Hokkaido Mathematical Journal Vol. 31 (2002) p. 343–348

A note on solvability of factorizable finite groups

Angel CAROCCA

(Received August 28, 2000; Revised February 14, 2001)

Abstract. Using theorems on the classification of finite simple groups, we give an extension of some results on the solvability of factorizable finite groups that are generalizations of a well known theorem due to O. Kegel and H. Wielandt.

Key words: factorizable groups, 2-decomposable and 2-nilpotent subgroups.

1. Introduction

Groups that can be written as a product $G=HK$ of two of its subgroups H and K have been studied by many authors. Based on Kegel-Wielandt's theorem [6, Satz 4.3, p.674], which states that a finite group is solvable if it is the product of two nilpotent subgroups. Similar problems on factorizable groups have been studied by various authors.

In [3] and [8], factorizable groups $G=HK$ are studied, where H is 2-decomposable and K is nilpotent of odd order. Here a finite group H is called 2-*decomposable* if it is the direct product of a Sylow 2-subgroup, with $\mathbf{O}(H)$ the largest normal subgroup of H of odd order. When H is only a product of $\mathbf{O}(H)$ with a Sylow 2-subgroup, it is called 2-*nilpotent*. In [1] (also see [9]), we attempted to generalize the 2-decomposability of H to 2-nilpotency. However, we did not succeed completely. Imposing a stronger restriction on K, we obtain the following result.

Let $G=HP$ be a group such that H is 2-nilpotent and P is a p-group of odd order. Then G is solvable.

The following is a generalization of the result above, which we obtain by removing the restriction on K.

Theorem 1 *Let $G=HK$ be a finite group such that H is 2-nilpotent and K nilpotent of odd order. Then G is solvable.*

2000 Mathematics Subject Classification : Primary 20D40, 20D08; Secondary 20F16, 20F18.

Part of this work was supported by FONDECYT Grant # 1000623.

In Section 3, we give the proof which is based on the following outline. Suppose the classification of finite simple groups (CFSG). The elaborate work of Kazarin restricts the possible isomorphism type of the simple direct factor N_1 of a minimal normal subgroup N of a minimal counterexample $G = HK$. Also, we may show that $N_1 \cap H$ is a 2-nilpotent subgroup of N_1 with nontrivial odd core. Using the divisibility of $|\mathbf{Out}(N_1)|\,|H \cap N_1|\,|K \cap N_1|$ by $|N_1|$ (see Lemma 2.3 (b)), it is not difficult to reach a contradiction with the self-centralizing property of a certain Sylow subgroup of N_1.

2. Preliminaries

In this section, we recall some results that are used in the proof of the main theorem. If G is the product of two solvable subgroups, then it is known that G is not necessarily solvable. Particular cases of finite groups that are factorizable by two subgroups were studied by various authors.

We next state Kazarin's result for the general case.

Lemma 2.1 (Kazarin [7]) *Let $G = HK$ be a group, where H and K are solvable subgroups of G. If all composition factors of G are known groups, then the nonabelian simple composition factors of G belong to the following list of groups:*

(a) $\mathbf{PSL}(2, q)$, *with* $q > 3$;
(b) $\mathbf{M}_{11}$;
(c) $\mathbf{PSL}(3, q)$, *with* $q < 9$;
(d) $\mathbf{PSp}(4, 3)$;
(e) $\mathbf{PSU}(3, 8)$;
(f) $\mathbf{PSL}(4, 2)$.

Remark 1 Consider a nontrivial r-subgroup of $\mathbf{PSL}(2, p^n)$, with p an odd prime. When $r = p$ or r divides $\frac{(q\pm1)}{2}$, it is well known that the centralizer is, respectively, elementary abelian of order p^n or a cyclic group of order $\frac{(q\pm1)}{2}$; analogously, for the normalizer, it is, respectively, the semidirect product of an elementary abelian group of order p^n by a cyclic group of order $\frac{(q-1)}{2}$ or a dihedral group of order $q \pm 1$.

Lemma 2.2 *Let $G = HK = HN = KN$ be a group, where H and K are solvable subgroups of G and N is the unique minimal normal subgroup of G and N is nonsolvable. Then:*

(a) *H is transitive on the set of direct factors of N, which are non-abelian simple groups and $H \cap N = \prod_i^m L_i$, where $N = \prod_i^m N_i$ (with $N_i \cong N_j$ simple) for every $i, j \in \{1, \dots, m\}$ and $L_i = N_i \cap H$.*

(b) *$|N_1|$ divides $|Out(N_1)|\,|N_1 \cap H|\,|N_1 \cap K|$.*

Proof. See Lemmas 2.3 and 2.5 of [4]. □

Remark 2 The structure of $\mathbf{Aut}(\mathbf{PSL}(2,p^n))$ is well known to be isomorphic to the semidirect product of $\mathbf{PGL}(2,p^n)$ by a cyclic group $\mathbf{A}$ of order n (see [5, p.462]). Also, $|\mathbf{Out}(\mathbf{PSL}(2,p^n))| = (2,p-1)n$.

