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Finsler metrics of positive constant flag curvature on Sasakian space forms

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Abstract. Let $M(c)$ be a Sasakian space form of constant φ-sectional curvature $c \in (-3, 1)$. We prove that for any $K > 0$ there exists a Randers metric on $M(c)$ of constant flag curvature K. Moreover, we show that such a Randers metric is not projectively flat. In particular, this means that every odd dimensional sphere admits such metrics.

Key words: Finsler manifolds of constant flag curvature, Sasakian space forms, Randers metric.

1. Introduction

Finsler metrics of constant flag curvature are unanimously considered to be of great interest in Finsler geometry. Under some growth constraints on the Cartan tensor, Akbar-Zadeh [1] proved that a closed Finsler manifold with constant flag curvature K is locally Minkowskian if $K = 0$, and Riemannian if $K = -1$. The case $K > 0$ is the least understood. Bryant [8] constructed many interesting non-Riemannian examples on the sphere S^2 with $K = 1$. Important results on the geometric structure of Finsler manifolds of constant flag curvature $K = 1$ have been obtained by Shen [11].

Recently, Bao and Shen [5] constructed Randers metrics of constant flag curvature $K > 1$ on the Lie group S^3. They have also proved that these Finsler space forms are not projectively flat. We would like to thank professors Bao and Shen for sending this reprint [5] to us. Their results have inspired our work, thus producing the present paper.

Our purpose here is to construct Randers metrics of positive constant flag curvature on a Sasakian space form subject to some constraints on the φ-sectional curvature. More precisely, we prove the following theorem.

Theorem 1 *Let $M(c)$ be a Sasakian space form of constant φ-sectional curvature $c \in (-3, 1)$. Then for any constant $K > 0$ there exists a Randers metric F on $TM(c)$ such that $(M(c), F)$ has constant flag curvature K and*

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is not projectively flat.

2. Sasakian space forms

Let M be a real $(2m+1)$-dimensional differentiable manifold and φ, ξ, and η be a tensor field of type $(1,1)$, a vector field and a 1-form respectively on M satisfying

$$\text{(a)}\quad \varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \text{(b)}\quad \eta(\xi) = 1. \tag{2.1}$$

Then we say that M has a (φ, ξ, η)-structure. It is proved (see Blair [7], pp.20, 21) that we have

$$\text{(a)}\quad \varphi\xi = 0 \quad \text{and} \quad \text{(b)}\quad \eta o \varphi = 0. \tag{2.2}$$

Also, there exists a Riemannian metric a on M such that

$$a(\varphi X, \varphi Y) = a(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X and Y on M. Taking $Y = \xi$ in (2.3) and by using (2.2a) and (2.1b) we obtain

$$\eta(X) = a(X, \xi). \tag{2.4}$$

Similarly, replace Y by φY in (2.3) and by using (2.1a), (2.2b) and (2.4) we deduce that

$$a(\varphi X, Y) + a(X, \varphi Y) = 0. \tag{2.5}$$

Throughout the paper we denote by $\mathcal{F}(M)$ the algebra of differentiable functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$-module of sections of a vector bundle E over M. Also, we make use of the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range.

The manifold M endowed with a (φ, ξ, η)-structure is called a *Sasakian manifold* if the above tensor fields satisfy

$$(\nabla_X \varphi)Y = a(X, Y)\xi - \eta(Y)X, \quad \forall X, Y \in \Gamma(TM), \tag{2.6}$$

where ∇ is the Levi-Civita connection on M with respect to the Riemannian metric a. Also, on the Sasakian manifold M we have

$$\nabla_X \xi = -\varphi X, \quad \forall X \in \Gamma(TM). \tag{2.7}$$

By direct calculations using (2.4), (2.7) and (2.5) we obtain

$$(\nabla_X \eta)Y = a(X, \varphi Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.8}$$

Thus (2.8) and (2.5) imply that

$$(\nabla_X \eta)Y + (\nabla_Y \eta)X = 0, \quad \forall X, Y \in \Gamma(TM). \tag{2.9}$$

