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Real hypersurfaces in complex space forms which are warped products

Bang-Yen CHEN

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Abstract. It is proved in [6] that there do not exist real hypersurfaces in nonflat complex space forms which are Riemannian products of Riemannian manifolds. By contrast, using Legendre curves we construct in this article many examples of real hypersurfaces in complex space forms which are warped products of Riemannian manifolds. Conversely, we prove that our examples are the only real hypersurfaces in complex space forms which are warped products of a complex hypersurface and a curve.

Key words: warped products, real hypersurfaces, complex space forms, Legendre curve.

1. Introduction

The study of real hypersurfaces in complex projective space CP^n and complex hyperbolic space CH^n has been an active field over the past three decades. Although these ambient spaces might be regarded as the simplest after the spaces of constant curvature, they impose significant restrictions on the geometry of their hypersurfaces. For instance, they do not admit totally umbilical hypersurfaces and Einstein hypersurfaces.

On the other hand, several important classes of real hypersurfaces in complex projective space have been constructed and investigated by many geometers. For instance, H.B. Lawson investigated real hypersurfaces of CP^n which lift to Clifford minimal hypersurfaces of S^{n+1} via Hopf fibration. R. Takagi [12] gave the list of homogeneous real hypersurfaces of CP^n. Many geometers then study the geometry from the list of Takagi and obtained various interesting geometric characterizations of homogeneous real hypersurfaces in CP^n.

Another important class of real hypersurfaces in CP^n which contains the list of R. Takagi is the class of Hopf hypersurfaces. Such hypersurfaces are real hypersurfaces whose structure vector $J\xi$ is a principal curvature vector, where J is the complex structure and ξ is the unit normal vector field. Examples and geometric characterizations of Hopf hypersurfaces have

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also been obtained by various geometers. It is known that in CP^n, M is a homogeneous real hypersurface if and only if M is a Hopf hypersurface with constant principal curvatures [8, 12].

The study of real hypersurfaces in complex hyperbolic space CH^n has followed developments in CP^n, often with similar results, but sometimes with differences (see, for instance, [1, 10]).

Recently, B.Y. Chen and S. Maeda proved in [6] that there do not exist real hypersurfaces which are the Riemannian products of Riemannian manifolds, both in complex projective space and complex hyperbolic space. By contrast, using Legendre curves in a hypersphere of the complex plane, we construct explicitly in this article many real hypersurfaces in complex space forms which are warped products of Riemannian manifolds. Conversely, we prove that our examples are the only real hypersurfaces in complex space forms which are warped products of a complex hypersurface and a curve.

2. Preliminaries

If N is a Riemannian k-manifold isometrically immersed in a Kaehler manifold $\tilde{M}$ with complex structure J. Then the formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for vector fields X, Y tangent to N and ξ normal to N, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$, σ the second fundamental form, D the normal connection, and A the shape operator of N in $\tilde{M}$. The second fundamental form and the shape operator are related by $\langle A_\xi X, Y\rangle = \langle \sigma(X,Y), \xi\rangle$, where $\langle\ ,\ \rangle$ denotes the inner product on M as well as on $\tilde{M}$. The mean curvature vector H is given by $H = (1/k)\,\mathrm{trace}\,\sigma$. For an orthonormal basis $e_1, \ldots, e_k$ of the tangent bundle of N, the scalar curvature τ is defined by $\tau = \sum_{i<j} K_{ij}$, where K_{ij} denotes the sectional curvature of the plane section spanned by e_i and e_j.

For a submanifold N of a Kaehler manifold $\tilde{M}$, the *equation of Gauss* is given by

$$\begin{aligned} R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) &+ \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ &- \langle \sigma(X,Z), \sigma(Y,W)\rangle, \end{aligned} \tag{2.3}$$

for X, Y, Z, W tangent to M and ξ, η normal to M, where R and $\tilde{R}$ denote the curvature tensors of N and $\tilde{M}^m$, respectively.

For the second fundamental form σ, we define its covariant derivative $\bar{\nabla}\sigma$ with respect to the connection on $TM \oplus T^{\perp}M$ by

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.4}$$

The *equation of Codazzi* is given by

$$(\tilde{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{2.5}$$

where $(\tilde{R}(X, Y)Z)^{\perp}$ denotes the normal component of $\tilde{R}(X, Y)Z$.

Let $\tilde{M}^n(4c)$ denote a complex m-dimensional Kaehler manifold with constant holomorphic sectional curvature $4c$. Such Kaehler manifolds are called *complex space forms*. It is known that the universal covering of a complete complex space form $\tilde{M}^n(4c)$ is the complex projective n-space $CP^n(4c)$, the complex Euclidean n-space $\mathbf{C}^n$, or the complex hyperbolic space $CH^n(4c)$, according to $c > 0$, $c = 0$, or $c < 0$. The Riemann curvature tensor of $\tilde{M}^n(4c)$ satisfies

$$\begin{aligned} &\tilde{R}(X, Y; Z, W) \\ &\quad = c\{ \langle X, W\rangle \langle Y, Z\rangle - \langle X, Z\rangle \langle Y, W\rangle + \langle JX, W\rangle \langle JY, Z\rangle \\ &\qquad\quad - \langle JX, Z\rangle \langle JY, W\rangle + 2\langle X, JY\rangle \langle JZ, W\rangle \}. \end{aligned} \tag{2.6}$$

Let B and F be Riemannian manifolds endowed with Riemannian metrics g_B and g_F, respectively, and $f > 0$ a differentiable function on B. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$||X||^2 = ||\pi_*(X)||^2 + f^2(\pi(x))||\eta_*(X)||^2 \tag{2.7}$$

for any tangent vector $X \in T_x M$. Thus, we have $g = g_B + f^2 g_F$. The function f is called the warping function of the warped product.

A submanifold N in a Kaehler manifold $\tilde{M}$ is called a CR-submanifold if there exists on N a differentiable holomorphic distribution $\mathcal{D}$ such that its orthogonal complement $\mathcal{D}^{\perp}$ is a totally real distribution, i.e., $J\mathcal{D}_x^{\perp} \subset T_x^{\perp}N$. Real hypersurfaces of a Kaehler manifold are CR-submanifolds. A CR-submanifold is called a *CR-warped product* if it is the warped product $N_T \times_f N_{\perp}$ of a holomorphic submanifold N_T and a totally real submanifold $N_{\perp}$ of $\tilde{M}$, where f is the warping function (cf. [5]).

We recall the following lemma from [2] for later use.

Lemma 2.1 *Let M be a CR-submanifold in a Kaehler manifold $\tilde{M}$. Then we have*

(1) $\langle \nabla_U Z, X\rangle = \langle JA_{JZ}U, X\rangle$,

(2) $A_{JZ}W = A_{JW}Z$, *and*

(3) $A_{J\xi}X = -A_\xi JX$,

for any vectors U tangent to M, X, Y in $\mathcal{D}$, Z, W in $\mathcal{D}^\perp$, and ξ in ν.

