] \times \{1\})$ are contained in the same $\mathcal{F}$-leaf L.

We can construct C as follows. First we take an $\mathcal{F}$-curve $F : [0,1] \to S^1 \times (0,1)$ such that $F(0) = z_1$, $F(1) = p$ and $\frac{d}{dt}F \neq 0$. Similarly we take an $\mathcal{H}$-curve $H : [0,1] \to S^1 \times (0,1)$ such that $H(0) = z_1$, $H(1) = z'_1$ and $\frac{d}{dt}H \neq 0$. Next we take lifts $\hat{F}$ and $\hat{H}$ of F and H onto the universal covering $\pi : \mathbf{R} \times (0,1) \to S^1 \times (0,1)$ so that $\hat{F}(0) = \hat{H}(0)$. Consider the rectangle defined by $\hat{H}([0,1])$ and $\hat{F}([0,1])$. We define $\hat{C}$ by $\hat{C}(t,s) = (\hat{F}(s), \hat{H}(t))$. Thus $\hat{C} : [0,1] \times [0,1] \to \mathbf{R} \times (0,1)$ is a C^∞ embedding. We get the C^∞

immersion C by $C = \pi \circ \hat{C}$.

Now we define $\tau_m \in [0,1]$ by $C(\tau_m, 0) = z_m$. Thus $\lim_{m\to\infty} \tau_m = 1$. By the construction of τ_m and C, we have $C(\tau_m, 1) = z'_m$. A curve $k(t) = C(1,t)$ with parameter $t \in [0,1]$ is an $\mathcal{H}$-curve. We have

$$k(0) = p, \quad k(1) = C(1,1) = C\Big(\lim_{m\to\infty} \tau_m, 1\Big) = \lim_{m\to\infty} z'_m = p.$$

Therefore the $\mathcal{H}$-curve passing through p is closed. This contradicts the fact that all the leaves of $\mathcal{H}|_{S^1\times(0,1]}$ are non-closed.

Case 2: $p < \cdots < z_2 < z'_2 < z_1 < z'_1$.

We can prove it by a similar argument. This proves Lemma 4.4. □

Fix $2\epsilon > 0$ satisfying the condition of Lemma 4.4. Denote by $V_{h(0)}$ the closure of $V_\epsilon(p)$ in $V_{2\epsilon}(p)$. By considering $\mathrm{Sat}_{\mathcal{H}} V_{h(0)}$, we can construct an element of holonomy along h

$$\{\psi_t : V_{h(0)} \to V_{h(t)}\}_{t\in[0,\infty)}.$$

Lemma 4.5 *If $t \neq t'$ then $V_{h(t)} \cap V_{h(t')} = \phi$.*

Proof. Assume that there exist t and t' $(t > t')$ such that $V_{h(t)} \cap V_{h(t')} \neq \phi$. Fix a point $x \in V_{h(t)} \cap V_{h(t')}$ and put $t'' = t - t'$. By the definition of an element of holonomy, there exist y and $y' \in V_{h(0)}$ such that $\psi_t(y) = x$ and $\psi_{t'}(y') = x$. Thus

$$\psi_{t'+t''}(y) = \psi_t(y) = \psi_{t'}(y').$$

Hence $\psi_{t''}(y) = y'$. By the definition, $\psi_s(z)$ with parameter $s \in [0,\infty)$ is an $\mathcal{H}$-curve for each $z \in V_{h(0)}$. Therefore there exists an $\mathcal{H}$-curve from y to y'. This contradicts Lemma 4.4. This proves Lemma 4.5. □

Lemma 4.6

$$\lim_{t\to\infty} \mathrm{diam}_{V_{h(t)}} V_{h(t)} = +\infty.$$

Proof. First we take the unit curve (about the metric $g|_L$) $c(s) = (x(s), y(s))$ with parameter $s \in [-\epsilon, \epsilon]$ on $V_{h(0)}$ such that $\frac{d}{ds}x > 0$ and $\frac{d}{ds}y > 0$ for all $s \in [-\epsilon, \epsilon]$. Thus

$$\mathrm{diam}_{V_{h(0)}} V_{h(0)} = \int_{-\epsilon}^{\epsilon} \sqrt{g_R\left(\frac{d}{ds}c, \frac{d}{ds}c\right)}ds.$$

