

On Lagrangian surfaces in $CP^2(\tilde{c})$

(Dedicated to Professor Koichi Ogiue on his sixtieth birthday)

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Abstract. Chen and Ogiue completely classified totally umbilical submanifolds in a non-flat complex-space-form. However, the classification problem of pseudo-umbilical submanifolds in a non-flat complex-space-form is still open. Very recently, Chen introduced the notion of Lagrangian H -umbilical submanifolds which is the simplest totally real submanifolds next to the totally geodesic ones in a complex-space-form and classified Lagrangian H -umbilical submanifolds in a complex-space-form. The author proved that a Lagrangian H -umbilical surface M in a complex 2-dimensional complex projective space $CP^2(\tilde{c})$ is an isotropic surface in $CP^2(\tilde{c})$ if and only if M is a minimal surface in $CP^2(\tilde{c})$. In this paper, firstly, we prove that a Lagrangian surface M in $CP^2(\tilde{c})$ is an isotropic surface in $CP^2(\tilde{c})$ if and only if M is a minimal surface in $CP^2(\tilde{c})$. Secondly, we classify Lagrangian non-totally geodesic pseudo-umbilical surfaces in $CP^2(\tilde{c})$.

Key words: totally real, isotropic, pseudo-umbilical, Lagrangian H -umbilical.

1. Introduction

Let M be an n -dimensional submanifold of a complex m -dimensional Kaehler manifold \tilde{M} with complex structure J and Kaehler metric g . A submanifold M of a Kaehler manifold \tilde{M} is said to be a *totally real* if each tangent space of M is mapped into the normal space by the complex structure of \tilde{M} (see Chen and Ogiue [5]). The totally real submanifold M of \tilde{M} is called *Lagrangian* if $n = m$. A Kaehler manifold of constant holomorphic sectional curvature \tilde{c} is called a *complex-space-form* and will be denoted by $\tilde{M}(\tilde{c})$. Let $CP^m(\tilde{c})$ be a complex m -dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature \tilde{c} .

Chen and Ogiue [6] classified totally umbilical submanifolds in a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$ ($\tilde{c} \neq 0$) and proved that a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$ ($\tilde{c} \neq 0$) ($m \geq 2$) admits no totally umbilical, Lagrangian submanifolds except the totally geodesic ones.

Very recently, Chen [1] introduced the notion of Lagrangian H -umbilical submanifolds which is the simplest totally real submanifolds next to

the totally geodesic ones in a complex-space-form $\tilde{M}^m(\tilde{c})$ and classified Lagrangian H -umbilical submanifolds in a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$. Further, Chen [2] [4] completely classified Lagrangian H -umbilical submanifolds in a complex Euclidean space.

A Lagrangian H -umbilical submanifold of a Kaehler manifold \tilde{M}^n is a non-totally geodesic Lagrangian submanifold whose second fundamental form takes the following simple form;

$$\begin{cases} \sigma(e_1, e_1) = \lambda J e_1, & \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu J e_1 \\ \sigma(e_1, e_j) = \mu J e_j, & \sigma(e_j, e_k) = 0, \quad j \neq k, \quad j = k = 2, \dots, n \end{cases} \quad (1.1)$$

for some suitable functions λ, μ with respect to some suitable orthonormal local frame field $\{e_i\}$.

Now, Matsuyama [9] proved that any non-totally geodesic, minimal totally real submanifold M^n (n : even) in $CP^n(\tilde{c})$ which has at most two principal curvatures in the direction of any normal is constant isotropic submanifold in $CP^n(\tilde{c})$ ($n \geq 4$) or minimal Lagrangian H -umbilical surface in $CP^2(\tilde{c})$. So, the author [13] showed the following.

Theorem 1.1 *Let M be a Lagrangian H -umbilical surface in $CP^2(\tilde{c})$. Then M is an isotropic surface in $CP^2(\tilde{c})$ if and only if M is a minimal surface in $CP^2(\tilde{c})$.*

Firstly, the aim of this paper is to show the following result which is a generalization of Theorem 1.1.