For CR-warped products in Kaehler manifolds we have the following.

Lemma 2.2 *For a CR-warped product $M = N_T \times_f N_\perp$ in any Kaehler manifold $\tilde{M}$, we have*

(1) $\langle \sigma(\mathcal{D},\mathcal{D}), J\mathcal{D}^\perp\rangle = 0$;

(2) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$;

(3) $\langle \sigma(JX, Z), JW\rangle = (X\ln f)\langle Z, W\rangle$;

where X, Y are vector fields on N_T and Z, W are on $N_\perp$

Proof. Since $\tilde{M}$ is Kaehlerian, we have

$$J\nabla_X Z + J\sigma(X,Z) = -A_{JZ}X + D_X JZ, \tag{2.8}$$

for any vector fields X, Y on N_T and Z in $N_\perp$. Thus, by taking the inner product of (2.8) with JY, we find

$$\langle \nabla_X Z, Y\rangle = -\langle A_{JZ}X, JY\rangle = -\langle \sigma(X, JY), JZ\rangle. \tag{2.9}$$

On the other hand, since $M = N_T \times_f N_\perp$ is a warped product, N_T is a totally geodesic submanifold of M. Thus, we also have $\langle \nabla_X Z, Y\rangle = 0$. Combining this with (2.9), we obtain statement (1).

Statement (2) can be found in [11].

By applying statement (2), Lemma 2.1 and statement (2), we get

$$\begin{aligned}\langle \sigma(JX, Z), JW\rangle &= -\langle JA_{JW}Z, X\rangle = -\langle \nabla_Z W, X\rangle \\ &= (X\ln f)\langle Z, W\rangle\end{aligned} \tag{2.10}$$

for X on N_T and Z, W on $N_\perp$. This proves statement (3). □

A real hypersurface M of a complex space form $\tilde{M}$ is called *ruled* if it is foliated by complex totally geodesic hypersurfaces of $\tilde{M}$.

3. Legendre curves and differential equations

A contact manifold is an odd-dimensional manifold M^{2n+1} with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. A curve $\gamma = \gamma(t)$ in a contact manifold is called a *Legendre curve* if $\eta(\beta'(t)) = 0$ along β. Let $S^{2n+1}(c)$ denote the hypersphere in $\mathbf{C}^{n+1}$ with curvature c centered at the origin. Then $S^{2n+1}(c)$ is a contact manifold endowed with a canonical contact structure which is the dual 1-form of $J\xi$, where J is the complex structure and ξ the unit normal vector on $S^{2n+1}(c)$.

Legendre curves are known to play an important role in the study of contact manifolds, e.g. a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves.

Let $\mathbf{C}^{n+1}$ denote the complex Euclidean $(n+1)$-space endowed with metric $g = \sum_{j=1}^{n+1} dz_j d\bar{z}_j$, $z_j = x_j + iy_j$. We put

$$S^{2n+1}(c) = \left\{(z_1, \ldots, z_{n+1}) \in \mathbf{C}^{n+1} : \langle z, z \rangle = c^{-1} > 0\right\},$$

where $\langle \ , \ \rangle$ denotes the inner product induced from the metric.

The following lemma from [3] provide a simple relationship between Legendre curves and a second order differential equation.

Lemma 3.1 *Let c be a positive number and $z = (z_1, z_2) : I \to S^3(c) \subset \mathbf{C}^2$ be a unit speed curve where I is either an open interval or a circle. If $z : I \to \mathbf{C}^2$ satisfies*

$$z''(t) - i\lambda\gamma(t)z'(t) + cz(t) = 0 \tag{3.1}$$

for some nonzero real-valued function λ on I, it defines a Legendre curve in $S^3(c)$.

Conversely, if z defines a Legendre curve in $S^3(c)$, it satisfies differential equation (3.1) *for some real-valued function λ.*

Remark 3.1 For any nonzero function $\lambda(t)$ there is a (unit speed) Legendre curve $z = z(t)$ in $S^3(c)$ satisfying equation (3.1). Such Legendre curve is unique if one imposes the initial conditions: $z(0) = z_0 \in S^3(c)$ and $z'(0) = u$ for some unit vector tangent to $S^3(c)$ (see [4, p.14]).

4. Warped product real hypersurfaces in $\mathbf{C}^{n+1}$

The following theorem classifies completely real hypersurfaces of complex Euclidean space which are warped products of a complex hypersurface and a real curve.

Theorem 4.1 *Let a be a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ be a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbf{C}^2$ defined on an open interval I. Then*

$$\mathbf{x}(z_1, \dots, z_n, t) = \big(a\Gamma_1(t)z_1, a\Gamma_2(t)z_1, z_2, \dots, z_n\big), \quad z_1 \neq 0 \tag{4.1}$$

defines a real hypersurface which is the warped product $\mathbf{C}^n_* \times_{a|z_1|} I$ of a complex n-plane and I, where $\mathbf{C}^n_* = \{(z_1, \dots, z_n) : z_1 \neq 0\}$.

Conversely, up to rigid motions of $\mathbf{C}^{n+1}$, every real hypersurface in $\mathbf{C}^{n+1}$ which is the warped product $N \times_f I$ of a complex hypersurface N and an open interval I is either obtained in the way described above or given by the product submanifold $\mathbf{C}^n \times C \subset \mathbf{C}^n \times \mathbf{C}^1$ of $\mathbf{C}^n$ and a real curve C in $\mathbf{C}^1$.

Proof. Let $N \times_f I$ be a real hypersurface in $\mathbf{C}^{n+1}$ which is the warped product $N \times_f I$ of a complex hypersurface N of $\mathbf{C}^{n+1}$ and an open interval I. Without loss of generality, we may assume that I contains 0.

We obtain from Lemma 2.2 that

$$\sigma(\mathcal{D}, \mathcal{D}) = 0. \tag{4.2}$$

Since N is totally geodesic in $N \times_f I$, (4.2) implies that N is immersed as a totally geodesic complex submanifold in $\mathbf{C}^{n+1}$. Hence, N is holomorphically isometric to a complex Euclidean n-space $\mathbf{C}^n$.