Lemma 2.3 *Let $G = HK$ be a finite group such that every proper quotient as well as every subgroup containing H or K is solvable. Assume that there is a prime divisor of $(|H|,|K|)$. If H and K have each a normal Sylow p-subgroup, then G is solvable.*

Proof. See [2]. □

Lemma 2.4 *Let $G = HK$ be a finite group, with H and K subgroups such that $(|H|,|K|) = 1$. Let $R \trianglelefteq H$ be a 2-decomposable subgroup with $|H : R| = 2^i$. Assume further that H/R is abelian or dihedral. If K is nilpotent of odd order, then G is solvable.*

Proof. See [3]. □

3. Proof of main result

Proof of Theorem 1. Suppose to the contrary that G need not be solvable and let $G = HK$ be a counterexample of smallest order. Let N be a minimal normal subgroup of G. Since the hypothesis is inherited by factor groups, as well as by every subgroup which contains H or K, we conclude that G/N is solvable and that $G = HK = NH = NK$ by the minimality of $|G|$. Hence $G/N \cong K/(K \cap N) \cong H/(H \cap N)$ is nilpotent of odd order and H contains a Sylow 2-subgroup of G. Also, $H \cap N$ contains a Sylow 2-subgroup of H and if $\mathbf{O}(H)$ is the normal 2-complement of H, then $G = N\mathbf{O}(H)$. Therefore $G/N \cong \mathbf{O}(H)/(\mathbf{O}(H) \cap N)$. Furthermore, N is the unique minimal normal subgroup of G (obviously N is nonsolvable). □

We have that $\mathbf{O}(H) \cap N = \mathbf{O}(H \cap N) \neq 1$. For otherwise, $\mathbf{O}(H) \cong \mathbf{O}(H)/(\mathbf{O}(H) \cap N) \cong G/N$ would be nilpotent and H would be the direct product of $\mathbf{O}(H)$ with a Sylow 2-subgroup. Thus if $(|H|,|K|) = 1$, then we could apply Lemma 2.4, and if $(|H|,|K|)\neq 1$, then we could apply Lemma 2.3, a contradiction.

Let N_1 be a simple direct factor of the minimal normal subgroup N. Since $N_1\cap H$ contains a Sylow 2-subgroup S of N_1, and since it follows, according to Lemma 2.2 (a), that $H\cap N=\prod_{i=1}^{m}(N_i\cap H)$, we have that the 2-nilpotency of $H\cap N_1$ implies $H\cap N_1=\mathbf{O}(H\cap N_1)S$ with $\mathbf{O}(H\cap N_1)\neq 1$.

Since $(|S|,|\mathbf{O}(H\cap N_1)|)=1$, and since S acts on the solvable subgroup $\mathbf{O}(H\cap N_1)$, we conclude there is a Sylow p-subgroup P of $H\cap N_1=\mathbf{O}(H\cap N_1)S$, with $S\leq \mathbf{N}_{N_1}(P)$ for every odd prime divisor p of $|H\cap N_1|$, by [10, Proposition 5.20, p.113].

We suppose the classification of finite simple groups. Thus the hypothesis of Kazarin's result (Lemma 2.1) is satisfied by our minimal counterexample G, and hence our simple group N_1 is isomorphic to one of the groups in the list of Lemma 2.1.

(a) Suppose $N_1\cong \mathbf{PSL(2,q)}$, with $q=p^n$.

(i) Assume that $q=p^n>3$, with p odd.

We choose $\epsilon=\pm1$, so that $q\equiv\epsilon \pmod 4$. Hence $\frac{(q+\epsilon)}{2}$ is odd. The fact that the odd number $q\frac{(q+\epsilon)}{2}$ divides $n|H\cap N_1|\ |K\cap N_1|$ follows from Lemma 2.2 (a), since $|\mathbf{Out}(N_1)|=2n$.

We claim that $q\frac{(q+\epsilon)}{2}$ is prime to $|H\cap N_1|$. For otherwise there would be a prime r that divides both $q\frac{(q+\epsilon)}{2}$ and $|H\cap N_1|$. Then there would exist a Sylow r-subgroup R $(\neq 1)$ of $\mathbf{O}(H\cap N_1)$ normalized by a Sylow 2-subgroup S of N_1. If $r=p$, then $\mathbf{N}_{N_1}(R)$ would be contained in the normalizer of a Sylow p-subgroup that is of order $q\frac{(q-1)}{2}$ and does not contain a Sylow 2-subgroup of N_1. Therefore r divides $\frac{(q+\epsilon)}{2}$. However, in this case, $\mathbf{N}_{N_1}(R)$ is contained in the normalizer of a cyclic group of order $\frac{(q+\epsilon)}{2}$, and whose order is $q+\epsilon$ and does not contain a Sylow 2-subgroup.

Hence both q and $\frac{(q+\epsilon)}{2}$ divide $n|K\cap N_1|$.

If q is prime to $|K\cap N_1|$, then q divides n. In particular,

$$n\geq q=p^n>(1+1)^n\geq 1+n,$$

which is impossible. Hence p divides $|K\cap N_1|$.

