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On the Milnor fiber of a real map-gem

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Abstract. We give an algebraic formula for a topological invariant of real analytic singularities. We deduce from this formula a new proof of the topological invariance of the Milnor number mod 2.

Key words: real Milnor fiber, local algebra, Euler characteristic.

1. Introduction

Let $f=(f_1,\ldots,f_k):(\mathbf{K}^n,0)\to(\mathbf{K}^k,0)$, with $1\leq k<n$ and $\mathbf{K}=\mathbf{C}$ or $\mathbf{K}=\mathbf{R}$, be an analytic germ defined in a neighborhood of the origin. We are interested in computing topological invariants associated to the mapping f.

Let $B_\varepsilon\subset\mathbf{K}^n$ be a small closed ball centered at the origin and let $\delta\in\mathbf{K}^k$ be a small regular value of f. The Milnor fiber of f is $f^{-1}(\delta)\cap B_\varepsilon$. If $k=1$, $\mathbf{K}=\mathbf{C}$ and f has an isolated critical point at 0, Milnor [Mi2] proved that $f^{-1}(\delta)\cap B_\varepsilon$ has the homotopy type of a bouquet of μ spheres of dimension $n-1$. This number of spheres is called the Milnor number of f, and according to Milnor [Mi2] and Palamodov [Pa],

$$\mu=\dim_{\mathbf{C}}\frac{\mathcal{O}_{\mathbf{C}^n,0}}{\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)},\tag{1}$$

where $\mathcal{O}_{\mathbf{C}^n,0}$ is the ring of germs of analytic functions defined at the origin.

This result was extended to the case $1<k<n$ by Hamm, who proved that the Milnor fiber has the homotopy type of a bouquet of μ spheres of dimension $n-k$, and by Lê [Le] and Greuel [Gr] who obtained the following formula

$$\mu(f')+\mu(f)=\dim_{\mathbf{C}}\mathcal{O}_{\mathbf{C}^n,0}/I,\tag{2}$$

where $f'=(f_1,\ldots,f_{k-1})$ and I is the ideal generated by $f_1,\ldots,f_{k-1}$ and all $k\times k$ minors $\frac{\partial(f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_k})}$.

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In the real case, it is difficult to give such precise information about the topology of the Milnor fiber. Nevertheless, it is possible to compute some Euler characteristics. For example, if $k = 1$ and f has an isolated critical point at the origin, the Khimshiasvili's formula ([Ar], [Fu2], [Kh], [Wa]) states that

$$\chi(f^{-1}(\delta) \cap B_\varepsilon) = 1 - \operatorname{sign}(-\delta)^n \deg_0 \nabla f,$$

where $\deg_0 \nabla f$ is the topological degree of the gradient of f at the origin. This formula can be viewed as a real version of the formula (1) above. The aim of this paper is to give a real version of the Lê-Greuel formula, i.e. the formula (2) above.

We first introduce the situation. Let $f = (f_1, \ldots, f_k) : (\mathbf{R}^n, 0) \to (\mathbf{R}^k, 0)$, with $n > k$, be an analytic map and let $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be an analytic function. Let I be the ideal generated by $f_1, \ldots, f_k$ and all $(k+1) \times (k+1)$ minors $\frac{\partial(g, f_1, \ldots, f_k)}{\partial(x_{i_1}, \ldots, x_{i_{k+1}})}$ in $\mathcal{O}_{\mathbf{R}^n,0}$, the ring of germs of analytic functions at the origin. Let $\delta \in \mathbf{R}^k$ be a regular value of f and let $\alpha \in \mathbf{R}$ such that $|\alpha| \ll |\delta|$ and (δ, α) is a regular value of (f, g). Assuming that $\dim_{\mathbf{R}} \mathcal{O}_{\mathbf{R}^n,0}/I < +\infty$, we will prove the following result (see Theorem 3.8)

$$\begin{aligned} &\chi\left(f^{-1}(\delta) \cap \{g \geq \alpha\} \cap B_\varepsilon\right) + \chi\left(f^{-1}(\delta) \cap \{g \leq \alpha\} \cap B_\varepsilon\right) \\ &\quad \equiv \dim_{\mathbf{R}} \frac{\mathcal{O}_{\mathbf{R}^n,0}}{I} \mod 2. \end{aligned}$$

This theorem generalizes the case $g = x_1^2 + \cdots + x_n^2$ which was already proved by Dudzinski *et al.* in [DLNS], using fixed point theory and the Lê-Greuel formula. It is also a mod 2 generalization of the formulas for counting the number of branches of a one-dimensional semi-analytic set given by Aoki *et al.* ([AFN1], [AFN2], [AFS]) and by Szafraniec ([Sz1]).