Let $z = (z_1, \dots, z_n)$ be a natural complex coordinate system on $\mathbf{C}^n$. We put $z_j = x_j + iy_j$, $j = 1, \dots, n$. The warped product metric on $N \times_f I$ is given by

$$g = \sum_{k=1}^{h} (dx_k^2 + dy_k^2) + f^2 dt^2. \tag{4.3}$$

From (4.3) and a straightforward computation we know that the Riemannian connection on $N \times_f I$ satisfies

$$\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial y_k} = \nabla_{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial y_k} = 0, \quad j, k = 1, \dots, n, \tag{4.4}$$

$$\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial t} = \frac{f_{x_j}}{f} \frac{\partial}{\partial t}, \quad j = 1, \dots, n, \tag{4.5}$$

$$\nabla_{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial t} = \frac{f_{y_j}}{f} \frac{\partial}{\partial t}, \quad j = 1, \ldots, n, \tag{4.6}$$

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = -f \sum_{k=1}^{n} \Big(f_{x_k} \frac{\partial}{\partial x_k} + f_{y_k} \frac{\partial}{\partial y_k} \Big), \tag{4.7}$$

where $f_{x_j} = \partial f/\partial x_j$, $f_{y_j} = \partial f/\partial y_j$.

Using (4.4)–(4.7) we know that the Riemann curvature tensor of satisfies

$$\begin{aligned} R\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t}\Big)\frac{\partial}{\partial x_k} &= \Big(\frac{\partial^2 \phi}{\partial x_j \partial x_k} + \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_k}\Big)\frac{\partial}{\partial t} \\ R\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t}\Big)\frac{\partial}{\partial y_k} &= \Big(\frac{\partial^2 \phi}{\partial x_j \partial y_k} + \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial y_k}\Big)\frac{\partial}{\partial t} \\ R\Big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial t}\Big)\frac{\partial}{\partial y_k} &= \Big(\frac{\partial^2 \phi}{\partial y_j \partial y_k} + \frac{\partial \phi}{\partial y_j}\frac{\partial \phi}{\partial y_k}\Big)\frac{\partial}{\partial t}, \end{aligned} \tag{4.8}$$

for $j, k = 1, \ldots, n$, where $\phi = \ln f$.

From (3) of Lemma 2.2 we have

$$\begin{aligned} \sigma\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t}\Big) &= -\frac{\partial \phi}{\partial y_j} J\Big(\frac{\partial}{\partial t}\Big), \\ \sigma\Big(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial t}\Big) &= \frac{\partial \phi}{\partial x_j} J\Big(\frac{\partial}{\partial t}\Big), \quad j = 1, \ldots, n. \end{aligned} \tag{4.9}$$

We put

$$\sigma\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = \lambda J\Big(\frac{\partial}{\partial t}\Big), \tag{4.10}$$

for some function $\lambda = \lambda(z_1, \ldots, z_n, t)$.

By applying the equation of Gauss, (4.8), and (4.9), we obtain

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x_j \partial x_k} &= \frac{\partial \phi}{\partial y_j}\frac{\partial \phi}{\partial y_k} - \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_k}, \\ \frac{\partial^2 \phi}{\partial x_j \partial y_k} &= -\frac{\partial \phi}{\partial y_j}\frac{\partial \phi}{\partial x_k} - \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial y_k}, \quad j, k = 1, \ldots, n, \\ \frac{\partial^2 \phi}{\partial y_j \partial y_k} &= \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_k} - \frac{\partial \phi}{\partial y_j}\frac{\partial \phi}{\partial y_k} \end{aligned} \tag{4.11}$$

Clearly, every constant function ϕ is a solution of the PDE system (4.11). However, if ϕ is constant, the warping function f is constant. In this case, the hypersurface is a CR-product in the sense of [2]. Hence, by applying Theorem 4.6 of [2], we know that the real hypersurface is locally the product submanifold obtained by the product of $\mathbf{C}^n$ and a curve in $\mathbf{C}^1$.

Now, we search for non-constant solutions of the PDE system.

Lemma 4.2 *The non-constant solutions* $\phi = \phi(x_1, y_1, \ldots, x_n, y_n)$ *of the system of partial differential equations*:

$$\frac{\partial^2 \phi}{\partial x_j \partial x_k} = \frac{\partial \phi}{\partial y_j}\frac{\partial \phi}{\partial y_k} - \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_k}, \tag{4.12}$$

$$\frac{\partial^2 \phi}{\partial x_j \partial y_k} = -\frac{\partial \phi}{\partial y_j}\frac{\partial \phi}{\partial x_k} - \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial y_k}, \quad j,k = 1, \ldots, n, \tag{4.13}$$

$$\frac{\partial^2 \phi}{\partial y_j \partial y_k} = \frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_k} - \frac{\partial \phi}{\partial y_j}\frac{\partial \phi}{\partial y_k} \tag{4.14}$$

are the functions given by

$$\phi = \frac{1}{2} \ln \{ \langle \alpha, z \rangle^2 + \langle i\alpha, z \rangle^2 \}, \tag{4.15}$$

where $z = (x_1 + iy_1, \ldots, x_n + iy_n)$, $\langle \ , \ \rangle$ *is the standard Euclidean inner product on* $\mathbf{C}^n$, *and* α *is a constant nonzero vector in* $\mathbf{C}^n$.

Proof. From (4.13) we get

$$\frac{\partial}{\partial x_1}\left(\ln \frac{\partial \phi}{\partial y_1}\right) = -2\frac{\partial \phi}{\partial x_1}. \tag{4.16}$$

Solving (4.16) yields

$$\frac{\partial \phi}{\partial y_1} = e^{-2\phi}\psi(y_1, x_2, y_2, \ldots, x_n, y_n), \tag{4.17}$$

for some function $\psi = \psi(y_1, x_2, y_2, \ldots, x_n, y_n)$. Therefore

$$\phi = \frac{1}{2} \ln \left(\eta(x_1, x_2, y_2, \ldots, x_n, y_n) + \mu(y_1, x_2, y_2, \ldots, x_n, y_n) \right), \tag{4.18}$$

for some function $\eta = \eta(x_1, x_2, y_2, \ldots, x_n, y_n)$, where $\mu = 2 \int^{y_1} \psi dy_1$. From (4.18) we find

$$\phi_{x_1} = \frac{\eta_{x_1}}{2(\eta + \mu)}, \quad \phi_{y_1} = \frac{\mu_{y_1}}{2(\eta + \mu)}, \tag{4.19}$$

$$\phi_{x_1 x_1} = \frac{\eta_{x_1 x_1}(\eta + \mu) - \eta_{x_1}^2}{2(\eta + \mu)^2}. \tag{4.20}$$

