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Harmonic univalent functions with fixed second coefficient

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Abstract. We define and investigate a family of harmonic univalent functions that have fixed second coefficient. Extreme points, convolution conditions, convex combinations, and distortion bounds are obtained for these functions.

Key words: harmonic, univalent, starlike, convex.

1. Introduction

Denote by $\mathcal{H}$ the family of functions $f = h + \bar{g}$ which are harmonic univalent and sense-preserving in the open unit disk $\Delta = \{z : |z| < 1\}$ where h and g are given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

Note that the family $\mathcal{H}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions whenever the co-analytic part of $f = h + \bar{g}$ is zero, i.e., $g \equiv 0$. Also, note that if $f = h + \bar{g} \in \mathcal{H}$, then $h'(0) = 1 > |g'(0)| = |b_1|$. It, therefore, follows that $(f - \overline{b_1 f})/(1 - |b_1|^2) \in \mathcal{H}$ whenever $f \in \mathcal{H}$. With this in mind, we consider $\mathcal{H}^o \subset \mathcal{H}$ for which $b_1 = f_{\bar{z}}(0) = 0$. Clunie and Sheil-Small [8] observed that both $\mathcal{H}$ and $\mathcal{H}^o$ are normal families. They also found that $\mathcal{H}^o$ is compact, while $\mathcal{H}$ is not.

Let $\mathcal{TH}$ be the class of functions in $\mathcal{H}$ that may be expressed as $f = h + \bar{g}$ where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{2}$$

For $0 \le \alpha < 1$, also let $\mathcal{THS}^*(\alpha)$ and $\mathcal{THK}(\alpha)$ be the subclasses of $\mathcal{TH}$ consisting, respectively, of functions starlike of order α and convex of order α in Δ.

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The following two theorems were proved by the second author and we shall need them throughout this paper.

Theorem A [10] *Let $f = h + \bar{g}$ be so that h and g are given by (1). Then f is harmonic starlike of order α, denoted by $\mathcal{S}_{\mathcal{H}}^*(\alpha)$, if*

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha}|b_n| \le 1, \quad 0 \le \alpha < 1. \tag{3}$$

The condition (3) is also necessary for the family $\mathcal{THS}^*(\alpha)$.

Theorem B [9] *Let $f = h + \bar{g}$ be so that h and g are given by (1). Then f is harmonic convex of order α, denoted by $\mathcal{K}_{\mathcal{H}}(\alpha)$, if*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|b_n| \le 1, \quad 0 \le \alpha < 1. \tag{4}$$

The condition (4) is also necessary for the family $\mathcal{THK}(\alpha)$.

Silverman [12] obtained results analogous to Theorems A and B for the special case $b_1 = \alpha = 0$ and Silverman and Silvia [14] improved the results of [12] to the case b_1 not necessarily zero.

Several authors, such as [1], [2], and [13], studied the subclasses of analytic univalent functions with a fixed second coefficient. In [5] and [6] it is noted that such functions are often the only extreme points for the closed convex hull of the respective family. Often a function that maximizes a particular coefficient also maximizes the remaining coefficients and so provides sharp distortion bounds. There is a challenge in fixing the second coefficient which is due to the removal of natural extremal functions from the class. This challenge is even greater when it comes to harmonic functions. In this paper we investigate sense-preserving harmonic univalent functions of the form (2) with a fixed coefficient and determine extreme points, convolution conditions, and convex combinations for these types of functions.

First we define the family of harmonic univalent functions with fixed second coefficient.

Definition 1 For $0 \le p \le 1$, a function $f = h + \bar{g}$ where

$$h(z) = z - \frac{p}{c_2} z^2 - \sum_{n=3}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \tag{5}$$

is said to be in the family $\mathcal{F}^p_{\mathcal{H}}(\{c_n\},\{d_n\})$ if there exists sequences $\{c_n\}$ and $\{d_n\}$ of positive real numbers such that

$$p+\sum_{n=3}^{\infty} c_n|a_n|+\sum_{n=1}^{\infty} d_n|b_n| \leq 1, \quad d_1|b_1|<1. \tag{6}$$

Also let $\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\},\{d_n\}) \equiv \mathcal{F}^p_{\mathcal{H}}(\{c_n\},\{d_n\}) \cap \mathcal{H}^o$.

Note that if the co-analytic part of f is zero then $\mathcal{F}^p_{\mathcal{H}}(\{c_n\},\{d_n\})$ reduces to the family $\mathcal{F}_p(\{c_n\})$, studied by Ahuja and Silverman [1].