Morover, by using (2.8), (2.6) and (2.4) we deduce that the second order covariant derivative of η is given by

$$(\nabla_Z \nabla_X \eta)Y = a(Y, Z)\eta(X) - a(X, Z)\eta(Y), \quad \forall X, Y, Z \in \Gamma(TM). \tag{2.10}$$

Next, we denote by D the contact distribution on M, that is, D is the orthogonal complementary distribution to the 1-dimensional distribution spanned by ξ on M. A plane section in the tangent space T_xM is said to be a *φ-section* if it is spanned by X and φX where $X \in D_x$. The sectional curvature $K(\Pi)$ determined by a φ-section Π is called a **φ-sectional curvature**. A Sasakian manifold of constant φ-sectional curvature c is called a **Sasakian space form** and it is denoted by $M(c)$. The curvature tensor R of $M(c)$ is given by (cf. Blair [7], p.97)

$$\begin{aligned}
&a(R(X, Y)Z, W)\\
&= \frac{c+3}{4}\{a(Y, Z)a(X, W) - a(X, Z)a(Y, W)\}\\
&+ \frac{1-c}{4}\{\eta(Y)\eta(Z)a(X, W) + \eta(X)\eta(W)a(Y, Z)\\
&\quad - \eta(X)\eta(Z)a(Y, W) - \eta(Y)\eta(W)a(X, Z)\\
&\quad - a(Y, \varphi Z)a(X, \varphi W) + a(X, \varphi Z)a(Y, \varphi W)\\
&\quad + 2a(W, \varphi Z)a(Y, \varphi X)\}, \quad \forall X, Y, Z, W \in \Gamma(TM).
\end{aligned} \tag{2.11}$$

We need some of those formulas expressed in local coordinates. To this end we set

$$a_{ij} = a\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right); \quad \eta_i = \eta\left(\frac{\partial}{\partial x^i}\right); \quad \eta^j = \eta_i a^{ij}$$

$$\eta_{i|j} = \left(\nabla_{\frac{\partial}{\partial x^j}} \eta\right)\left(\frac{\partial}{\partial x^i}\right); \quad \eta_{i|j|k} = \left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \eta\right)\left(\frac{\partial}{\partial x^i}\right);$$

$$R_{hijk} = a\left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^i}\right).$$

Then, from (2.4) and (1.16) we deduce that

$$\|\eta\|^2 = \eta_i \eta^i = 1. \tag{2.12}$$

Also, (2.9) and (2.10) become

$$\eta_{j|i} + \eta_{i|j} = 0, \tag{2.13}$$

and

$$\eta_{i|j|k} = a_{ik}\eta_j - a_{jk}\eta_i, \tag{2.14}$$

respectively. Finally, we substitute X, Y, Z, W by $\partial/\partial x^k$, $\partial/\partial x^j$, $\partial/\partial x^h$, $\partial/\partial x^i$ in (2.11) and by using (2.8) and the above notations we obtain

$$\begin{aligned} R_{hijk} = {} & \frac{c+3}{4}\{a_{jh}a_{ik} - a_{kh}a_{ij}\} \\ & + \frac{1-c}{4}\{\eta_j\eta_h a_{ki} + \eta_k\eta_i a_{jh} - \eta_k\eta_h a_{ij} - \eta_i\eta_j a_{kh} \\ & - \eta_{h|j}\eta_{i|k} + \eta_{h|k}\eta_{i|j} + 2\eta_{h|i}\eta_{k|j}\}. \end{aligned} \tag{2.15}$$

3. Finsler metrics of constant flag curvature

Let $\mathbb{F}^n = (M, F)$ be a Finsler manifold, where M is an n-dimensional C^∞ manifold and F is the **Finsler metric** of $\mathbb{F}^n$. Here, F is supposed to be a C^∞ function on the slit tangent bundle $TM^\circ = TM\backslash\{0\}$ satisfying

(i) $F(x, ky) = kF(x, y)$, for any $x \in M$, $y \in T_xM$ and $k > 0$ (positive homogeneity).