By (4.12), (4.19) and (4.20) we obtain

$$2(\eta+\mu)\eta_{x_1x_1} = \eta_{x_1}^2 + \mu_{y_1}^2. \tag{4.21}$$

Similarly, from (4.14) with $j = k = 1$ and (4.18), we also have

$$2(\eta+\mu)\mu_{y_1y_1} = \eta_{x_1}^2 + \mu_{y_1}^2. \tag{4.22}$$

By combining (4.21) and (4.22) we find $\eta_{x_1x_1} = \mu_{y_1y_1}$. Since η and μ are independent of y_1 and x_1 respectively, we find

$$\eta_{x_1x_1} = \mu_{y_1y_1} = 2F(x_2, y_2, \dots, x_n, y_n), \tag{4.23}$$

for some positive function $F = F(x_2, y_2, \dots, x_n, y_n)$. Thus, after solving (4.23), we obtain

$$\begin{aligned} \eta &= F(x_2,\dots,y_n)x_1^2 + G(x_2,\dots,y_n)x_1 + H(x_2,\dots,y_n), \\ \mu &= F(x_2,\dots,y_n)y_1^2 + K(x_2,\dots,y_n)y_1 + L(x_2,\dots,y_n), \end{aligned} \tag{4.24}$$

for some functions G, H, K, L of $2n-2$ variables. Substituting (4.24) into (4.21) gives $4F(H+L) = G^2 + K^2$. Hence, by (4.24), we get

$$\eta + \mu = \frac{1}{4F}\left\{(2Fx_1 + G)^2 + (2Fy_1 + K)^2\right\}. \tag{4.25}$$

Combining (4.18) and (4.25) yields

$$\phi = \frac{1}{2}\ln\left\{(ax_1+\beta)^2 + (ay_1+\delta)^2\right\}, \tag{4.26}$$

where $a(x_2,\dots,y_n) = \sqrt{F} > 0$, $\beta(x_2,\dots,y_n) = \frac{G}{2\sqrt{F}}$, and $\delta(x_2,\dots,y_n) = \frac{K}{2\sqrt{F}}$.

From (4.26) we find

$$\begin{aligned} \phi_{x_1} &= \frac{a(ax_1+\beta)}{(ax_1+\beta)^2 + (ay_1+\delta)^2}, \\ \phi_{y_1} &= \frac{a(ay_1+\delta)}{(ax_1+\beta)^2 + (ay_1+\delta)^2}, \\ \phi_{x_j} &= \frac{(ax_1+\beta)(a_{x_j}x_1 + \beta_{x_j}) + (ay_1+\delta)(a_{x_j}y_1 + \delta_{x_j})}{(ax_1+\beta)^2 + (ay_1+\delta)^2}, \\ \phi_{y_j} &= \frac{(ax_1+\beta)(a_{y_j}x_1 + \beta_{y_j}) + (ay_1+\delta)(a_{y_j}y_1 + \delta_{y_j})}{(ax_1+\beta)^2 + (ay_1+\delta)^2}, \end{aligned} \tag{4.27}$$

for $j = 2, \ldots, n$. Hence, by applying (4.12) for $\phi_{x_1 x_j}$, we obtain

$$\begin{aligned}&(a_{x_j}(2ax_1+\beta)+a\beta_{x_j})((ax_1+\beta)^2+(ay_1+\delta)^2)\\&-a(ax_1+\beta)\left[(ax_1+\beta)(a_{x_j}x_1+\beta_{x_j})+(ay_1+\delta)(a_{x_j}y_1+\delta_{x_j})\right]\\&\quad=a(ay_1+\delta)\left[(ax_1+\beta)(a_{y_j}x_1+\beta_{y_j})+(ay_1+\delta)(a_{y_j}y_1+\delta_{y_j})\right].\end{aligned} \tag{4.28}$$

By comparing the coefficients of x_1^3 and y_1^3 in (4.17) we find $\partial a/\partial x_j = \partial a/\partial y_j = 0$ for $j = 2, \ldots, n$. Hence a is constant. Thus, (4.27) and (4.28) imply

$$\begin{aligned}\phi_{x_j}&=\frac{(ax_1+\beta)\beta_{x_j}+(ay_1+\delta)\delta_{x_j}}{(ax_1+\beta)^2+(ay_1+\delta)^2},\\ \phi_{y_j}&=\frac{(ax_1+\beta)\beta_{y_j}+(ay_1+\delta)\delta_{y_j}}{(ax_1+\beta)^2+(ay_1+\delta)^2},\end{aligned} \tag{4.29}$$

$$(ay_1+\delta)\beta_{x_j}-(ax_1+\beta)\delta_{x_j}=(ax_1+\beta)\beta_{y_j}+(ay_1+\delta)\delta_{y_j}. \tag{4.30}$$

Since a is constant, (4.29) implies

$$\begin{aligned}\phi_{x_jy_k}=&\frac{\beta_{x_j}\beta_{y_k}+\delta_{x_j}\delta_{y_k}+(ax_1+\beta)\beta_{x_jy_k}+(ay_1+\delta)\delta_{x_jy_k}}{(ax_1+\beta)^2+(ay_1+\delta)^2}\\&-2\frac{[(ax_1+\beta)\beta_{x_j}+(ay_1+\delta)\delta_{x_j}]\cdot[(ax_1+\beta)\beta_{y_k}+(ay_1+\delta)\delta_{y_k}]}{[(ax_1+\beta)^2+(ay_1+\delta)^2]^2},\end{aligned} \tag{4.31}$$

for $2 \leq j,\ k \leq n$. Hence, by applying (4.13) with $j = k$, (4.29) and (4.31), we get

$$\beta_{x_j}\beta_{y_j}+\delta_{x_j}\delta_{y_j}+(ax_1+\beta)\beta_{x_jy_j}+(ay_1+\delta)\delta_{x_jy_j}=0. \tag{4.32}$$

Thus, we obtain $\beta_{x_jy_j} = \delta_{x_jy_j} = 0$ for $j = 2, \ldots, n$, by comparing the coefficients of x_1^3 and y_1^3 in (4.32), respectively.

Similarly, by using (4.29) and by comparing the coefficients of x_1^3 and y_1^3 in other equations from (4.12)–(4.14), we may also obtain $\beta_{x_jx_j} = \beta_{x_ky_j} = \beta_{x_jy_k} = \delta_{x_jx_j} = \delta_{x_ky_j} = \delta_{x_jy_k} = 0$ for $2 \leq j,\ k \leq n$. Therefore, there exist constants $a_3, \ldots, a_{2n}, b_3, \ldots, b_{2n}$ such that

$$\begin{aligned}\beta&=a_3x_2+a_4y_2+\cdots+a_{2n-1}x_n+a_{2n}y_n,\\ \delta&=b_3x_2+b_4y_2+\cdots+b_{2n-1}x_n+b_{2n}y_n.\end{aligned} \tag{4.33}$$

Combining (4.15) and (4.22) yields

$$\phi = \frac{1}{2} \ln \{(ax_1 + a_3x_2 + a_4y_2 + \cdots + a_{2n}y_n)^2 \\ + (ay_1 + b_3x_2 + \cdots + b_{2n}y_n)^2\}. \tag{4.34}$$

Finally, from (4.12) with $j = 1$ and (4.34), we obtain $a_{2k-1} = b_{2k}$ and $a_{2k} = -b_{2k-1}$. Thus, we obtain from (4.34) that

$$\phi = \frac{1}{2} \ln \{(ax_1 + a_3x_2 + a_4y_2 + \cdots + a_{2n-1}x_n + a_{2n}y_n)^2 \\ + (ay_1 - a_4x_2 + a_3y_2 - \cdots - a_{2n}x_n + a_{2n-1}y_n)^2\}. \tag{4.35}$$