Theorem 1.2 *Let M be a Lagrangian surface in $CP^2(\tilde{c})$. Then M is an isotropic surface in $CP^2(\tilde{c})$ if and only if M is a minimal surface in $CP^2(\tilde{c})$.*

Very recently, Chen [3] showed that non-totally geodesic minimal Lagrangian surfaces in any Kaehler surface are Lagrangian H -umbilical. Thus we get

Corollary 1.1 *A nonzero isotropic Lagrangian surface in $CP^2(\tilde{c})$ is a minimal Lagrangian H -umbilical surface in $CP^2(\tilde{c})$.*

Now, the class of totally umbilical submanifolds in a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$ ($\tilde{c} \neq 0$) was completely classified by Chen and Ogiue. However, it is well known that the class of pseudo-umbilical submanifolds in a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$ ($\tilde{c} \neq 0$) is too wide to classify.

The classification problem of pseudo-umbilical submanifolds in a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$ ($\tilde{c} \neq 0$) is still open. Thus, it is reasonable to study pseudo-umbilical submanifolds in a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$ ($\tilde{c} \neq 0$) under some additional condition. Recently, the author [12] proved that any pseudo-umbilical submanifold M^n ($n \geq 2$) with nonzero parallel mean curvature vector in a non-flat complex-space-form $\tilde{M}^m(\tilde{c})$ ($\tilde{c} \neq 0$) is a totally real submanifold and satisfies $m > n$. Immediately, we see that there exist no pseudo-umbilical surfaces with nonzero parallel mean curvature vector in $CP^2(\tilde{c})$. Thus, it is very interesting to study Lagrangian pseudo-umbilical surfaces in $CP^2(\tilde{c})$.

Secondly, the aim of this paper is to classify Lagrangian pseudo-umbilical surfaces in $CP^2(\tilde{c})$.

Theorem 1.3 *Let M be a Lagrangian non-totally geodesic pseudo-umbilical surface in $CP^2(\tilde{c})$. Then M is a non-isotropic Lagrangian H -umbilical surface in $CP^2(\tilde{c})$ with the Gauss curvature $K = \tilde{c}/4$ and the scalar normal curvature $K_N = 0$.*

Remark 1.1 Without loss of generality, $CP^2(\tilde{c})$ is equipped with the Fubini-Study metric of constant holomorphic sectional curvature $\tilde{c} = 4$. By Remark 6.1 in [3], we see that the warped metric tensor of such a Lagrangian H -umbilical surface of $CP^2(\tilde{c})$ in Theorem 1.3 is given by

$$g = (dx)^2 + (1/H)^2(dy)^2$$

with respect to a coordinate system $\{x, y\}$, where H denotes the mean curvature of M in $CP^2(4)$.

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2. Preliminaries

Let M be an n -dimensional submanifold of a complex m -dimensional Kaehler manifold \tilde{M} with complex structure J and Kaehler metric g . Let ∇ (resp. $\tilde{\nabla}$) be the covariant differentiation on M (resp. \tilde{M}). We denote by σ the second fundamental form of M in \tilde{M} . Then the Gauss formula and the Weingarten formula are given respectively by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$ for all vector fields X, Y tangent to M and a

vector field ξ normal to M , where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X\xi$. The covariant derivative $\tilde{\nabla}\sigma$ of the second fundamental form σ is defined by $(\tilde{\nabla}_X\sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ for all vector fields X, Y and Z tangent to M . Let $\zeta = (1/n)\text{trace } \sigma$ and $H = |\zeta|$ denote the mean curvature vector and the mean curvature of M in \tilde{M} , respectively. The submanifold M of \tilde{M} is said to be a λ -isotropic submanifold if $|\sigma(X, X)| = \lambda$ for all unit tangent vectors X at each point.

Let R (resp. \tilde{R}) be the Riemannian curvature for ∇ (resp. $\tilde{\nabla}$). Then the Gauss equation is given by

$$g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(\sigma(X, Z), g(\sigma(Y, W))) - g(\sigma(Y, Z), \sigma(X, W)) \tag{2.1}$$

for all vector fields X, Y, Z and W tangent to M .

The Riemannian curvature \tilde{R} of $\tilde{M}(\tilde{c})$ is given by

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = (\tilde{c}/4)\{g(\tilde{Y}, \tilde{Z})\tilde{X} - g(\tilde{X}, \tilde{Z})\tilde{Y} + g(J\tilde{Y}, \tilde{Z})J\tilde{X} - g(J\tilde{X}, \tilde{Z})J\tilde{Y} + 2g(\tilde{X}, J\tilde{Y})J\tilde{Z}\} \tag{2.2}$$

for all vector fields \tilde{X}, \tilde{Y} and \tilde{Z} on $\tilde{M}(\tilde{c})$.