(ii) The $n \times n$ Hessian matrix

$$[g_{ij}(x,y)] = \left[\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)\right],$$

is positive definite at every point (x, y) of TM°. We denote by (x^i, y^i) the coordinates on TM where (x^i) are the coordinates on M. The local frame field on TM is $\{\partial/\partial x^i,\ \partial/\partial y^i\}$. Then the Liouville vector field $L = y^i(\partial/\partial y^i)$ is a global section of the vertical vector bundle TM°. Moreover, $\ell = (1/F)L$ is a unit vector field,

$$g_{ij}(x,y)\ell^i\ell^j = 1, \quad \text{where } \ell^i = \frac{y^i}{F}.$$

A complementary vector bundle to the vertical vector bundle VTM° in TTM° is called a **non-linear connection**. The canonical non-linear connection of $\mathbb{F}^n$ is the distribution GTM° whose local frame field is given by (see Bejancu-Farran ([6], p.37)

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \tag{3.1}$$

where we set

$$\text{(a)} \quad G_i^j = \frac{\partial G^j}{\partial y^i}; \quad \text{(b)} \quad G^j = \frac{1}{4} g^{jh} \left(\frac{\partial^2 F^2}{\partial y^h \partial x^k} y^k - \frac{\partial F^2}{\partial x^h} \right). \tag{3.2}$$

By means of the local coefficients G_i^j there are defined the following Finsler tensor fields (cf. Bao-Chern-Shen [4], p.66)

$$\text{(a)} \quad R^k{}_j = \ell^h \left(\frac{\delta}{\delta x^j} \left(\frac{G_h^k}{F} \right) - \frac{\delta}{\delta x^h} \left(\frac{G_j^k}{F} \right) \right); \quad \text{(b)} \quad R_{ij} = g_{ik} R^k{}_j. \tag{3.3}$$

Next, we consider a flag $\ell \wedge V$ at $x \in M$ determined by ℓ and $V = V^i(\partial/\partial x^i)$. Then according to Bao-Chern-Shen [4], p.69 the **flag curvature** for the flag $\ell \wedge V$ is the number

$$K(\ell, V) = \frac{V^i R_{ij} V^j}{g_{ij} V^i V^j - (g_{ij} \ell^i V^j)^2}.$$

When the flag curvature K depends neither on (y^i) nor on (V^i) it is proved that K must be a constant (see Matsumoto [10], p.169). Also, it is proved that $\mathbb{F}^n$ has constant flag curvature K if and only if (see Bao-Chern-Shen [4], p.313).

$$R_{ij} = K h_{ij}, \tag{3.4}$$

where h_{ij} are the components of the angular metric on $\mathbb{F}^n$ given by

$$h_{ij} = g_{ij} - \ell_i \ell_j. \tag{3.5}$$

4. Randers metrics of positive constant flag curvature

In the present section we prove Theorem 1. To this end we first consider a real n-dimensional manifold M endowed with a Riemannian metric $a =$

$(a_{ij}(x))$ and a 1-form $b = (b_i(x))$. Then we define on TM° the function

$$F(x,y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i. \tag{4.1}$$

It can be proved that F is positive-valued on the whole TM° if and only if the length $\|b\|$ of b satisfies (see Antonelli-Ingarden-Matsumoto [2], p.43)

$$\|b\|^2 = b_i(x)b^i(x) < 1, \tag{4.2}$$

where $b^i(x) = a^{ij}(x)b_j(x)$ and $[a^{ij}(x)]$ is the inverse matrix of $[a_{ij}(x)]$. A Finsler metric given by (4.1) is called a **Randers metric** and $\mathbb{F}^n = (M, F)$ is called a **Randers manifold**. Now we prove the following.

Lemma 1 *Let $\mathbb{F}^n = (M, F)$ be a Randers manifold of constant flag curvature $K = 1$. Then for any constant $K^* > 0$ there exists a Randers metric F^* on TM° such that $\mathbb{F}^{n*} = (M, F^*)$ is a Randers manifold of constant flag curvature K^*.*