Now let us consider the complexification $f_{\mathbf{C}} : (\mathbf{C}^n, 0) \to (\mathbf{C}^k, 0)$ of f. Let $\mu(f)$ be the Milnor number of $f_{\mathbf{C}}$. Let $L(f)$ be the link of f and let $\psi(f)$ be the semi-characteristic of $L(f)$. We recall that the semi-characteristic is defined to be half the sum of the mod 2 Betti numbers. C.T.C Wall ([Wa]) showed that

$$\psi(f) \equiv 1 + \mu(f) \mod 2.$$

As a corollary, one gets that $\mu(f) \bmod 2$ is a topological invariant of f. Wall's proof is straightforward for the case $k = 1$. The case of arbitrary k is more complicated; Wall gives a sophisticated topological argument using spectral sequence from fixed point theory. At the end of his paper, he asks if there is a proof of this result more like the case for $k = 1$. Using Theorem 3.8, we will supply a proof of this type (see Theorem 4.5).

The paper is organized as follows: in Section 2, we recall some facts about Morse theory for manifolds with boundary; Section 3 is devoted to the proof of our main formula; in Section 4, we give a new proof of the topological invariance of the Milnor number mod 2. The author is very grateful to Karim Bekka for his helpful remarks and comments.

2. Morse theory for manifolds with boundary

We recall the results of Morse theory for manifolds with boundary. Our reference is [HL] where the results are given for a C^∞ manifold M with boundary ∂M. For simplicity we will present the results for manifolds with boundary of type $M \cap \{g * 0\}$, $* \in \{\geq, \leq\}$, where M is a C^∞ manifold and $g : M \to \mathbf{R}$ a C^∞ function such that $M \cap g^{-1}(0)$ is smooth. In fact this is the case we need in the following sections.

Let M be a C^∞ manifold of dimension n. Let $g : M \to \mathbf{R}$ be a C^∞ function such that $\nabla g(x) \neq 0$ for all $x \in g^{-1}(0)$. This implies that $M \cap g^{-1}(0)$ is a smooth manifold of dimension $n-1$ and that $M \cap \{g \geq 0\}$ and $M \cap \{g \leq 0\}$ are smooth manifolds with boundary. Let $f : M \to \mathbf{R}$ be a smooth function. A critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) is a critical point of $f_{|M\cap\{g>0\}}$ (resp. $f_{|M\cap\{g<0\}}$) or a critical point of $f_{|M\cap g^{-1}(0)}$.

Definition 2.1 Let $q \in M \cap g^{-1}(0)$. We say that q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) if q is a critical point of $f_{|M\cap g^{-1}(0)}$ and q is not a critical point of $f_{|M}$.

We say that q is a correct non-degenerate critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) if q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) and q is a non-degenerate critical point of $f_{|M\cap g^{-1}(0)}$.

If q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) then $\nabla f(q) \neq \vec{0}$, $\nabla f(q)$ and $\nabla g(q)$ are colinear and there is $\tau(q) \in \mathbf{R}^*$ with $\nabla f(q) = \tau(q) \cdot \nabla g(q)$.

Definition 2.2 If q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ then

- $\nabla f(q)$ points inwards if and only if $\tau(q) > 0$,
- $\nabla f(q)$ points outwards if and only if $\tau(q) < 0$.

If q is a correct critical point of $f_{|M\cap\{g\leq 0\}}$ then

- $\nabla f(q)$ points inwards if and only if $\tau(q)<0$,
- $\nabla f(q)$ points outwards if and only if $\tau(q)>0$.

Definition 2.3 A C^∞ function $f: M\cap\{g\geq 0\}\to \mathbf{R}$ (resp. $M\cap\{g\leq 0\}\to\mathbf{R}$) is a correct function if all critical points of $f_{|M\cap g^{-1}(0)}$ are correct. A C^∞ function $f: M\cap\{g\geq 0\}\to\mathbf{R}$ (resp. $M\cap\{g\leq 0\}\to\mathbf{R}$) is a Morse correct function if $f_{|M\cap\{g>0\}}$ (resp. $f_{|M\cap\{g<0\}}$) admits only non-degenerate critical points and if f admits only non-degenerate correct critical points.

Proposition 2.4 *For all C^∞ manifold M and for all function $g: M\to\mathbf{R}$ such that $\nabla g(x)\neq 0$ for all $x\in g^{-1}(0)$, the set of C^∞ functions $f: M\to \mathbf{R}$ such that $f_{|M\cap\{g\geq 0\}}$ and $f_{|M\cap\{g\leq 0\}}$ are Morse correct functions is dense in $C^\infty(M,\mathbf{R})$.*

We will denote $\chi\big(M\cap\{g*0\}\cap\{f?0\}\big)$, where $*,?\in\{\leq,=,\geq\}$, by $\chi_{*,?}$ and we will use the following result:

Theorem 2.5 *Let M be a C^∞ compact manifold of dimension n and let $g: M\to\mathbf{R}$ be a C^∞ function such that $\nabla g(x)\neq 0$ for all $x\in g^{-1}(0)$. Let $f: M\to\mathbf{R}$ be a C^∞ function such that $f_{|M\cap\{g\geq 0\}}$ and $f_{|M\cap\{g\leq 0\}}$ are Morse correct. Let $\{p_i\}$ be the set of critical points of $f_{|M}$ and $\{\lambda_i\}$ be the set of their respective indices. Let $\{q_j\}$ be the set of critical points of $f_{|M\cap g^{-1}(0)}$ and $\{\mu_j\}$ be the set of their respective indices. Then we have*

$$\chi_{\geq,\geq}-\chi_{\geq,=}=\sum_{\substack{i/f(p_i)>0\\ g(p_i)>0}}(-1)^{\lambda_i}+\sum_{\substack{j/f(q_j)>0\\ \tau(q_j)>0}}(-1)^{\mu_j},$$

$$\chi_{\geq,\leq}-\chi_{\geq,=}=(-1)^n\sum_{\substack{i/f(p_i)<0\\ g(p_i)>0}}(-1)^{\lambda_i}+(-1)^{n-1}\sum_{\substack{j/f(q_j)<0\\ \tau(q_j)<0}}(-1)^{\mu_j},$$

and

$$\chi_{\leq,\geq}-\chi_{\leq,=}=\sum_{\substack{i/f(p_i)>0\\ g(p_i)<0}}(-1)^{\lambda_i}+\sum_{\substack{j/f(q_j)>0\\ \tau(q_j)<0}}(-1)^{\mu_j},$$

$$\chi_{\leq,\leq}-\chi_{\leq,=}=(-1)^n\sum_{\substack{i/f(p_i)<0\\ g(p_i)<0}}(-1)^{\lambda_i}+(-1)^{n-1}\sum_{\substack{j/f(q_j)<0\\ \tau(q_j)>0}}(-1)^{\mu_j}.$$

3. The main formula

In this section, we prove the result announced in the introduction for a real map germ. We recall that $f = (f_1, \ldots, f_k) : (\mathbf{R}^n, 0) \to (\mathbf{R}^k, 0)$ is an analytic germ, that $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is an analytic function germ, that I is the ideal generated by $f_1, \ldots, f_k$ and all $(k+1) \times (k+1)$ minors $\frac{\partial(g, f_1, \ldots, f_k)}{\partial(x_{i_1}, \ldots, x_{i_{k+1}})}$ in $\mathcal{O}_{\mathbf{R}^n,0}$ and that $\dim_{\mathbf{R}} \mathcal{O}_{\mathbf{R}^n,0}/I < +\infty$. Our proof differs from [DLNS] because we shall not use neither fixed point theory nor the Lê-Greuel formula.

3.1. Characterization of a non-degenerate critical point

We study the following local situation. Let

$$\begin{array}{rcl} f : (\mathbf{K}^n, p) & \to & (\mathbf{K}^k, \delta) \\ x & \mapsto & (f_1(x), \ldots, f_k(x)) \end{array}$$

be an analytic germ defined near p ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$); δ is a regular value of f so that $f^{-1}(\delta)$ is an analytic manifold of dimension $n-k$ and $p \in f^{-1}(\delta)$. Let $g : \mathbf{K}^n \to \mathbf{K}$ be defined near p. We will find a condition for the point p to be a non-degenerate critical point of $g_{|f^{-1}(\delta)}$.

Let

$$M(x) = \det \left[\frac{\partial f_i}{\partial x_j}(x) \right]_{1 \leq i, j \leq k}.$$

For shortness we write $x = (x', x'') = (x_1, \ldots, x_k; x_{k+1}, \ldots, x_n)$. We can assume that $M(p) \neq 0$ and apply the implicit function theorem in the neighborhood of p. There exists an analytic mapping

$$\varphi : \mathbf{K}^{n-k} \to \mathbf{K}^k, \ x'' \mapsto \varphi(x'')$$

such that $\varphi(p'') = p'$ and $f(\varphi(x''), x'') = f(p) = \delta$. We write

$$\begin{aligned} G(x'') &= g(\varphi(x''), x''), \\ m_j(x) &= \frac{\partial(g, f_1, \ldots, f_k)}{\partial(x_1, \ldots, x_k, x_j)} \quad \text{for } j \geq k+1. \end{aligned}$$

Let $\mathcal{O}_{\mathbf{K}^n,p}$ be the ring of germs of analytic functions defined near p. Let I_p be the ideal in $\mathcal{O}_{\mathbf{K}^n,p}$ generated by all $(k+1) \times (k+1)$ minors $\frac{\partial(g, f_1, \ldots, f_k)}{\partial(x_{i_1}, \ldots, x_{i_{k+1}})}$. Let $C_p = V(I_p)$ be the set of the zeros of I_p. Let J_p be the ideal generated by $m_{k+1}, \ldots, m_n$. We have the following lemmas:

Lemma 3.1 *The function $g_{|f^{-1}(\delta)}$ has a critical point at p if and only if* $\frac{\partial G}{\partial x_{k+1}} = \cdots = \frac{\partial G}{\partial x_n} = 0$.