The families $\mathcal{F}^p_{\mathcal{H}}(\{c_n\},\{d_n\})$ and $\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\},\{d_n\})$ incorporate many subfamilies of $\mathcal{TH}$ consisting of functions with a fixed second coefficient. For example, for functions $f=h+\bar{g}$ of the form (5), we have

(i) $\quad \mathcal{F}^p_{\mathcal{H}}(\{n\},\{n\}) \equiv \mathcal{TH}_p := \{f : f \in \mathcal{TH}\}$,

(ii) $\quad \mathcal{F}^p_{\mathcal{H}}\Big(\Big\{\frac{n-\alpha}{1-\alpha}\Big\},\Big\{\frac{n+\alpha}{1-\alpha}\Big\}\Big) \equiv \mathcal{THS}^*_p(\alpha) := \{f : f \in \mathcal{THS}^*(\alpha)\}$,

(iii) $\quad \mathcal{F}^p_{\mathcal{H}}\Big(\Big\{\frac{n(n-\alpha)}{1-\alpha}\Big\},\Big\{\frac{n(n+\alpha)}{1-\alpha}\Big\}\Big) \equiv \mathcal{THK}_p(\alpha) := \{f : f \in \mathcal{THK}(\alpha)\}$,

(iv)

$$\mathcal{F}^p_{\mathcal{H}}\Big(\Big\{\frac{n}{1-\alpha}\Big\},\Big\{\frac{n}{1-\alpha}\Big\}\Big) \equiv \mathcal{N}^p_{\mathcal{H}}(\alpha) := \Bigg\{f : \Re\frac{\frac{\partial}{\partial\theta}f(z)}{\frac{\partial}{\partial\theta}z} \geq \alpha,\ z=re^{i\theta}\in\Delta\Bigg\},$$

(v)

$$\mathcal{F}^p_{\mathcal{H}}\Big(\Big\{\frac{n^2}{1-\alpha}\Big\},\Big\{\frac{n^2}{1-\alpha}\Big\}\Big) \equiv \mathcal{R}^p_{\mathcal{H}}(\alpha) := \Bigg\{f : \Re\frac{\frac{\partial^2}{\partial\theta^2}f(z)}{\frac{\partial^2}{\partial\theta^2}z} \geq \alpha,\ z=re^{i\theta}\in\Delta\Bigg\}.$$

We note that if $d_n \equiv 0$ and the co-analytic part of $f=h+\bar{g}$ is zero then the above conditions (iv) and (v) reduce, respectively, to the well-known conditions $\Re(f')=\Re(h')\geq\alpha$, (see [3] and [11]), and $\Re(zf')'=\Re(zh')'\geq\alpha$, (see [4], [7], and [15]).

2. Main Results

First we justify the sense-preserving and univalence of the family $\mathcal{F}^p_{\mathcal{H}}(\{c_n\},\{d_n\})$.

Theorem 1 *If $c_n \geq n$ and $d_n \geq n$ for all n, then $\mathcal{F}^p_{\mathcal{H}}(\{c_n\}, \{d_n\})$ consists of starlike sense-preserving harmonic mappings in Δ.*

Proof. Note that for $|a_2| = p/c_2$ and $c_2 \geq 2$ we have $2|a_2| = 2p/c_2 \leq p$. Now the proof is complete by Theorem A, since

$$2|a_2| + \sum_{n=3}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq p + \sum_{n=3}^{\infty} c_n|a_n| + \sum_{n=1}^{\infty} d_n|b_n| \leq 1.$$

We will henceforth, unless otherwise stated, assume that $c_n \geq n$ and $d_n \geq n$. □

Next we give the radius of the convexity of the functions in $\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$.

Theorem 2 *Each function in $\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$ maps the disk $|z| = r < 1/2$ onto a convex domain.*

Proof. For $f \in \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$ we have $\frac{f(rz)}{r} \in \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$ where $0 < r < 1$. Then we can write

$$\begin{aligned} p &+ \sum_{n=3}^{\infty} n^2|a_n|r^{n-1} + \sum_{n=2}^{\infty} n^2|b_n|r^{n-1} \\ &= p + \sum_{n=3}^{\infty} n|a_n|nr^{n-1} + \sum_{n=2}^{\infty} n|b_n|nr^{n-1} \\ &\leq p + \sum_{n=3}^{\infty} n|a_n| + \sum_{n=2}^{\infty} n|b_n| \leq p + \sum_{n=3}^{\infty} c_n|a_n| + \sum_{n=2}^{\infty} d_n|b_n|. \end{aligned}$$

This last expression is never less than 1 if $nr^{n-1} \leq 1$, or if $r \leq 1/2$. Therefore, the convexity follows by Theorem B. □

In the following theorem we examine the extreme points of the closed convex hull of $\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$, which is denoted by $\operatorname{clco} \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$.