Proof. By using (3.4) on $\mathbb{F}^n$ we have

$$R_{ij} = h_{ij}. \tag{4.3}$$

Now, define on M the Riemannian metric

$$a^*_{ij}(x) = \frac{1}{K^*}a_{ij}(x), \tag{4.4}$$

and the 1-form

$$b^*_i(x) = \frac{1}{\sqrt{K^*}}b_i(x). \tag{4.5}$$

Then it is easy to check (4.2) for the pair $(b^*_i(x), a^*_{ij}(x))$. Thus the function

$$F^*(x,y) = \sqrt{a^*_{ij}(x)y^i y^j} + b^*_i(x)y^i = \frac{1}{\sqrt{K^*}}F(x,y), \tag{4.6}$$

is a new Randers metric on $\mathbb{F}^n$. Also, we have

$$\text{(a)} \quad g^*_{ij}(x,y) = \frac{1}{K^*}g_{ij}(x,y) \quad \text{and} \quad \text{(b)} \quad g^{ij*}(x,y) = K^* g^{ij}(x,y). \tag{4.7}$$

Then by using (3.2), (4.6) and (4.7b) we deduce that F and F^* define the same canonical non-linear connection i.e., $G^{j*}_i = G^j_i$. As a consequence, (3.3), (4.6) and (4.7a) imply

$$R^*_{ij} = R_{ij}. \tag{4.8}$$

Finally, by using (3.5) for both F and F^* and taking into account (4.6) and (4.7a) we infer that

$$h_{ij} = K^* h^*_{ij}. \tag{4.9}$$

Then (4.3), (4.8) and (4.9) yield $R^*_{ij} = K^* h^*_{ij}$, that is, $\mathbb{F}^{n*}$ is a Randers manifold of constant flag curvature K^*. □

Now, we recall the following important result.

Theorem 2 (Yasuda-Shimada [13]) *A Randers manifold of dimension $n > 2$ is of positive constant flag curvature K if and only if the following conditions are satisfied:*

(i) *The length $\|b\|$ of b is constant.*

(ii) *The covariant derivative of b with respect to the Levi-Civita connection on M satisfies*

$$b_{i|j} + b_{j|i} = 0. \tag{4.10}$$

(iii) *The curvature tensor of the Riemannian manifold $(M, a_{ij}(x))$ is given by*

$$\begin{aligned} R_{hijk} = K(1 - \|b\|^2) &\{a_{hj}a_{ik} - a_{hk}a_{ij}\} \\ &+ K\{b_i b_k a_{hj} + b_h b_j a_{ik} - b_i b_j a_{hk} - b_h b_k a_{ij}\} \\ &\qquad - b_{h|j}b_{i|k} + b_{h|k}b_{i|j} + 2b_{h|i}b_{k|j}. \end{aligned} \tag{4.11}$$

Now, we may complete the proof of Theorem 1. Suppose $M(c)$ is a Sasakian space form of φ-sectional curvature $c \in (-3, 1)$ and (φ, ξ, η, a) is the Sasakian structure on $M(c)$. Then let $\alpha = \sqrt{1-c}/2$ and define a new 1-form $b = \alpha\eta$ on $M(c)$. By using (2.12) we deduce that

$$\|b\|^2 = b_i b^i = \alpha^2 < 1. \tag{4.12}$$

Therefore F given by (4.1) with $a = (a_{ij}(x))$ from the Sasakian structure and the above $b = (b_i(x))$ is a Randers metric on $M(c)$. Then we prove the following.

Lemma 2 *The Randers manifold $\mathbb{F}^{2m+1} = (M(c), F)$ is a Finsler manifold of constant flag curvature $K = 1$.*

Proof. We check the conditions from Theorem 2 for $K = 1$. By (4.12) we see that (i) is satisfied. Then by using (2.13) we obtain (4.10). Finally, from (2.15) we obtain (4.11) since

$$\frac{c+3}{4} = 1 - \alpha^2 = 1 - \|b\|^2; \quad \frac{1-c}{4}\eta_j\eta_h = b_j b_h \quad \text{and}$$
$$\frac{1-c}{4}\eta_{h|j}\eta_{i|k} = b_{h|j}b_{i|k}.$$