Now, we prepare the following fundamental fact.

Lemma 2.1 *Let M^n be a totally real submanifold in $CP^m(\tilde{c})$. Then we have*

$$g(\sigma(X, Y), JZ) = g(\sigma(X, Z), JY)$$

for all vector fields X, Y and Z tangent to M .

Proof.

$$\begin{aligned} g(\sigma(X, Y), JZ) &= g(\tilde{\nabla}_X Y, JZ) \\ &= -g(J\tilde{\nabla}_X Y, Z) \\ &= -g(\tilde{\nabla}_X JY, Z) \\ &= g(JY, \tilde{\nabla}_X Z) \\ &= g(\sigma(X, Z), JY) \end{aligned}$$

□

Here, we prepare the following result.

Theorem 2.1 (Maeda [8]) *Let M be an n -dimensional real-space-form of constant curvature c . If M is an isotropic Lagrangian submanifold of $CP^n(\tilde{c})$, then M is parallel. Thus M is totally geodesic or $n = 2$ and M is locally congruent to a flat torus T^2 ($c = 0$).*

Now, we recall the following result.

Theorem 2.2 (Chen [3]) *Let M be a non-totally geodesic Lagrangian surface in a Kaehler manifold \tilde{M}^2 . If M is a minimal surface in \tilde{M}^2 , then M is a Lagrangian H -umbilical surface in \tilde{M}^2 .*

3. Proof of Theorems

First, we shall show Theorem 1.2.

Let M be a Lagrangian surface in $CP^2(\tilde{c})$. We choose a local orthonormal frame field

$$e_1, e_2, e_3 = Je_1, e_4 = Je_2$$

of $CP^2(\tilde{c})$ such that e_1, e_2 are tangent to M . The surface in $CP^2(\tilde{c})$ satisfies

$$\begin{cases} \sigma(e_1, e_1) = ae_3 + be_4 \\ \sigma(e_1, e_2) = ce_3 + de_4 \\ \sigma(e_2, e_2) = fe_3 + ge_4 \end{cases} \tag{3.1}$$

for some functions a, b, c, d, f and g with respect to the orthonormal local frame field $\{e_i\}$.

By Lemma 2.1, we get

$$g(\sigma(e_1, e_2), Je_1) = g(\sigma(e_1, e_1), Je_2) \tag{3.2}$$

$$g(\sigma(e_2, e_1), Je_2) = g(\sigma(e_2, e_2), Je_1) \tag{3.3}$$

Thus by (3.1), (3.2) and (3.3) we obtain $c = b$ and $f = d$. Therefore we have

$$\begin{cases} \sigma(e_1, e_1) = ae_3 + be_4 \\ \sigma(e_1, e_2) = be_3 + de_4 \\ \sigma(e_2, e_2) = de_3 + ge_4 \end{cases} \tag{3.4}$$

for some functions a, b, d and g with respect to the orthonormal local frame field $\{e_i\}$.

If the surface is isotropic, we get (see O'Neill [11])

$$g(\sigma(e_1, e_1), \sigma(e_1, e_2)) = g(\sigma(e_2, e_2), \sigma(e_1, e_2)) = 0 \quad (3.5)$$

By (3.4) and (3.5) we obtain

$$b(a + d) = 0 \quad (3.6)$$

$$d(b + g) = 0 \quad (3.7)$$

Now, for an isotropic surface we get

$$|\sigma(e_1, e_1)|^2 = |\sigma(e_2, e_2)|^2$$

Thus we have

$$a^2 + b^2 = d^2 + g^2 \quad (3.8)$$

Firstly, if $b \neq 0$, by (3.6) we get

$$a + d = 0 \quad (3.9)$$

The case (I) [$b \neq 0$ and $a + d = 0$]:

By (3.8) and (3.9), we have

$$b^2 = g^2 \quad (3.10)$$

The case (I) [i]: If $b = g (\neq 0)$ in (3.10), by (3.7) we get $db = 0$. Since $b \neq 0$, we have $d = 0$. So we obtain $a = 0$ by (3.9). Thus we get by (3.4)

$$\begin{cases} \sigma(e_1, e_1) = & be_4 \\ \sigma(e_1, e_2) = & be_3 \\ \sigma(e_2, e_2) = & be_4 \end{cases} \quad (3.11)$$

for some function b with respect to some suitable orthonormal local frame field $\{e_i\}$.