Proof. It is clear. □

Lemma 3.2 *The function $g_{|f^{-1}(\delta)}$ has a critical point at p if and only if $p \in f^{-1}(\delta) \cap C_p$.*

Proof. It is clear. □

Lemma 3.3 $J_p = I_p$

Proof. It is proved in [Sz2, p.349]. □

Remark Szafraniec's proof uses the Lê-Greuel formula. We give a direct proof in the appendix at the end of the paper.

Lemma 3.4 *The function $g_{|f^{-1}(\delta)}$ has a non-degenerate critical point if and only if C_p is a regular complete intersection at p of dimension k and C_p intersects $f^{-1}(\delta)$ transversally at p.*

Proof. Let $A(\theta)$ be the following matrix

$$A(\theta) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\theta) & \cdots & \frac{\partial f_1}{\partial x_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(\theta) & \cdots & \frac{\partial f_k}{\partial x_n}(\theta) \\ \frac{\partial m_{k+1}}{\partial x_1}(\theta) & \cdots & \frac{\partial m_{k+1}}{\partial x_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial m_n}{\partial x_1}(\theta) & \cdots & \frac{\partial m_n}{\partial x_n}(\theta) \end{bmatrix}.$$

As Szafraniec does in [Sz2, p.349–350], we obtain

$$\det A(p) = (-1)^{k(n-k)} M(p)^{n-k+1} \det \left[\frac{\partial^2 G}{\partial x_i \partial x_j}(p'') \right]_{k+1 \leq i, j \leq n}.$$

This result allows us to conclude because $g_{|f^{-1}(\delta)}$ has a non-degenerate critical point if and only if

$$\det \left[\frac{\partial^2 G}{\partial x_i \partial x_j} \right]_{k+1 \leq i,\, j \leq n} \neq 0.$$

□

3.2. A Morse approximation

We return to the real case and, using Morse theory for manifolds with boundary, we relate

$$\chi(f^{-1}(\delta) \cap \{g \geq \alpha\} \cap B_\varepsilon) + \chi(f^{-1}(\delta) \cap \{g \leq \alpha\} \cap B_\varepsilon) \mod 2$$

to the number of critical points of a Morse approximation of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$.

We need the following result about correct critical points of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$.

Lemma 3.5 *Let δ be a small regular value so that the manifold with boundary $f^{-1}(\delta) \cap B_\varepsilon$ is non-singular. Then*

- *At all correct critical points of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$ where $g > 0$, the gradient of $g_{|F_\delta}$ points outwards.*
- *At all correct critical points of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$ where $g < 0$, the gradient of $g_{|F_\delta}$ points inwards.*
- *There are no correct critical points of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$ with $g = 0$.*

Proof. We prove the first point. In order to prove the second one it is enough to replace g by $-g$. Let ω be the euclidian distance function and let

$$X = \Big\{ x \in \big(f^{-1}(0) \setminus \{0\}\big) \cap \{g(x) > 0\} \mid \exists \lambda(x) \text{ and } \mu(x)$$
$$\text{with } \nabla g(x) = \lambda(x) \nabla f(x) + \mu(x) \nabla \omega(x) \text{ and } \mu(x) < 0 \Big\}.$$

It is a subanalytic set. If $0 \in \overline{X}$, we apply the curve selection lemma (cf. [Mi2]). There exists an analytic arc $\gamma : [0, \varepsilon_0[\to \overline{X}$ such that $\gamma(0) = 0$. Then we have for all $t \in [0, \varepsilon_0[$

$$\frac{\partial (g \circ \gamma(t))}{\partial t} = \langle \nabla g(\gamma(t)), \gamma'(t) \rangle$$

and

$$\frac{\partial (g \circ \gamma(t))}{\partial t} = \lambda(\gamma(t)) \langle \nabla f(\gamma(t)), \gamma'(t) \rangle + \mu(\gamma(t)) \langle \nabla \omega(\gamma(t)), \gamma'(t) \rangle.$$

We have for all $t \in [0, \varepsilon_0[$, $\langle \nabla f(\gamma(t)), \gamma'(t) \rangle = 0$ for $\gamma([0, \varepsilon_0[) \subset f^{-1}(0)$ and since $\langle \nabla\omega(\gamma(t)), \gamma'(t)\rangle \geq 0$ for all $t \in [0, \varepsilon_0[$, we will have

$$\text{for all } \ t \in [0, \varepsilon_0[\quad \frac{\partial(g \circ \gamma(t))}{\partial t} \leq 0.$$