This completes the proof of the lemma. □

Combining the Lemmas 1 and 2 we deduce that for any $K^* > 0$, on a Sasakian space form $M(c)$ of constant φ-sectional curvature $c \in (-3, 1)$ there exists a Randers metric of constant flag curvature K^*. Finally we show that the above Randers manifolds are not projectively flat. First, from Douglas [9] we know that a Finlser manifold is projectively flat if and only if its projective Weyl and Douglas tensors both vanish. Then from Bacsó-Matsumoto [3] we know that the Douglas tensor of a Randers manifold vanishes if and only if the 1-form b is closed. But, in case of a Sasakian space form $M(c)$, by using (2.8) we deduce that

$$db(X,Y) = \alpha d\eta(X,Y) = \frac{\alpha}{2}\left((\nabla_X\eta)Y - (\nabla_Y\eta)X\right)$$
$$= \alpha a(X, \varphi Y), \forall X, Y \in \Gamma(TM(c)).$$

Thus for any non zero vector field Y from the contact distribution of $M(c)$ we have

$$db(\varphi Y, Y) = \alpha a(\varphi Y, \varphi Y) > 0.$$

This completes the proof of Theorem 1.

Corollary 1 *Let S^{2m+1} be the real $(2m+1)$-dimensional sphere with $m \geq 1$. Then, for any $K > 0$ there exists on S^{2m+1} a Finsler metric F of constant flag curvature K. Moreover $\mathbb{F}^{2m+1} = (S^{2m+1}, F)$ is not projectively flat.*

Proof. By a result of Tanno [12], for any $\varepsilon > 0$ there exists on S^{2m+1} a Sasakian structure of constant φ-sectional curvature $c = \frac{4}{\varepsilon} - 3$. Take $\varepsilon > 1$ and obtain $-3 < c < 1$. Thus we may apply Theorem 1 and complete the proof of the corollary. □

The above corollary confirms Bao and Shen's prediction [5] that odd dimensional spheres may admit such metrics.

Remark After submitting this paper for publication, we were informed by professor D. Bao that Collen Robles is in the process of constructing Randers metrics of constant positive curvature on spheres using a totally different approach.

Added in Proofs According to a recent paper: Bao, D. and Robles, C., On Randers metrics of constant curvature, preprint 2001, the Theorem 2 is true provided that the additional condition

$$b^i(b_{i|j} - b_{j|i}) = 0,$$

is satisfied. By using (2.8) and (2.2a) we deduce that this condition is satisfied on any Sasakian space form. Thus our main result stated in Theorem 1 is not affected by the above correction.

References

[1] Akbar-Zadeh H., *Sur les espaces de Finsler à courbures sectionelles constantes.* Acad. Roy. Belg. Bull. Sci. (5), **74** (1988), 281–322.

[2] Antonelli P.L., Ingarden R.S. and Matsumoto M., *The theory of sprays and Finsler spaces with applications in physics and biology.* Kluwer Academic Publishers, Dordrecht, 1993.

[3] Bacsó S. and Matsumoto M., *On Finsler spaces of Douglas type, a generalization of the notion of Berwald space.* Publ. Math. Debrecen **51** (1997), 385–406.

[4] Bao D., Chern S.S. and Shen Z., *An introduction to Riemann-Finsler geometry.* Graduate Text in Math. **200**, Springer, Berlin, 2000.

[5] Bao D. and Shen Z., *Finsler metrics of constant positive curvature on the Lie group* S^3. Preprint november 2000.

[6] Bejancu A. and Farran H.R., *Geometry of pseudo-Finsler submanifolds.* Kluwer Academic Publishers, Dordrecht, 2000.

[7] Blair D.E., *Contact manifolds in Riemannian geometry.* Lecture Notes in Math. **509**, Springer, Berlin, 1976.

[8] Bryant R., *Finsler structures on the 2-sphere satisfying* $K = 1$. Cont. Math. **196** (1996), 27–42.

[9] Douglas J., *The general geometry of paths.* Ann. Math. (2), **29** (1928), 143–168.

[10] Matsumoto M., *Foundations of Finsler geometry and special Finsler spaces.* Kaiseisha Press, Saikawa, Otsu, 1986.

[11] Shen Z., *Finsler manifolds of constant positive curvature.* Cont. Math. **196** (1996), 83–93.

[12] Tanno S., *Sasakiam manifolds with constant φ-holomorphic sectional curvature*. Tohoku Math. J. **21** (1969), 501–507.

[13] Yasuda H. and Shimada H., *On Randers space of scalar curvature*. Rep. Math. Phys. **11** (1977), 347–360.

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