Now, the Gauss curvature K is given by

$$K = g(R(e_1, e_2)e_2, e_1) \quad (3.12)$$

By (2.1), (2.2) and (3.12) we get the Gauss curvature

$$K = \tilde{c}/4 + \sum_{\alpha=3}^4 \{h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2\} \quad (3.13)$$

where $h_{ij}^\alpha = g(\sigma(e_i, e_j), e_\alpha)$ for $i, j = 1, 2$ and $\alpha = 3, 4$.

By (3.11) and (3.13), we have nonzero constant Gauss curvature $K = \tilde{c}/4$. Immediately from Theorem 2.1, we see that the surface is totally geodesic. This is a contradiction for $b \neq 0$.

The case (I) [ii]: If $b = -g (\neq 0)$ in (3.10), we get by (3.4) and (3.9)

$$\begin{cases} \sigma(e_1, e_1) = ae_3 + be_4 \\ \sigma(e_1, e_2) = be_3 - ae_4 \\ \sigma(e_2, e_2) = -ae_3 - be_4 \end{cases} \quad (3.14)$$

for some functions a, b with respect to the orthonormal local frame field $\{e_i\}$.

Thus we see that the surface is non-totally geodesic and minimal.

Secondly, if $a + d \neq 0$, by (3.6) we get $b = 0$.

The case (II) [$a + d \neq 0$ and $b = 0$]:

By (3.7) and (3.8) we obtain

$$dg = 0 \quad (3.15)$$

$$a^2 = d^2 + g^2 \quad (3.16)$$

The case (II) [i]: If $d = 0$ and $g \neq 0$ in (3.15), we get $b = d = 0$. So, by (3.4) and (3.13) we have $K = \tilde{c}/4$. Thus from Theorem 2.1, we see that the surface is totally geodesic. This is a contradiction for $g \neq 0$.

The case (II) [ii]: If $d \neq 0$ and $g = 0$ in (3.15), by (3.16) we get $a^2 = d^2$. Since $a + d \neq 0$, by (3.16) we get $a = d$. Since $d = a$ and $b = g = 0$, by (3.4) and (3.13) we have $K = \tilde{c}/4$. By the same argument as in the case (II) [i], we see that this case does not occur.

The case (II) [iii]: If $d = 0$ and $g = 0$ in (3.15), by (3.16) we get $a = 0$. This is a contradiction for $a + d \neq 0$.

Finally, we study the following case.

The case (III) [$b = 0$ and $a + d = 0$]:

By (3.8), we have $g = 0$. Since $d = -a$ and $b = g = 0$, we get

$$\begin{cases} \sigma(e_1, e_1) = ae_3 \\ \sigma(e_1, e_2) = -ae_4 \\ \sigma(e_2, e_2) = -ae_3 \end{cases} \quad (3.17)$$

for some function a with respect to the orthonormal local frame field $\{e_i\}$. Thus we see that the surface is minimal.

Conversely, if the surface is minimal, by Theorem 2.2 and Theorem 1.1 we see that the surface is isotropic.

This completes the proof of Theorem 1.2.

By (3.13) and (3.14), we get $K = \tilde{c}/4 - 2(a^2 + b^2)$, $b \neq 0$. And by (3.13) and (3.17), we have $K = \tilde{c}/4 - 2a^2$. Thus we get

Corollary 3.1 *An isotropic Lagrangian surface in $CP^2(\tilde{c})$ with the Gauss curvature $K < \tilde{c}/4$ is a non-totally geodesic minimal surface in $CP^2(\tilde{c})$.*

By Theorem 2.2, we obtain

Corollary 3.2 *An isotropic Lagrangian surface in $CP^2(\tilde{c})$ with the Gauss curvature $K < \tilde{c}/4$ is a minimal Lagrangian H -umbilical surface in $CP^2(\tilde{c})$.*

Remark 3.1 A flat torus T^2 in $CP^2(\tilde{c})$ has various properties (see Ludden-Okumura-Yano [7], Naitoh [10], Maeda [8]). In fact, a flat torus T^2 is a totally real isotropic, non-totally geodesic minimal surface with parallel second fundamental form in $CP^2(\tilde{c})$. A flat torus T^2 in $CP^2(4)$ satisfies (see [7])

$$\begin{cases} \sigma(e_1, e_1) = & \lambda e_4 \\ \sigma(e_1, e_2) = & -\lambda e_3 \\ \sigma(e_2, e_2) = & -\lambda e_4 \end{cases}$$

for $\lambda = 1/\sqrt{2}$.