The function $g \circ \gamma$ is decreasing and so, for all $t \in [0, \varepsilon_0[$, $g \circ \gamma(t) \leq g \circ \gamma(0) = 0$. But for all $t \neq 0$, $g \circ \gamma(t) > 0$ so $0 \notin \overline{X}$. We can choose ε sufficiently small so that in $f^{-1}(0) \setminus \{0\} \cap B_\varepsilon$, the gradient of $g_{|f^{-1}(0)\setminus\{0\}}$ and the gradient of $\omega_{|f^{-1}(0)\setminus\{0\}}$ do not point in opposite directions in $\{g > 0\}$. Choosing δ sufficiently close to 0, this implies that at all correct critical points of $g_{|F_\delta \cap \{g>0\}}$, the gradient of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$ will point outwards.

We prove the third point with the same ideas considering the sets

$$Y_> = \Big\{x \in \big(f^{-1}(0) \setminus \{0\}\big) \cap \{g(x) = 0\} \mid \exists \lambda(x) \text{ and } \mu(x)$$
$$\text{with } \nabla g(x) = \lambda(x)\nabla f(x) + \mu(x)\nabla\omega(x) \text{ and } \mu(x) > 0\Big\}$$

and

$$Y_< = \Big\{x \in \big(f^{-1}(0) \setminus \{0\}\big) \cap \{g(x) = 0\} \mid \exists \lambda(x) \text{ and } \mu(x)$$
$$\text{with } \nabla g(x) = \lambda(x)\nabla f(x) + \mu(x)\nabla\omega(x) \text{ and } \mu(x) < 0\Big\}$$

and proving that $0 \notin \overline{Y_>}$ and that $0 \notin \overline{Y_<}$. □

Now let $\tilde{g}$ be a perturbation of g such that $\tilde{g}_{|f^{-1}(\delta)\cap B_\varepsilon}$ is a Morse correct function then we have

Lemma 3.6 *Let δ be a small regular value so that the manifold with boundary $f^{-1}(\delta) \cap B_\varepsilon$ is non-singular. Let $\alpha \in \mathbf{R}$ such that $|\alpha| \ll |\delta|$ and (δ, α) is a regular value of (f, g) then*

$$\chi(f^{-1}(\delta) \cap \{g \geq \alpha\} \cap B_\varepsilon) + \chi(f^{-1}(\delta) \cap \{g \leq \alpha\} \cap B_\varepsilon)$$

is equal to the number of non-degenerate critical points of $\tilde{g}_{|f^{-1}(\delta)\cap B_\varepsilon}$ modulo 2.

Proof. For convenience, we denote $f^{-1}(\delta) \cap B_\varepsilon$ by F_δ and $f^{-1}(\delta) \cap \{g * \alpha\}$ by $F_\delta(g * \alpha)$ where $* \in \{=, \geq, \leq\}$.

Since $\dim_{\mathbf{R}} \mathcal{O}_{\mathbf{R}^n,0}/I < +\infty$, $g_{|f^{-1}(0)\setminus\{0\}\cap\{\omega\leq\varepsilon\}}$ admits no critical points. This implies that for ε sufficiently small, the levels of g intersect $f^{-1}(0)$ transversally on $f^{-1}(0) \cap \{\varepsilon/4 \leq \omega \leq \varepsilon\}$. If we choose δ such that $|\delta| \ll \varepsilon$, the levels of g will also intersect $f^{-1}(\delta)$ transversally on $f^{-1}(\delta) \cap \{\varepsilon/4 \leq \omega \leq$

$\varepsilon\}$, because transversality is an open property. Thus $g_{|f^{-1}(\delta)\cap\{\omega<\varepsilon\}}$ admits its critical points on $f^{-1}(\delta)\cap B_{\varepsilon/4}$ and the critical points of $g_{|f^{-1}(\delta)\cap\{\omega=\varepsilon\}}$ are correct critical points of $g_{|F_\delta}$, because on $\{\omega=\varepsilon\}$ the levels of g and $f^{-1}(\delta)$ are transversal. Moreover correct critical points of $g_{|F_\delta}$ where $g>0$ (resp. $g<0$) point outwards (resp. inwards) and there are no correct critical points near $\{g=0\}$ by the previous lemma.