The curve $\psi_t(c(s))$ with parameter $s \in [-\epsilon, \epsilon]$ is the unit curve on $V_{h(t)}$ about the distance induced from $g|_{L_{h(t)}}$ (Figure 5). So

$$\operatorname{diam}_{V_{h(t)}} V_{h(t)} = \int_{-\epsilon}^{\epsilon} \sqrt{g_R\Big(\frac{d}{ds}\left(\psi_t\left(c\left(s\right)\right)\right), \frac{d}{ds}\left(\psi_t\left(c\left(s\right)\right)\right)\Big)}ds.$$

Next we prove that for each $\delta > 0$ there exists a $t > 0$ such that $V_{h(t)} \subset S^1 \times (0, \delta)$. The $\mathcal{H}$-curve $\psi_t(c(\epsilon))$ approaches $S^1 \times \{0\}$ as $t \to \infty$. Hence for each $\delta > 0$ there exists a $t > 0$ such that $\{\psi_s(c(\epsilon)) \mid s \in [t, \infty)\} \subset S^1 \times (0, \delta)$. Because $\psi_t(c(\epsilon))$ has the largest y-component of all the points in $V_{h(t)}$, for each $\delta > 0$ there exists a $t > 0$ such that $V_{h(t)} \subset S^1 \times (0, \delta)$. Combining this with Lemma 4.3, we get that for each $A > 0$ there exists a $t > 0$ such that $\operatorname{diam}_{V_{h(t)}} V_{h(t)} \geq 2\epsilon A$. This proves Lemma 4.6. □

Lemma 4.7 *Let (x, y) be the canonical coordinates of $\mathbf{R}^2$. Put $g_R = dx^2 + dy^2$. Assume that a curve $c(t) = (x(t), y(t))$ with parameter $t \in [0, 1]$ satisfies*

$$\dot{x}(t) = \frac{d}{dt}x(t) > 0, \quad \dot{y}(t) = \frac{d}{dt}y(t) > 0 \quad \textit{for all} \ \ t \in [0, 1].$$

Then (see Figure 4)

$$\int_0^1 \sqrt{g_R(\dot{c}(t), \dot{c}(t))}dt \leq \int_0^1 \dot{x}(t)dt + \int_0^1 \dot{y}(t)dt.$$

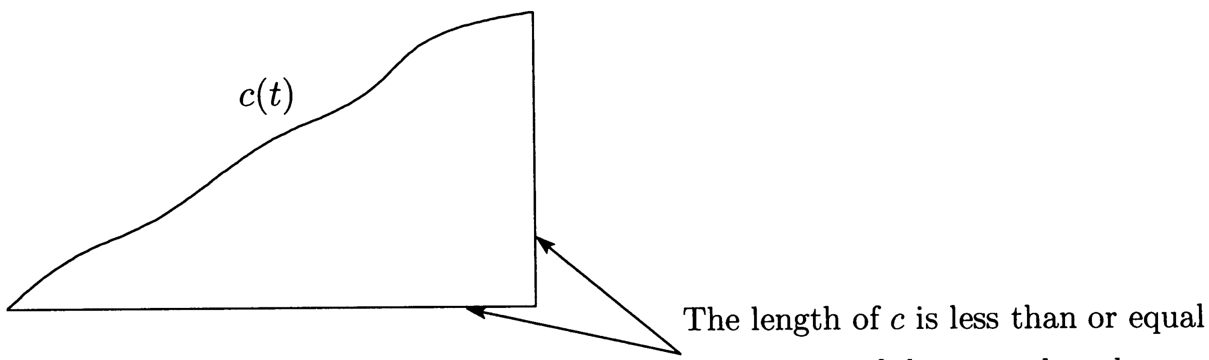

Fig. 4.

Proof. Notice that

$$\{\dot{x}(t) + \dot{y}(t)\}^2 - \{\dot{x}(t)\}^2 - \{\dot{y}(t)\}^2 = 2\dot{x}(t)\dot{y}(t) > 0.$$

Therefore

$$\int_0^1 \{\dot{x}(t)+\dot{y}(t)\}dt - \int_0^1 \sqrt{\dot{x}(t)^2+\dot{y}(t)^2}dt > 0.$$

This proves the lemma. □

Now we prove the proposition.