If we put $\alpha = (a, a_3 + ia_4, \ldots, a_{2n-1} + ia_{2n})$ and $z = (x_1 + iy_1, \ldots, x_n + iy_n)$, then (4.35) becomes (4.15) with $a_1 = a$ and $a_2 = 0$. After applying a suitable change of variable on z_1, we obtain Lemma 4.2. □

Lemma 4.2 implies that $\phi = \ln f$ is given by $\phi = \frac{1}{2} \ln\{\langle \alpha, z\rangle^2 + \langle i\alpha, z\rangle^2\}$ for some vector $\alpha \in \mathbf{C}^{n+1}$. By choosing a suitable Euclidean complex coordinates on $\mathbf{C}^{n+1}$, we may obtain $\alpha = (b_1 + ib_2, 0, \ldots, 0)$. With respect to this vector α, we have $f = \{(b_1x_1 + b_2y_1)^2 + (-b_2x_1 + b_1y_1)^2\}^{1/2}$, which is nothing but

$$f = a\sqrt{x_1^2 + y_1^2}, \quad a = \sqrt{b_1^2 + b_2^2}. \tag{4.36}$$

From the formula of Gauss, (4.2), (4.4)–(4.7), (4.9) and (4.36), we know that the immersion $\mathbf{x}$ of $N \times_f I$ in $\mathbf{C}^m$ satisfies

$$\mathbf{x}_{x_jx_k} = \mathbf{x}_{x_jy_k} = \mathbf{x}_{y_jy_k} = 0, \quad j, k = 1, \ldots, n, \tag{4.37}$$

$$\mathbf{x}_{x_1t} = \frac{x_1 - iy_1}{x_1^2 + y_1^2}\mathbf{x}_t, \quad \mathbf{x}_{y_1t} = \frac{y_1 + ix_1}{x_1^2 + y_1^2}\mathbf{x}_t \tag{4.38}$$

$$\mathbf{x}_{x_jt} = \mathbf{x}_{y_jt} = 0, \quad j = 2, \ldots, n, \tag{4.39}$$

$$\mathbf{x}_{tt} = -a^2(x_1\mathbf{x}_{x_1} + y_1\mathbf{x}_{y_1}) + i\lambda\mathbf{x}_t. \tag{4.40}$$

It is straightforward to verify from (4.37)–(4.40) that $\mathbf{x}_{x_jtt} = \mathbf{x}_{ttx_j}$, $\mathbf{x}_{y_jtt} = \mathbf{x}_{tty_j}$ hold for $j = 1, \ldots, n$ if and only if $\partial\lambda/\partial x_j = \partial\lambda/\partial y_j = 0$ and $\mathbf{x}_{y_j} = i\mathbf{x}_{x_j}$ for $j = 1, \ldots, n$. Hence $\lambda = \lambda(t)$ is a function of t and, moreover, (4.40) reduces to

$$\mathbf{x}_{tt} = -a^2z_1\mathbf{x}_{x_1} + i\lambda(t)\mathbf{x}_t, \tag{4.41}$$

Solving (4.37) gives

$$\mathbf{x} = \sum_{k=1}^{n} \hat{A}^k(t)x_k + \sum_{k=1}^{n} \hat{B}^k(t)y_k + C(t) \tag{4.42}$$

for some functions $\hat{A}^1, \dots, \hat{A}^n$, $\hat{B}^1, \dots, \hat{B}^n$, C of t. Substituting (4.42) into (4.38) gives

$$(x_1^2 + y_1^2)\hat{A}_t^1 = (x_1 - iy_1)\left\{ \sum_{k=1}^{n} \hat{A}_t^k x_k + \sum_{k=1}^{n} \hat{B}_t^k y_k + C_t \right\}, \tag{4.43}$$

for $j = 1, \dots, n$, which implies

$$\hat{A}_t^k = \hat{B}_t^k = C_t = 0, \quad k = 2, \dots, n, \tag{4.44}$$

$$\hat{B}_t^1 = i\hat{A}_t^1. \tag{4.45}$$

Condition (4.44) implies that $\hat{A}^2, \dots, \hat{A}^n$, $\hat{B}^2, \dots, \hat{B}^n$, C are constant vectors. We may choose $C = 0$ by applying a suitable translation if necessary.

Solving (4.45) implies that there is a vector δ_1 so that

$$\hat{B}^1 = i\hat{A}^1 + i\delta_1. \tag{4.46}$$

We put

$$\hat{A}^1 = a\gamma + \delta_1, \quad \hat{A}^k = \beta_k, \quad \hat{B}^k = i\delta_k, \quad k = 2, \dots, n. \tag{4.47}$$

(4.42), (4,46) and (4.47) imply

$$\mathbf{x}(z_1, \dots, z_n, t) = a\gamma(t)z_1 + \sum_{k=1}^{n} (\beta_k x_k + i\delta_k y_k), \tag{4.48}$$

where $\beta_1 = \delta_1$.

From $\mathbf{x}_{y_j} = i\mathbf{x}_{x_j}$ and (4.48) we find $\delta_k = \beta_k$, $k = 1, \dots, n$. Thus, (4.48) gives

$$\mathbf{x}(z_1, \dots, z_n, t) = a\gamma(t)z_1 + \sum_{k=1}^{n} \beta_k z_k \tag{4.49}$$

Substituting (4.49) into (4.41) yields $\beta_1 = \delta_1 = 0$ and

$$\gamma''(t) - i\lambda(t)\gamma'(t) + a^2\gamma(t) = 0. \tag{4.50}$$

If we choose the initial conditions:

$$\begin{aligned}&\mathbf{x}_{x_k}(1,0,\ldots,0)=(0,\ldots,0,\overbrace{1}^{k+1\text{-th}},0,\ldots,0),\quad k=1,\ldots,n,\\&\mathbf{x}_t(1,0,\ldots,0)=(a,0,\ldots,0),\end{aligned}\tag{4.51}$$

we obtain from (4.49) and (4.51) that

$$\begin{aligned}&\gamma(0)=(0,a^{-1},0,\ldots,0),\quad \gamma'(0)=(1,0,\ldots,0),\\&\beta_2=(0,0,1,0,\ldots,0),\ldots,\beta_n=(0,\ldots,0,1).\end{aligned}\tag{4.52}$$