We choose α close to 0 such that 0 is the only possible critical value of $g_{|F_\delta}$ in $[-|\alpha|,|\alpha|]$ and such that all correct critical points lie far from the level $\{g=\alpha\}$. We apply the result of Morse theory to the manifold with boundary $F_\delta(g=\alpha)$ (see Theorem 2.5) and we get

$$\chi(F_\delta(g\geq\alpha),F_\delta(g=\alpha))=n_+(\tilde{g}_\alpha)-n_-(\tilde{g}_\alpha),$$

where $n_+(\tilde{g}_\alpha)$ (resp. $n_-(\tilde{g}_\alpha)$) is the number of non-degenerate critical points with even (resp. odd) index of $\tilde{g}_{|F_\delta}$ lying in $F_\delta(g\geq\alpha)$. In the same way, we have

$$\chi(F_\delta(g\leq\alpha),F_\delta(g=\alpha))=(-1)^{n-k}(n_+(\tilde{g}_{-\alpha})-n_-(\tilde{g}_{-\alpha})),$$

where $n_+(\tilde{g}_{-\alpha})$ (resp. $n_-(\tilde{g}_{-\alpha})$) is the number of non-degenerate critical points with even (resp. odd) index of $\tilde{g}_{|F_\delta}$ lying in $F_\delta(g\leq\alpha)$. Finally we have

$$\begin{aligned}&\chi(F_\delta(g\geq\alpha))+\chi(F_\delta(g\leq\alpha))\\&\qquad=n_+(\tilde{g}_\alpha)+n_-(\tilde{g}_\alpha)+n_+(\tilde{g}_{-\alpha})+n_-(\tilde{g}_{-\alpha})\mod 2,\end{aligned}$$

hence

$$\begin{aligned}&\chi(F_\delta(g\geq\alpha))+\chi(F_\delta(g\leq\alpha))\\&\qquad=\text{number of non-degenerate critical points of }\ \tilde{g}_{|F_\delta}\mod 2.\end{aligned}$$

□

3.3. Study of $\mathcal{O}_{\mathbf{R}^n,0}/I$

We relate the dimension of $\mathcal{O}_{\mathbf{R}^n,0}/I$ to the number of non-degenerate critical points of a suitable Morse approximation of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$. Let $J_{\mathbf{C}}$ be the ideal generated in $\mathcal{O}_{\mathbf{C}^n,0}$ by all the $(k+1)\times(k+1)$ minors $\frac{\partial(g,f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_{k+1}})}$ and let $I_{\mathbf{C}}=(f_1,\ldots,f_k;J_{\mathbf{C}})$. Let $C=V(J_{\mathbf{C}})$. Saito has proved in [Sa] that $\frac{\mathcal{O}_{\mathbf{C}^n,0}}{J_{\mathbf{C}}}$ is a Cohen-Macaulay ring of dimension k and so C is equidimensional of dimension k. A result about multiplicity from Serre (see [Se]) gives the following relation

$$\dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,0}}{I_{\mathbf{C}}} = (f_{\mathbf{C}}^{-1}(0); C)_0 = \gamma,$$

where $(f_{\mathbf{C}}^{-1}(0); C)_0$ is the intersection multiplicity of $f_{\mathbf{C}}^{-1}(0)$ and C at 0. Replacing g by a suitable perturbation if necessary, we can assume that C and $f_{\mathbf{C}}^{-1}(\delta)$ intersect transversally at regular points. We have

Lemma 3.7 *The function $g_{|f^{-1}(\delta)\cap B_\varepsilon}$ is a Morse function and $\dim_{\mathbf{R}} \mathcal{O}_{\mathbf{R}^n,0}/I$ is equal to the number of non-degenerate critical points of $g_{|f^{-1}(\delta)\cap B_\varepsilon}$ modulo 2.*

Proof. Write

$$C \cap f_{\mathbf{C}}^{-1}(\delta) = \{p_1, \ldots, p_r\} \cup \{p_{r+1}, \overline{p_{r+1}}, \ldots, p_m, \overline{p_m}\},$$

where $p_1, \ldots, p_r$ are real points. Hence

$$\gamma = \sharp C \cap f_{\mathbf{C}}^{-1}(\delta) = \sharp\{p_1, \ldots, p_r\} \mod 2.$$

One can choose δ sufficiently small so that $\{p_1, \ldots, p_r\} \subset B_\varepsilon$. Let us examine the situation at p_i for $1 \leq i \leq r$. Since $f^{-1}(\delta) \cap B_\varepsilon$ is regular, we can assume that, for instance, $M(p_i) \neq 0$ with $M(x) = \det\left[\frac{\partial f_i}{\partial x_j}(x)\right]_{1\leq i,j\leq k}$. Using Lemma 3.3 and its proof, we have