If $\frac{(q+\epsilon)}{2}$ is prime to $|K\cap N_1|$, then similarly $\frac{(q+\epsilon)}{2}\leq n$. But this implies that

$$2n+1\geq 2n-\epsilon\geq q=p^n\geq (1+2)^n\geq 1+2n.$$

Hence the equality holds, and we have that $q = p = 3$ and that $n = 1 = -\epsilon$. But this contradicts $q > 3$. Thus there is a prime r dividing both $\frac{(q+\epsilon)}{2}$ and $|K \cap N_1|$.

Since q is prime to $\frac{(q+\epsilon)}{2}$, from the nilpotency of $K \cap N_1$ and the remarks above it follows that $K \cap N_1$ contains an element of order pr. But this is impossible, since the centralizer of an element of order p in $N_1 \cong \mathbf{PSL}(2, q)$ is of order q.

(ii) Assume that $q = 2^n$.

By the remark above, if r is a prime divisor of $\mathbf{O}(H \cap N_1)$, then there is a Sylow r-subgroup R of $H \cap N_1 = \mathbf{O}(H \cap N_1)S$ with $S \leq \mathbf{N}_{N_1}(R)$. But this contradicts the fact that $|\mathbf{N}_{N_1}(R)| = 2(q + \epsilon)$ (where $\epsilon = \pm 1$) which implies that a Sylow 2-subgroup of N_1 does not normalize the r-subgroup R.

(b) Suppose N_1 is isomorphic to some group of the following list: $\{\mathbf{M}_{11}$, $\mathbf{PSL}(3, q)$ with $q < 9$ (here $q \neq 2$ since $\mathbf{PSL}(3, 2) \cong \mathbf{PSL}(2, 7)$), $\mathbf{PSp}(4, 3)$, $\mathbf{PSL}(4, 2)$ or $\mathbf{PSU}(3, 8)\}$.

Let p and q be the largest prime divisor of $|N_1|$, and the second largest prime divisor of $|N_1|$, respectively. In each possible case for N_1, we may check the following (consulting for example ATLAS):

- Both p and q are odd primes.
- A Sylow p-subgroup P of N_1 is of order p and $\mathbf{C}_{N_1}(P) = P$.
- $|\mathbf{Out}(N_1)|$ is prime to both p and q.

By Lemma 2.2 (b), we have that $p|Q|$ divides $|\mathbf{O}(H \cap N_1)|\,|K \cap N_1|$, where Q is a Sylow q-subgroup of N_1. If p divides $|\mathbf{O}(H \cap N_1)|$, then a Sylow 2-subgroup S normalizes P, a Sylow p-subgroup of $\mathbf{O}(H \cap N_1)$ [by the fundamental property of coprime action and the fact $H \cap N_1 = \mathbf{O}(H \cap N_1)S$]. However, this implies that S is isomorphic to a subgroup of $\mathbf{N}_{N_1}(P)/P \leq \mathbf{Aut}(P)$, and hence S is cyclic because P is a cyclic group of odd order, which contradicts the simplicity of N_1. Therefore p divides $|K \cap N_1|$.

If q divides $|K \cap N_1|$, then the nilpotency of $K \cap N_1$ implies that $\mathbf{C}_{N_1}(P) = P$ contains an element of order q, which is a contradiction. Thus the order of a Sylow q -subgroup of N_1 divides $|\mathbf{O}(H \cap N_1)|$, and hence there is a Sylow q-subgroup of N_1 (contained in $\mathbf{O}(H \cap N_1)$) normalized by a Sylow 2-subgroup S of N_1. However, (consulting for example ATLAS) in each possible case for N_1, it is easy to verify that the normalizer of a Sylow

q-subgroup of N_1 does not contain a Sylow 2 -subgroup of N_1.

Acknowledgment The author wishes to thank the referee for his suggestions, which contributed in part to reorganize and improve considerably this article.

References

[1] Carocca A. and Matos H., *Some solvability criteria for finite Groups*. Hokkaido Math. Journal **26** (1997), 157–161.

[2] Finkel D., *On the solvability of certain factorizable groups*. J. Algebra **47** (1977), 223–230.

[3] Finkel D. and Ward M., *Products of supersolvable and nilpotent finite groups*. Arch. Math. **36** (1981), 385–393.

[4] Fisman E., *Product of solvable subgroups of a finite group*. Arch. Math. **61** (1993), 201–205.

[5] Gorenstein D., *Finite Groups*. Harper and Row, New York, 1968.

[6] Huppert B., *Endliche Gruppen* I. Springer-Verlag, Berlin / New York, 1967.

[7] Kazarin L., *Product of two solvable subgroups*. Comm. Algebra **14** (1986), 1001–1066.

[8] Kazarin L., *On the product of 2-decomposable and nilpotent groups*. Collect. Sci. Works (Sverdlovsk) (1988), 74–81.

[9] Matos H., *Thesis Subgrupos Syl-permutáveis em grupos finitos*. Universidade Federal do Rio de Janeiro, Brazil, 1996.

[10] Suzuki M., *Group Theory* II. Springer-Verlag, New York, 1986.

Departamento de Matematicas
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Casilla 306, Santiago 22, Chile
E-mail: acarocca@mat.puc.cl