Combining (4.49) and (4.53) yields

$$\mathbf{x}(z_1,\ldots,z_n,t)=a\gamma(t)z_1+(0,0,z_2,\ldots,z_n).\tag{4.53}$$

Since $\gamma(t)$ is a solution of the second order homogeneous linear equation (4.50), $\gamma(t)$ can be expressed as $\gamma(t)=c_1A_1(t)+c_2A_2(t)$, where c_1, c_2 are constant vectors in $\mathbf{C}^{n+1}$ and $A_1(t),A_2(t)$ are two independent solutions of (4.50). Thus, the image of $\gamma(t)$ must lie in the complex plane, say $\mathbf{C}^2$, spanned by c_1, c_2. Clearly, $\mathbf{C}^2$ is the complex plane defined by $\{(w_1,w_2,0,\ldots,0):w_1,w_2\in\mathbf{C}\}$. Hence, if we denote the curve γ by $\gamma(t)=(\Gamma_1(t),\Gamma_2(t))$, then the hypersurface is given by

$$\mathbf{x}(z_1,\ldots,z_n,t)=(a\Gamma(t)z_1,a\Gamma(t)z_1,z_2,\ldots,z_n).\tag{4.54}$$

From (4.53) we get $\langle\mathbf{x}_t,\mathbf{x}_t\rangle=f^2|\gamma'(t)|^2$. Comparing this with the warped metric (4.3) of the hypersurface yields $|\gamma'(t)|=1$. Thus, $\gamma(t)$ is of unit speed, so we have $\langle\gamma'(t),\gamma''(t)\rangle=0$. Now, by taking the inner product of $\gamma'(t)$ with (4.50), we obtain $\langle\gamma'(t),\gamma(t)\rangle=0$. Thus, $\gamma(t)$ has constant length. Therefore, by applying the first equation in (4.52), we get $|\gamma(t)|=1/a$. Thus γ defines a unit speed curve in $S^3(a^2)$:

$$\gamma:I\to S^3(a^2)\subset\mathbf{C}^2.\tag{4.55}$$

Case (a): $\lambda=0$. In this case, the solution of the differential equation (4.50) is given by $\gamma(t)=c_1\cos(at)+c_2\sin(at)$. So, if we choose the initial conditions: $\gamma(0)=(0,a^{-1},0,\ldots,0),\gamma'(0)=(1,0\ldots,0)$, we get

$$c_1=(0,a^{-1},0,\ldots,0),\quad c_2=(a^{-1},0,\ldots,0).$$

Hence, we find

$$\gamma(t)=\big(a^{-1}\sin(at),a^{-1}\cos(at),0,\ldots,0\big),\tag{4.56}$$

which is a unit speed Legendre curve in $S^3(a^2)$.

Case (b): $\lambda(t) \neq 0$. In this case, since $\gamma = \gamma(t)$ satisfies the differential equation (4.50) with $\lambda(t) \neq 0$. Thus, Lemma 3.1 implies that $\gamma = \gamma(t)$ is a unit speed Legendre curve in $S^3(a^2)$.

Conversely, since $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve in $S^3(a^2)$, it is easy to verify that the real hypersurface defined by (4.1) is the warped product $\mathbf{C}_*^n \times_{a|z_1|} I$ of $\mathbf{C}_*^n$ and an open interval I. $\square$

Remark 4.1 The real hypersurface of $\mathbf{C}^{n+1}$ defined by (4.1) is a *non-complete ruled hypersurface.* Moreover, a direct computation shows that the squared mean curvature and scalar curvature of the real hypersurface are given respectively by

$$|H|^2 = \frac{\lambda^2}{(2n+1)^2 a^2 |z_1|^2}, \quad \tau = -\frac{1}{|z_1|^2} \tag{4.57}$$

where λ is the curvature of the unit speed Legendre curve $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ in $S^3(a^2)$. It follows from (4.57) that the hypersurface has non-constant scalar curvature. Moreover, it has non-constant mean curvature unless the Legendre curve γ is a geodesic in $S^3(a^2)$.

Example 4.1 If λ is constant, then the unique Legendre curve obtained from the solution of the differential equation (4.50), which also satisfies the initial conditions: $\gamma(0) = (a^{-1}, 0)$, $\gamma'(0) = (0, 1)$, is given by

$$\gamma(t) = \left(\frac{e^{\frac{i}{2}\lambda t}}{a}\left(\cos\left(\frac{bt}{2}\right) - \frac{i\lambda}{b}\sin\left(\frac{bt}{2}\right)\right), \frac{2e^{\frac{i}{2}\lambda t}}{b}\sin\left(\frac{bt}{2}\right)\right)$$
$$b = \sqrt{4a^2 + \lambda^2}. \tag{4.58}$$

From (4.54) and (4.58) we see that the corresponding warped product hypersurface is given by

$$\mathbf{x} = \left(\frac{e^{\frac{i}{2}\lambda t}}{a}\left(\cos\left(\frac{bt}{2}\right) - \frac{i\lambda}{b}\sin\left(\frac{bt}{2}\right)\right)z_1, \frac{2e^{\frac{i}{2}\lambda t}}{b}\sin\left(\frac{bt}{2}\right)z_1, z_2, \dots, z_n\right). \tag{4.59}$$

5. Warped product real hypersurfaces in CP^{n+1}

Let S^{2n+3} denote the unit hypersphere in $\mathbf{C}^{n+2}$ centered at the origin and put $U(1) = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$. Then there is a $U(1)$-action on S^{2n+3} defined by $z \mapsto \lambda z$. At $z \in S^{2n+3}$ the vector $V = iz$ is tangent to the flow of the action. The quotient space $S^{2n+3}/\sim$, under the identification induced from the action, is a complex projective space $CP^{n+1}(4)$ which endows with the canonical Fubini-Study metric of constant holomorphic sectional curvature 4. The almost complex structure J on $CP^{n+1}(4)$ is induced from the complex structure J on $\mathbf{C}^{n+2}$ via the Hopf fibration: $\pi : S^{2n+3} \to CP^{n+1}(4)$. It is well-known that the Hopf fibration π is a Riemannian submersion such that $V = iz$ spans the vertical subspaces.

Let $\phi : M \to CP^{n+1}(4)$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers. The lift $\hat{\phi} : \hat{M} \to S^{2n+3}$ of ϕ is an isometric immersion so that the diagram:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\phi}} & S^{2n+3} \\ \pi\big\downarrow & & \big\downarrow\pi \\ M & \xrightarrow{\phi} & CP^{n+1}(4) \end{array} \tag{5.1}$$

commutes.

Conversely, if $\psi : \hat{M} \to S^{2n+3}$ is an isometric immersion which is invariant under the $U(1)$-action, then there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to CP^{n+1}(4)$ such that the associated diagram (5.1) commutes. We simply call the immersion $\psi_\pi : \pi(\hat{M}) \to CP^{n+1}(4)$ *the projection* of $\psi : \hat{M} \to S^{2n+3}$.