Proof of Proposition 4.1. Assume that $\mathcal{F}$ is a totally geodesic foliation which satisfies (1) and (2) of Proposition 4.1. We may assume that $f>0$ and all the leaves of $\mathcal{F}|_{S^1\times(0,1]}$ are spacelike. Define the Riemannian metric g_R on $S^1\times[0,1]$ by $g_R = dx^2+dy^2$. Let $\mathcal{H}$ be the foliation defined by the normal distribution of $\mathcal{F}$ with respect to g. Fix $L\in\mathcal{F}|_{S^1\times(0,1]}$ and $p\in L$. Define an $\mathcal{H}$-curve $h:[0,\infty)\to S^1\times[0,1]$ such that $h(0)=p$ and $h(t)$ approaches $S^1\times\{0\}$ as $t\to\infty$ (see Figure 3). Fix $2\epsilon>0$ satisfying the condition of Lemma 4.4. Denote by $V_{h(0)}$ the closure of $V_\epsilon(p)$ in $V_{2\epsilon}(p)$. By considering $\mathrm{Sat}_{\mathcal{H}}V_{h(0)}$, we can construct an element of holonomy along h

$$\left\{\psi_t : V_{h(0)}\to V_{h(t)}\right\}_{t\in[0,\infty)}.$$

We lift $V_{h(t)}$ to the universal cover $\mathbf{R}\times[0,1]\to S^1\times[0,1]$. We can write (see equation (1) for the definition of $\mathrm{diam}_{V_{h(t)}} V_{h(t)}$)

$$\mathrm{diam}_{V_{h(t)}} V_{h(t)} = \int_{-\epsilon}^{\epsilon}\sqrt{g_R\Big(\frac{d}{ds}\psi_t(c),\frac{d}{ds}\psi_t(c)\Big)}ds$$

as in the proof of Lemma 4.6 (Figure 5). We have $\lim_{t\to\infty}\mathrm{diam}_{V_{h(t)}} V_{h(t)} = +\infty$ by Lemma 4.6. Applying the above lemma, we have

$$\left|\int_{-\epsilon}^{\epsilon}\frac{d}{ds}y(s)ds\right| = |y(\epsilon)-y(-\epsilon)|\le 1.$$

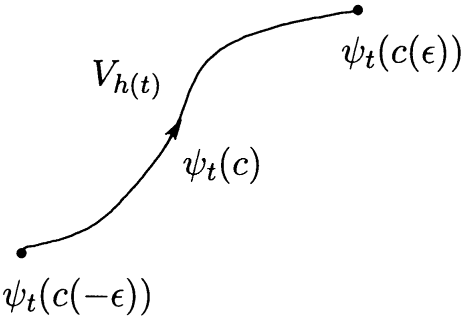

Fig. 5.

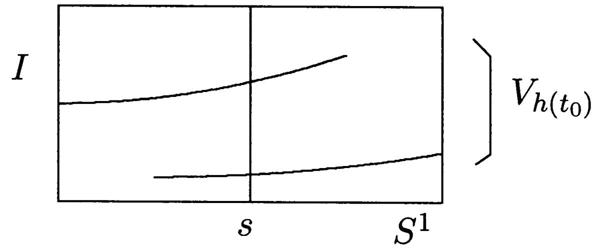

Fig. 6.

This means $|x(\epsilon) - x(-\epsilon)| \to \infty$. By this argument there exist $t_0 \in [0, \infty)$ and $s \in S^1$ such that $V_{h(t_0)} \cap (\{s\} \times [0,1])$ contains more than two points (Figure 6).

Choose distinct points $q_0, q_1 \in V_{h(t_0)} \cap (\{s\} \times I)$ so that q_0 is nearer to $S^1 \times \{0\}$ than q_1, and all the points in $V_{h(t_0)}$ between q_0 and q_1 do not intersect $\{s\} \times I$. Cutting $S^1 \times I$ along $\{s\} \times I$, we get $I \times I$. The $\mathcal{H}$-curve passing through q_1 must intersect $V_{h(t_0)}$ (Figure 7), because its tangent vectors must lie in the second and the fourth quadrants (Figure 1). Hence there exist $t_1 > t_0$ such that $V_{h(t_1)} \cap V_{h(t_0)} \neq \phi$. This contradicts Lemma 4.5, and Proposition 4.1 is proved. $\square$

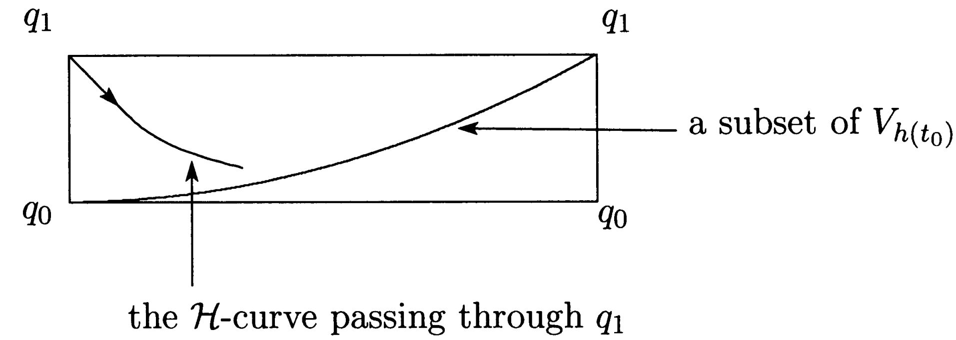

Fig. 7.