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\theta) & \cdots & \frac{\partial f_1}{\partial x_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(\theta) & \cdots & \frac{\partial f_k}{\partial x_n}(\theta) \\ \frac{\partial m_{k+1}}{\partial x_1}(\theta) & \cdots & \frac{\partial m_{k+1}}{\partial x_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial m_n}{\partial x_1}(\theta) & \cdots & \frac{\partial m_n}{\partial x_n}(\theta) \end{bmatrix} \neq 0,$$

because C and $f_{\mathbf{C}}^{-1}(\delta)$ intersect transversally. From this we deduce that $C \cap \mathbf{R}^n$, which is the set of real points of C, and $f^{-1}(\delta) \cap B_\varepsilon$ intersect transversally because all the coefficients involved in the above determinant are real and p_i is a real point. Finally we get that $g_{|F_\delta}$ is a Morse function and $p_1, \ldots, p_r$ are its non-degenerate critical points hence

$$\dim_{\mathbf{R}} \frac{\mathcal{O}_{\mathbf{R}^n,0}}{I} \equiv r \mod 2,$$

since $\dim_{\mathbf{R}} \frac{\mathcal{O}_{\mathbf{R}^n,0}}{I} = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,0}}{I_{\mathbf{C}}}$. □

3.4. Main result and some corollaries

Now we are ready to state our main theorem.

Theorem 3.8 *Let $\delta \in \mathbf{R}^k$ be regular value of f and let $\alpha \in \mathbf{R}$ such that $|\alpha| \ll |\delta|$ and (δ, α) is a regular value of (f, g). If $\dim_{\mathbf{R}} \mathcal{O}_{\mathbf{R}^n,0}/I < +\infty$ then*

$$\begin{aligned}\chi\left(f^{-1}(\delta) \cap \{g \geq \alpha\} \cap B_\varepsilon\right) + \chi\left(f^{-1}(\delta) \cap \{g \leq \alpha\} \cap B_\varepsilon\right) \\ \equiv \dim_{\mathbf{R}} \frac{\mathcal{O}_{\mathbf{R}^n,0}}{I} \mod 2.\end{aligned}$$

Proof. It is a combination of Lemma 3.6 and Lemma 3.7. □

Corollary 3.9 *Let $\delta \in \mathbf{R}^k$ be regular value of f and let $\alpha \in \mathbf{R}$ such that $|\alpha| \ll |\delta|$ and (δ, α) is a regular value of (f, g). If $\dim_{\mathbf{R}} \mathcal{O}_{\mathbf{R}^n,0}/I < +\infty$ then:*

$$\begin{aligned}\chi\left(f^{-1}(\delta) \cap B_\varepsilon\right) + \chi\left(f^{-1}(\delta) \cap \{g = \alpha\} \cap B_\varepsilon\right) \\ \equiv \dim_{\mathbf{R}} \frac{\mathcal{O}_{\mathbf{R}^n,0}}{I} \mod 2.\end{aligned}$$

Proof. Using the Mayer-Vietoris sequence, we find

$$\begin{aligned}\chi\left(f^{-1}(\delta) \cap B_\varepsilon\right) = \chi\left(f^{-1}(\delta) \cap \{g \geq \alpha\} \cap B_\varepsilon\right) \\ + \chi\left(f^{-1}(\delta) \cap \{g \leq \alpha\} \cap B_\varepsilon\right) \\ - \chi\left(f^{-1}(\delta) \cap \{g = \alpha\} \cap B_\varepsilon\right)\end{aligned}$$

and it is easy to conclude. □

Let L_f and $L_{(f,g)}$ be the respective links of f and (f, g). We define, following Wall's notation

$$\psi(f) = \frac{1}{2} \dim_{\mathbf{Z}_2} \left(H_*(L_f, \mathbf{Z}_2)\right),$$

$$\psi((f, g)) = \frac{1}{2} \dim_{\mathbf{Z}_2} \left(H_*(L_{(f,g)}, \mathbf{Z}_2)\right).$$

We have

Corollary 3.10

$$\psi(f) + \psi((f,g)) \equiv \dim_{\mathbf{R}} \frac{\mathcal{O}_{\mathbf{R}^n,0}}{I} \mod 2.$$

Proof. We have $L_f = f^{-1}(0) \cap S_\varepsilon$. For δ sufficiently small, L_f is diffeomorphic to $f^{-1}(\delta) \cap S_\varepsilon$. Now it is enough to apply the following relation

$$\frac{1}{2} \dim_{\mathbf{Z}_2} \big(H_*(f^{-1}(\delta) \cap S_\varepsilon, \mathbf{Z}_2)\big) \equiv \chi\big(f^{-1}(\delta) \cap S_\varepsilon\big) \mod 2,$$

which is well-known if $n - k$ is odd and which is proved, for instance, in [ASV] Proposition 3.4, if $n - k$ is even. □

4. Topological invariance of the Milnor number mod 2

We give here an alternative proof of the topological invariance of the Milnor number mod 2 proved in [Wa]. Our proof does not use the methods of fixed point theory.