For a given vector $X \in T_z(CP^{n+1})$ and a point $u \in S^{2n+2}$ with $\pi(u) = z$, we denote by X_u^* the horizontal lift of X at u via π. There exists a canonical orthogonal decomposition:

$$T_u S^{2n+3} = (T_{\pi(u)} CP^{n+1})_u^* \oplus \mathrm{Span}\,\{V_u\}. \tag{5.2}$$

Since π is a Riemannian submersion, X and X_u^* have the same length.

We put

$$S_*^{2n+1} = \Big\{(z_0, \ldots, z_n) : \sum_{k=0}^{n} z_k\bar{z}_k = 1,\ z_0 \neq 0\Big\}, \quad CP_0^n = \pi(S_*^{2n+1}).$$

The following theorem classifies completely real hypersurfaces of complex projective space which are warped products of a complex hypersurface and a real curve.

Theorem 5.1 *Suppose that a is a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbf{C}^2$ defined on an open interval I. Let $\mathbf{x} : S_*^{2n+1} \times I \to \mathbf{C}^{n+2}$ be the map defined by*

$$\mathbf{x}(z_0, \ldots, z_n, t) = \big(a\Gamma_1(t)z_0, a\Gamma_2(t)z_0, z_1, \ldots, z_n\big), \quad \sum_{k=0}^{n} z_k\bar{z}_k = 1. \tag{5.3}$$

Then

(i) $\mathbf{x}$ *induces an isometric immersion* $\psi : S_*^{2n+1} \times_{a|z_0|} I \to S^{2n+3}$.

(ii) *The image* $\psi(S_*^{2n+1} \times_{a|z_0|} I)$ *in* S^{2n+3} *is invariant under the action of* $U(1)$.

(iii) *the projection* $\psi_\pi : \pi(S_*^{2n+1} \times_{a|z_0|} I) \to CP^{n+1}(4)$ *of* ψ *via* π *is a warped product hypersurface* $CP_0^n \times_{a|z_0|} I$ *in* $CP^{n+1}(4)$.

Conversely, if a real hypersurface in $CP^{n+1}(4)$ is a warped product $N \times_f I$ of a complex hypersurface N of $CP^{n+1}(4)$ and an open interval I, then, up to rigid motions, it is locally obtained in the way described above.

Proof. Statement (i) is easy to verify, since $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve in $S^3(a^2)$.

Statement (ii) follows from (5.3) and the definition of the $U(1)$-action.

Since $\gamma = \gamma(t)$ is a Legendre curve in $S^3(a^2)$, (5.3) implies that, for each $z = (z_0, \ldots, z_n)$ with $\sum z_k\bar{z}_k = 1$, the curve Γ_z defined by

$$\Gamma_z(t) = \big(a\Gamma_1(t)z_0, a\Gamma_2(t)z_0, z_1, \ldots, z_n\big)$$

is a horizontal curve in S^{2n+3}. Thus $\pi : \Gamma_z \to \pi(\Gamma_z)$ is isometric. Clearly, the restriction of π on S_*^{2n+1} is also a Riemannian submersion. Hence, the projection of $\psi_\pi : \psi(S_*^{2n+1} \times_{a|z_0|} I) \to CP^{n+1}(4)$ of $\psi : S_*^{2n+1} \times_{a|z_0|} I \to S^{2n+3}$ is a warped product hypersurface $CP^{n+1}(4) \times_{a|z_0|} I$ in $CP^{n+1}(4)$.

Conversely, assume that $M = N \times_f I$ is a warped product hypersurface of $CP^{n+1}(4)$, where N is a complex hypersurface of $CP^{n+1}(4)$ and I is an open interval. Then, according to [6], the warping function f cannot be a constant function.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the distributions on M spanned by vectors tangent to the N and I, respectively. Trivially, $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable distributions. From (1) of Lemma 2.2 we know that the second fundamental form σ of M in $CP^{n+1}(4)$ satisfies $\sigma(\mathcal{D}_1, \mathcal{D}_1) = 0$. Since N is totally geodesic in $N \times_f I$, N is thus totally geodesic in $CP^{n+1}(4)$. Hence, N is holomorphically isometric to an open part of a $CP^{n+1}(4)$.

Let $\hat{\nabla}$ and ∇ denote the Riemannian connections of $\hat{M}$ and M respectively. And let $\hat{\sigma}$ denote the second fundamental form of $\hat{M}$ in S^{2n+3}. Then we have

$$\hat{\nabla}_{X^*}Y^* = (\nabla_X Y)^* - \langle PX, Y\rangle V, \tag{5.4}$$

$$\hat{\nabla}_V X^* = \hat{\nabla}_{X^*} V = (PX)^*, \tag{5.5}$$

$$\hat{\nabla}_V V = 0, \tag{5.6}$$

$$\hat{\sigma}(X^*, Y^*) = (\sigma(X, Y))^*, \tag{5.7}$$

$$\hat{\sigma}(X^*, V) = (FX)^*, \tag{5.8}$$

$$\hat{\sigma}(V, V) = 0, \tag{5.9}$$

for vector fields X, Y tangent to M, where PX and FX denote the tangential and the normal components of JX, respectively.

Let $\hat{\mathcal{D}}_1$ denote the distribution on $\hat{M} = \pi^{-1}(M)$ spanned by $\mathcal{D}_1^*$ and $V = iz$, where $\mathcal{D}_1^* = \{X^* : X \in \mathcal{D}_1\}$. Since $\mathcal{D}_1$ is integrable, (5.4)–(5.6) implies that $\hat{\mathcal{D}}_1$ is also integrable. From (5.7)–(5.9) we know that each leaf of $\hat{\mathcal{D}}_1$ is totally geodesic in S^{2n+3}. Thus, each leaf of $\hat{\mathcal{D}}$ is isometric to an open portion of the unit sphere S^{2n+1}.

Clearly, $\mathcal{D}_2^* = \{Z^* \in T\hat{M} : Z \in \mathcal{D}_2\}$ is the orthogonal complementary distribution of $\hat{\mathcal{D}}_1$ in $T\hat{M}$. Since $\mathcal{D}_2^*$ is of rank one, $\mathcal{D}_2^*$ is also integrable and $P\mathcal{D}_2 = \{0\}$. Thus, it follows from (5.4) that

$$\hat{\nabla}_{Z^*} W^* = (\nabla_Z W)^*, \quad Z, W \in \mathcal{D}_2. \tag{5.10}$$

On the other hand, for any vector field X in $\mathcal{D}_1$, and Z, W in $\mathcal{D}_2$, we have [11]

$$\langle \nabla_Z W, X\rangle = -(X \ln f)\,\langle Z, W\rangle. \tag{5.11}$$

Applying (5.10), (5.11), $(\nabla_Z W)^* \perp V$, and the fact that the Hopf fibration is a Riemannian submersion, we obtain

$$\langle \hat{\nabla}_{Z^*} W^*, X^*\rangle = -(X \ln f)\,\langle Z^*, W^*\rangle, \;\; \langle \hat{\nabla}_{Z^*} W^*, V\rangle = 0$$

which implies that each integral curve of $\hat{\mathcal{D}}_2$ is a circle in $\hat{M}$, i.e., a order two Frenet curve with constant curvature in $\hat{M}$. Thus, a result of [7] implies that locally $\hat{M}$ is a warped product $S^{2n+1} \times_{\hat{f}} I$, with warping function $\hat{f}$.