5. The proof of main theorem

In this section we show that a totally geodesic foliation of a lightlike complete Lorentzian manifold T^2 has only one kind of leaves among spacelike, timelike, and lightlike ones. Examples B and C show that there is an example of a totally geodesic foliation of an incomplete Lorentzian 2-torus which contains more than or equal to two kinds of leaves.

We review the main theorem.

Theorem 5.1 *Let (T^2, g) be a lightlike complete Lorentzian 2-torus. There exists no totally geodesic foliation which contains more than or equal to two kinds of leaves among spacelike, timelike, and lightlike ones.*

First we recall some results, which will be used in the proof. Let (M, g) be a two-dimensional Lorentzian manifold. Because the set of all lightlike vectors on M can be described as a union of two line bundles, there exist two foliations whose leaves are lightlike submanifolds. Only they are the lightlike totally geodesic foliations. We call them the *canonical lightlike totally geodesic foliations* of M. The following is known.

Theorem 5.2 (Carrière-Rozoy [CR]) *Let (T^2, g) be a C^2 Lorentzian torus. If g is lightlike complete, then the canonical lightlike totally geodesic foliations are C^0-linearizable.*

We are going to prove the following lemmas.

Lemma 5.3 *Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be foliations of T^2. If all the leaves of $\mathcal{G}_0$ and $\mathcal{G}_1$ are compact and $\mathcal{G}_0$ and $\mathcal{G}_1$ are transverse each other, then $L_0 \cap L_1 \neq \phi$ for each $L_0 \in \mathcal{G}_0$ and each $L_1 \in \mathcal{G}_1$.*

Proof. We assume that there exist $L_0 \in \mathcal{G}_0$ and $L_1 \in \mathcal{G}_1$ such that $L_0 \cap L_1 = \phi$. By the standard Euler class argument, both of the homology classes $[L_0]$ and $[L_1]$ are non-zero. Cutting T^2 along L_0, we obtain a manifold diffeomorphic to $[0,1] \times S^1$. From $\mathcal{G}_1$ we obtain a foliation of $[0,1] \times S^1$ which is transverse to the boundaries. Since $L_1 \subset [0,1] \times S^1$ is a compact leaf, $[L_1] = \pm 1$ in $H_1([0,1] \times S^1)$. Hence $[0,1] \times S^1 - L_1$ has two connected components. So the $\mathcal{G}_1$-leaves which intersect L_0 are not closed. Therefore this contradicts that all the leaves of $\mathcal{G}_1$ are compact. This proves the lemma. □

By using the theorem of Carrière-Rozoy, we have the following lemma.

Lemma 5.4 *Let (T^2, g) be a lightlike complete Lorentzian 2-torus and $\mathcal{F}$ a totally geodesic foliation of T^2. Denote the canonical lightlike totally geodesic foliations by $\mathcal{G}_0$ and $\mathcal{G}_1$. Let $T^2 = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ denote the STL-decomposition of T^2 by $\mathcal{F}$. Assume that*

$$\phi \neq \mathbf{L} \subsetneqq T^2.$$

Then $\mathbf{L}$ consists of either $\mathcal{G}_0$-leaves or $\mathcal{G}_1$-leaves. (And all the leaves contained in $\mathbf{L}$ are compact.)

Proof. By Theorem 5.2, we can consider the following cases for each of $\mathcal{G}_0$ and $\mathcal{G}_1$:

(c) all leaves are compact;
(d) all leaves are dense.

Thus we have the following cases:

	(i)	(ii)	(iii)	(iv)
$\mathcal{G}_0$	d	d	c	c
$\mathcal{G}_1$	d	c	d	c

Consider the STL-decomposition of T^2 by $\mathcal{F}$:

$$T^2 = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}.$$

Since $\mathbf{L}$ is the set of all lightlike leaves of $\mathcal{F}$, the set $\mathbf{L}$ is a union of leaves of $\mathcal{G}_0$ and $\mathcal{G}_1$.

In the case (i), all leaves of $\mathcal{G}_0$ and $\mathcal{G}_1$ are dense in T^2. Thus $\mathbf{L}$ must contain a dense leaf. So $\mathbf{L} = \overline{\mathbf{L}} = T^2$. Therefore this contradicts the assumption. Hence this case does not occur.