Secondly, we shall show Theorem 1.3. Now, we prepare the following fact.

Proposition 3.1 *Let M be a Lagrangian surface of a Kaehler manifold \tilde{M}^2 . Then the following two conditions are equivalent.*

- (1) *M is a non-totally geodesic pseudo-umbilical surface of \tilde{M}^2 .*
- (2) *M is a Lagrangian H -umbilical surface with $\lambda = \mu$ in (1.1) of \tilde{M}^2 .*

Proof. Let M be a Lagrangian pseudo-umbilical surface in \tilde{M}^2 . We choose a local orthonormal frame field

$$e_1, e_2, e_3 = Je_1, e_4 = Je_2$$

of \tilde{M}^2 such that e_1, e_2 are tangent to M and e_3 in such a way that its direction coincides with that of the mean curvature vector ζ . Since M is a pseudo-umbilical surface, it is umbilic with respect to the direction of the

mean curvature vector ζ . Thus, the surface satisfies

$$\begin{cases} \sigma(e_1, e_1) = He_3 + ae_4 \\ \sigma(e_1, e_2) = be_4 \\ \sigma(e_2, e_2) = He_3 - ae_4 \end{cases} \tag{3.18}$$

for some functions a, b with respect to the orthonormal local frame field $\{e_i\}$.

By Lemma 2.1, we get

$$g(\sigma(e_1, e_2), Je_1) = g(\sigma(e_1, e_1), Je_2) \tag{3.19}$$

$$g(\sigma(e_2, e_1), Je_2) = g(\sigma(e_2, e_2), Je_1) \tag{3.20}$$

Thus by (3.18), (3.19) and (3.20) we obtain $a = 0$ and $b = H$. Therefore we have

$$\begin{cases} \sigma(e_1, e_1) = He_3 \\ \sigma(e_1, e_2) = He_4 \\ \sigma(e_2, e_2) = He_3 \end{cases} \tag{3.21}$$

By (3.21), we see that M is a Lagrangian H -umbilical surface with $\lambda = \mu$ in \tilde{M}^2 .

Conversely, let M be a Lagrangian H -umbilical surface with $\lambda = \mu$ in \tilde{M}^2 . We choose a local orthonormal frame field

$$e_1, e_2, e_3 = Je_1, e_4 = Je_2$$

of \tilde{M}^2 such that e_1, e_2 are tangent to M . By (1.1), the surface in \tilde{M}^2 satisfies

$$\begin{cases} \sigma(e_1, e_1) = \mu e_3 \\ \sigma(e_1, e_2) = \mu e_4 \\ \sigma(e_2, e_2) = \mu e_3 \end{cases} \tag{3.22}$$

for some function μ with respect to the orthonormal local frame field $\{e_i\}$.

By (3.22), we see that $\mu = H$ and M is a Lagrangian pseudo-umbilical surface in \tilde{M}^2 . □

The scalar normal curvature K_N is given by

$$K_N = \sum_{\alpha, \beta=3}^4 \left\{ \sum_{i=1}^2 (h_{1i}^\alpha h_{2i}^\beta - h_{1i}^\beta h_{2i}^\alpha) \right\}^2 \tag{3.23}$$

where $h_{ij}^\alpha = g(\sigma(e_i, e_j), e_\alpha)$ for $i, j = 1, 2$ and $\alpha = 3, 4$. By (3.21), (3.13) and (3.23), we get $K = \tilde{c}/4$ and $K_N = 0$.

By (3.21), for any unit vector $e = (ke_1 + le_2)/\sqrt{k^2 + l^2}$, where k, l are some real numbers, we get

$$|\sigma(e, e)|^2 = H^2 + 4k^2l^2H^2/(k^2 + l^2)^2 \quad (3.24)$$

$$|\sigma(e_1, e_1)|^2 = H^2 \quad (3.25)$$

If M is an isotropic surface, by (3.24) and (3.25) we have $H = 0$, i.e., M is a totally geodesic surface in $CP^2(\tilde{c})$ by (3.21).

This completes the proof of Theorem 1.3.

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