Theorem 3 $f \in \operatorname{clco} \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$ *if and only if*

$$f(z) = \sum_{n=2}^{\infty} (\lambda_n h_n(z) + \mu_n g_n(z)) \tag{7}$$

where $h_2(z) = z - \frac{p}{c_2} z^2$,

$$h_n(z) = z - \frac{p}{c_2}z^2 - \frac{1-p}{c_n}z^n, \quad n = 3, 4, \dots,$$

$$g_n(z) = z - \frac{p}{c_2}z^2 + \frac{1-p}{d_n}\bar{z}^n, \quad n = 2, 3, 4, \dots,$$

$\lambda_n \geq 0$, $\mu_n \geq 0$, and $\sum_{n=2}^{\infty}(\lambda_n + \mu_n) = 1$. In particular, the extreme points of $\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose f is expressed by (7). Then

$$\begin{aligned} f(z) &= \lambda_2\Big(z - \frac{p}{c_2}z^2\Big) + \mu_2\Big(z - \frac{p}{c_2}z^2 + \frac{1-p}{d_2}\bar{z}^2\Big) \\ &\quad + \sum_{n=3}^{\infty}\Big[\lambda_n\Big(z - \frac{p}{c_2}z^2 - \frac{1-p}{c_n}z^n\Big) + \mu_n\Big(z - \frac{p}{c_2}z^2 + \frac{1-p}{d_n}\bar{z}^n\Big)\Big] \\ &= z - \frac{p}{c_2}z^2 - \sum_{n=3}^{\infty}\Big(\frac{1-p}{c_n}\lambda_n\Big)z^n + \sum_{n=2}^{\infty}\Big(\frac{1-p}{d_n}\mu_n\Big)\bar{z}^n. \end{aligned}$$

Therefore, $f \in \operatorname{clco}\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$, since

$$\begin{aligned} p &+ \sum_{n=3}^{\infty}\Big(\frac{1-p}{c_n}\lambda_n\Big)c_n + \sum_{n=2}^{\infty}\Big(\frac{1-p}{d_n}\mu_n\Big)d_n \\ &= p + (1-p)\sum_{n=3}^{\infty}\lambda_n + (1-p)\sum_{n=2}^{\infty}\mu_n \\ &= p + (1-p)\Big(\sum_{n=3}^{\infty}\lambda_n + \sum_{n=2}^{\infty}\mu_n\Big) \\ &= p + (1-p)(1-\lambda_2) = 1 - \lambda_2(1-p) \leq 1. \end{aligned}$$

Conversely, assume that $f = h + \bar{g} \in \operatorname{clco}\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$ where h and g are given by

$$h(z) = z - \frac{p}{c_2}z^2 - \sum_{n=3}^{\infty}|a_n|z^n, \quad g(z) = \sum_{n=2}^{\infty}|b_n|\bar{z}^n.$$

Since, by (6), $|a_n| \leq \frac{1-p}{c_n}$ $(n = 3, 4, \dots)$, and $|b_n| \leq \frac{1-p}{d_n}$ $(n = 2, 3, 4, \dots)$, we may set $\lambda_n = \frac{|a_n|c_n}{1-p}$ $(n = 3, 4, \dots)$, $\mu_n = \frac{|b_n|d_n}{1-p}$ $(n = 2, 3, 4, \dots)$, and $\lambda_2 = 1 - \sum_{n=3}^{\infty}\lambda_n - \sum_{n=2}^{\infty}\mu_n$. Then the proof is complete by noting that

$$
\begin{aligned}
f(z) &= z - \frac{p}{c_2}z^2 - \sum_{n=3}^{\infty}|a_n|z^n + \sum_{n=2}^{\infty}|b_n|\bar{z}^n \\
&= h_2(z) + \sum_{n=3}^{\infty}(h_n(z) - h_2(z))\lambda_n + \sum_{n=3}^{\infty}(g_n(z) - g_2(z))\mu_n \\
&= \lambda_2 h_2(z) + \sum_{n=3}^{\infty}(\lambda_n h_n(z) + \mu_n g_n(z)) + \mu_2 g_2(z) \\
&= \sum_{n=2}^{\infty}(\lambda_n h_n(z) + \mu_n g_n(z)).
\end{aligned}
$$