Let $\breve{M}$ denote the punctured cone over $\hat{M}$ with 0 as its vertex defined by

$$\breve{M} = \{tw \in \mathbf{C}^{n+2} : w \in \hat{M} = S^{2n+1} \times_{\hat{f}} I \subset S^{2n+3} \subset \mathbf{C}^{n+2},\ t > 0\}.$$

Since the tangent vector field $\partial/\partial t$ on $\breve{M}$ is parallel to the position vector field of $\breve{M}$ in $\mathbf{C}^{n+2}$ and V is tangent to the first component of $S^{2n+1} \times_{\hat{f}} I$, we see that locally $\breve{M}$ is the warped product $\mathbf{C}^{n+1} \times_{t\hat{f}} I$, where $\mathbf{C}^{n+1}$ is a complex hyperplane of $\mathbf{C}^{n+2}$. Since the warping function is non-constant, Theorem 4.1 implies that, up to rigid motions, $\breve{M}$ is given by

$$\mathbf{x}(z_1, \ldots, z_n, t) = \big(a\Gamma_1(t)z_1, a\Gamma_2(t)z_1, z_2, \ldots, z_n\big), \quad z_1 \neq 0, \quad (5.12)$$

for some positive number a and a unit speed Legendre curve $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ in $S^3(a^2)$. Consequently, up to rigid motions, the warped product hypersurface in $CP^{n+1}(4)$ is the projection of ψ given by (5.3) via the Hopf fibration. □

Remark 5.1 The real hypersurface of $CP^{n+1}(4)$ defined by (5.3) is a *non-complete ruled hypersurface*. Moreover, a direct computation shows that the squared mean curvature and scalar curvature of the real hypersurface are given respectively by $|H|^2 = \lambda^2/(2n+1)^2 a^2 |z_0|^2$, $\tau = -1/|z_0|^2$ where λ is the curvature of the unit speed Legendre curve $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ in $S^3(a^2)$. Therefore, the hypersurface has non-constant mean curvature unless the Legendre curve γ is a geodesic of $S^3(a^2)$; moreover, the real hypersurface defined by (5.3) has non-constant scalar curvature.

6. Warped product real hypersurfaces in CH^{n+1}

In the complex pseudo-Euclidean space $\mathbf{C}_1^{n+2}$ endowed with pseudo-Euclidean metric $g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^{n+1} dz_j d\bar{z}_j$, we define the the anti-de Sitter space-time by

$$H_1^{2n+3} = \{(z_0, z_1, \ldots, z_{n+1}) : \langle z, z \rangle = -1\}.$$

It is known that H_1^{2n+3} has constant sectional curvature -1. There is a $U(1)$-action on H_1^{2n+3} defined by $z \mapsto \lambda z$. At a point $z \in H_1^{2n+3}$, iz is tangent to the flow of the action. The orbit is given by $z_t = e^{it}z$ with $\frac{dz_t}{dt} = iz_t$ which lies in the negative-definite plane spanned by z and iz. The quotient space $H_1^{2n+3}/\sim$ is the complex hyperbolic space $CH^{n+1}(-4)$ which endows a canonical Kähler metric of constant holomorphic sectional curvature -4. The complex structure J on $CH^{n+1}(-4)$ is induced from the canonical complex structure J on $\mathbf{C}_1^{n+2}$ via the totally geodesic fibration: $\pi : H_1^{2n+3} \to CH^{n+1}(-4)$.

Let $\phi : M \to CH^{n+1}(-4)$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers. The lift $\hat{\phi} : \hat{M} \to H_1^{2n+3}$ of ϕ is an isometric immersion such that the diagram:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\phi}} & H_1^{2n+3} \\ \pi\big\downarrow & & \big\downarrow\pi \\ M & \xrightarrow{\phi} & CH^{n+1}(-4) \end{array} \tag{6.1}$$

commutes.

Conversely, if $\psi : \hat{M} \to H_1^{2n+3}$ is an isometric immersion which is invariant under the $U(1)$-action, there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to CH^{n+1}(-4)$, called the *projection of* ψ so that the associated diagram commutes.

We put

$$\begin{aligned} &H_{1*}^{2n+1} = \{(z_0, \ldots, z_n) \in H_1^{2n+1} : z_n \neq 0\}, \\ &CH_*^n = \pi(H_{1*}^{2n+1}). \end{aligned} \tag{6.2}$$

For warped product hypersurfaces in CH^{n+1}, we have the following classification theorem.

Theorem 6.1 *Suppose that a is a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbf{C}^2$. Let $\mathbf{y} : H_{1*}^{2n+1} \times I \to \mathbf{C}_1^{n+2}$ be the map defined by*

$$\begin{aligned} &\mathbf{y}(z_0, \ldots, z_n, t) = (z_0, \ldots, z_{n-1}, a\Gamma_1(t)z_n, a\Gamma_2(t)z_n), \\ &z_0\bar{z}_0 - \sum_{k=1}^{n} z_k\bar{z}_k = 1. \end{aligned} \tag{6.3}$$

Then

(i) $\mathbf{y}$ *induces an isometric immersion* $\psi : H^{2n+1}_{1*} \times_{a|z_n|} I \to H^{2n+3}_1$.

(ii) *The image* $\psi(H^{2n+1}_{1*} \times_{a|z_n|} I)$ *in* H^{2n+3}_1 *is invariant under the* $U(1)$-*action.*

(iii) *the projection* $\psi_\pi : \pi(H^{2n+1}_{1*} \times_{a|z_n|} I) \to CH^{n+1}(-4)$ *of* ψ *via* π *is a warped product hypersurface* $CH^n_* \times_{a|z_n|} I$ *in* $CH^{n+1}(-4)$.

Conversely, if a real hypersurface in $CH^{n+1}(-4)$ *is a warped product* $N \times_f I$ *of a complex hypersurface* N *and an open interval* I*, then, up to rigid motions, it is locally obtained in the way described above.*

We omit the proof since it is very similar to the proof of Theorem 5.1.

Remark 6.1 A direct computation shows that the hypersurface in $CH^{n+1}(-4)$ obtained from (6.3) is a non-complete ruled hypersurface which has non-constant scalar curvature $\tau = -1/|z_n|^2$ and non-constant mean curvature unless the Legendre curve γ is a geodesic in $S^3(a^2)$.

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Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
U. S. A.
E-mail: bychen@math.msu.edu