In the case (ii), the set $\mathbf{L}$ cannot contain $\mathcal{G}_0$-leaves, because leaves of $\mathcal{G}_0$ are dense in T^2. Therefore $\mathbf{L}$ is a union of some $\mathcal{G}_1$-leaves.

In the case (iii), by changing the numbering of $\mathcal{G}_0$ and $\mathcal{G}_1$, we can reduce the case (iii) to the case (ii).

In the case (iv), the set $\mathbf{L}$ cannot contain $\mathcal{G}_0$-leaves and $\mathcal{G}_1$-leaves together by Lemma 5.3. Therefore $\mathbf{L}$ contains either $\mathcal{G}_0$-leaves or $\mathcal{G}_1$-leaves. □

Now we prove the main theorem.

Proof of Theorem 5.1. We prove Theorem 5.1 by contradiction. Assume that there exists a totally geodesic foliation $\mathcal{F}$ of a lightlike complete Lorentzian 2-torus T^2 which contains more than or equal to two kinds of leaves among spacelike, timelike, and lightlike ones. Notice that $\phi \neq \mathbf{L} \subsetneqq T^2$. Since $\mathcal{F}$ satisfies the assumption of Lemma 5.4, the set $\mathbf{L}$ consists of either $\mathcal{G}_0$-leaves or $\mathcal{G}_1$-leaves. By the proof of Lemma 5.4 and renumbering $\mathcal{G}_0$ and $\mathcal{G}_1$, we can assume that all the leaves of $\mathcal{G}_0$ are compact and that $\mathbf{L}$ is a union of some $\mathcal{G}_0$-leaves.

Case 1: $\mathbf{S} \neq \phi$.

Fix a Riemannian metric ζ on T^2, and consider the completion of $\mathbf{S}$ with respect to $\zeta|_{\mathbf{S}}$ (see [CC], for example) and immerse it to T^2. Denote the immersed image by $\hat{\mathbf{S}}$. Since $\hat{\mathbf{S}}$ is a closed saturated set, $\hat{\mathbf{S}} - \mathbf{S}$ is a closed saturated set contained in $\mathbf{L}$. Fix an arbitrary compact leaf $L_0 \subset \hat{\mathbf{S}} - \mathbf{S}$.

Take an embedding $\varphi : S^1 \times [0,1] \to T^2$ such that

(a) $\varphi(S^1 \times \{*\}) \in \mathcal{G}_0$,
(b) $\varphi(\{*\} \times [0,1])$ is contained in a leaf of $\mathcal{G}_1$,
(c) $\varphi(S^1 \times \{0\}) = L_0$, and
(d) $\varphi(S^1 \times (0,1]) \subset \mathbf{S}$.

Let (x,y) be a coordinate of $S^1 \times [0,1]$. Notice that the vector fields $\varphi_*(\frac{\partial}{\partial x})$ and $\varphi_*(\frac{\partial}{\partial y})$ are lightlike. Hence we can write

$$\varphi^* g = f(dx \otimes dy + dy \otimes dx),$$

where f is a function whose range is bounded away from 0. Notice that $\varphi : (S^1 \times [0,1], \varphi^* g) \hookrightarrow (T^2, g)$ is an isometry. Therefore $\varphi^* \mathcal{F}$ is a totally geodesic foliation of $(S^1 \times [0,1], \varphi^* g)$ which satisfies the condition (1) and (2) in Proposition 4.1. Therefore the existence of such a foliation contradicts Proposition 4.1.

Case 2: $\mathbf{T} \neq \phi$.

We are brought to a contradiction in the same way as above. This proves Theorem 5.1. □

The foliation in Example D has Reeb components. In contrast with this foliation, we have the following.

Corollary 5.5 *Let (T^2, g) be a lightlike complete Lorentzian 2-torus. Let $\mathcal{F}$ be a totally geodesic foliation of T^2. Then $\mathcal{F}$ has no Reeb components.*

Proof. If all the leaves of $\mathcal{F}$ are spacelike or timelike, then the foliation $\mathcal{H}$ defined by the normal distribution of $\mathcal{F}$ is a Riemannian foliation. According to [CG], $\mathcal{F}$ is geodesible in Riemannian sense. By [G], $\mathcal{F}$ has no Reeb components. If all the leaves of $\mathcal{F}$ are lightlike, then $\mathcal{F}$ has no Reeb components by [CR]. This proves the corollary. □

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