□

Corollary 1 *The extreme points of* clco $\mathcal{TH}^o\mathcal{S}_p^*(\alpha)$ *are the functions given by* $h_2(z) = z - \frac{p(1-\alpha)}{2-\alpha}z^2$,

$$h_n(z) = z - \frac{p(1-\alpha)}{2-\alpha}z^2 - \frac{(1-p)(1-\alpha)}{n-\alpha}z^n \quad (n = 3, 4, \ldots),$$

$$g_n(z) = z - \frac{p(1-\alpha)}{2-\alpha}z^2 v + \frac{(1-p)(1-\alpha)}{n+\alpha}\bar{z}^n \quad (n = 2, 3, 4, \ldots).$$

Corollary 2 *The extreme points of* clco $\mathcal{TH}^o\mathcal{K}_p(\alpha)$ *are the functions given by* $h_2(z) = z - \frac{p(1-\alpha)}{2(2-\alpha)}z^2$,

$$h_n(z) = z - \frac{p(1-\alpha)}{2(2-\alpha)}z^2 - \frac{p(1-\alpha)}{n(n-\alpha)}z^n \quad (n = 3, 4, \ldots),$$

and

$$g_n(z) = z - \frac{p(1-\alpha)}{2(2-\alpha)}z^2 + \frac{p(1-\alpha)}{n(n+\alpha)}\bar{z}^n \quad (n = 2, 3, 4, \ldots).$$

We remark that results analogous to Corollaries 1 *and* 2 *for the analytic case are proved by Silverman and Silvia* [13].

For the harmonic functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \bar{z}^n$ we define the convolution of f and F by

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \bar{z}^n.$$

Theorem 4 *The family* $\mathcal{F}_{\mathcal{H}^o}^p(\{c_n\}, \{d_n\})$ *is closed under convolution.*

Proof. For $f, F \in \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\},\{d_n\})$, we may write

$$f(z) * F(z) = z - \frac{p^2}{c_2^2} z^2 - \sum_{n=3}^{\infty} |a_n A_n| z^n + \sum_{n=2}^{\infty} |b_n B_n| \bar{z}^n$$

where

$$f(z) = z - \frac{p}{c_2} z^2 - \sum_{n=3}^{\infty} |a_n| z^n + \sum_{n=2}^{\infty} |b_n| \bar{z}^n$$

and

$$F(z) = z - \frac{p}{c_2} z^2 - \sum_{n=3}^{\infty} |A_n| z^n + \sum_{n=2}^{\infty} |B_n| \bar{z}^n.$$

Note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Therefore, we obtain

$$p + \sum_{n=3}^{\infty} c_n |a_n A_n| + \sum_{n=1}^{\infty} d_n |b_n B_n| \leq p + \sum_{n=3}^{\infty} c_n |a_n| + \sum_{n=2}^{\infty} d_n |b_n| \leq 1.$$

This shows that $f * F \in \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\},\{d_n\})$. □

Theorem 5 *The family $\mathcal{F}^p_{\mathcal{H}^o}(\{c_n\},\{d_n\})$ is closed under convex combination.*

Proof. Since

$$f_i(z) = z - \frac{p}{c_2} z^2 - \sum_{n=3}^{\infty} |a_{i_n}| z^n + \sum_{n=2}^{\infty} |b_{i_n}| \bar{z}^n \in \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\},\{d_n\})$$

we have

$$p + \sum_{n=3}^{\infty} c_n |a_{i_n}| + \sum_{n=2}^{\infty} d_n |b_{i_n}| \leq 1, \quad i = 1, 2, 3, \dots . \tag{8}$$

Note that

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \frac{p}{c_2} z^2 - \sum_{n=3}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) z^n + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \bar{z}^n.$$

Using the condition (8) we obtain

$$
\begin{aligned}
p &+ \sum_{n=3}^{\infty} c_n \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) + \sum_{n=2}^{\infty} d_n \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \\
&= p + \sum_{i=1}^{\infty} t_i \left(\sum_{n=3}^{\infty} c_n |a_{i_n}| \right) + \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} d_n |b_{i_n}| \right) \\
&= p + \sum_{i=1}^{\infty} t_i \left(\sum_{n=3}^{\infty} c_n |a_{i_n}| \right) + \sum_{n=2}^{\infty} d_n |b_{i_n}| \le p + \sum_{i=1}^{\infty} t_i (1-p) \\
&= p + (1-p) \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}
$$

Hence, by an application of (6), it follows that $\sum_{i=1}^{\infty} t_i f_i \in \mathcal{F}^p_{\mathcal{H}^o}(\{c_n\}, \{d_n\})$. □

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