Let $f = (f_1, \ldots, f_k) : (\mathbf{R}^n, 0) \to (\mathbf{R}^k, 0)$, with $n > k$, be an analytic germ of finite singularity type and let $f_{\mathbf{C}}$ be the complexification of f. Then according to Wall [Wa], $f_{\mathbf{C}}$ has an isolated singularity and we can define its Milnor number $\mu(f_{\mathbf{C}})$. We define the Milnor number of f by $\mu(f) = \mu(f_{\mathbf{C}})$ and we will prove that

$$\psi(f) = 1 + \mu(f) \mod 2,$$

where $\psi(f)$ is defined as in the previous section.

We need the following version of Sard's lemma.

Lemma 4.1 *Let $N \subset M \subset \mathbf{R}^N$ be analytic sets and let $N_{\mathbf{C}}$ and $M_{\mathbf{C}}$ be their respective complexifications. Assume that $M_{\mathbf{C}} \setminus N_{\mathbf{C}}$ is a smooth complex manifold of dimension K. Let $\pi : \mathbf{R}^N \to \mathbf{R}^P$, with $P \leq K$, be an analytic mapping and let $\pi_{\mathbf{C}}$ be its complexification. Then for almost all $\beta \in \mathbf{R}^P$, $\pi_{\mathbf{C}}^{-1}(\beta) \cap M_{\mathbf{C}} \setminus N_{\mathbf{C}}$ is a smooth manifold of dimension $K - P$.*

Proof. Let $\Sigma_{\mathbf{C}}$ be the critical set of $\pi_{\mathbf{C}|M_{\mathbf{C}} \setminus N_{\mathbf{C}}}$ and let Σ be the critical set of $\pi_{|M \setminus N}$. Then $\pi_{\mathbf{C}}(\Sigma_{\mathbf{C}})$ has at most dimension $P-1$ and $\pi(\Sigma) \subset \pi_{\mathbf{C}}(\Sigma_{\mathbf{C}}) \cap \mathbf{R}^P$ is a subanalytic set of dimension at most $P - 1$, so for $\beta \in \mathbf{R}^P \setminus \pi(\Sigma)$, $\beta \notin \pi_{\mathbf{C}}(\Sigma_{\mathbf{C}})$ which means that β is a regular value of $\pi_{\mathbf{C}} : M_{\mathbf{C}} \setminus N_{\mathbf{C}} \to \mathbf{C}^P$. □

Lemma 4.2 *Let $f = (f_1, \dots, f_k) : (\mathbf{R}^n, 0) \to (\mathbf{R}^k, 0)$, with $n > k$, be an analytic germ of finite singularity type. There exists an analytic germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ such that $F_{\mathbf{C}} = (f_{\mathbf{C}}, g_{\mathbf{C}}) : (\mathbf{R}^n, 0) \to (\mathbf{R}^{k+1}, 0)$ has an isolated singularity or, equivalently, (f, g) is of finite singularity type.*

Proof. Consider the following analytic map

$$\begin{array}{rcl} H : (\mathbf{R}^n \times \mathbf{R}^n, (0,0)) & \to & (\mathbf{R}^{k+1}, 0) \\ (x, a) & \mapsto & (f_1, \dots, f_k, a_1 x_1 + \dots + a_n x_n). \end{array}$$

The derivative of H at (x, a) is given by the following matrix

$$DH(x,a) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_k}{\partial x_1} & \cdots & \dfrac{\partial f_k}{\partial x_n} & 0 & \cdots & 0 \\ a_1 & \cdots & a_n & x_1 & \cdots & x_n \end{pmatrix}.$$

If $(x, a) \in H^{-1}(0)$ and $x \neq 0$ then there exists $i \in \{1, \dots, n\}$ with $x_i \neq 0$ and then rank $DH(x, a) = k + 1$ since

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \dfrac{\partial f_k}{\partial x_1} & \cdots & \dfrac{\partial f_k}{\partial x_n} & 0 \\ a_1 & \cdots & a_n & x_i \end{pmatrix} = k + 1.$$

So $X = H^{-1}(0) \setminus (\{0\} \times \mathbf{R}^n)$ is an analytic manifold of dimension $2n - (k + 1)$. Consider

$$\begin{array}{rcl} \pi : H^{-1}(0) & \to & (\mathbf{R}^n, 0) \\ (x, a) & \mapsto & a \end{array}$$

the projection map on the second component. Using the above lemma, we can find $a \in \mathbf{R}^n$ such that $X_{\mathbf{C}} \cap \pi_{\mathbf{C}}^{-1}(a)$ is a smooth analytic manifold of dimension $n - (k + 1)$, where $X_{\mathbf{C}} = H_{\mathbf{C}}^{-1}(0) \setminus (\{0\} \times \mathbf{C}^n)$ and $\pi_{\mathbf{C}}$ is the complexification of π. This exactly means that $(f_{\mathbf{C}}, a_1 x_1 + \dots + a_n x_n) : (\mathbf{C}^n, 0) \to (\mathbf{C}^{k+1}, 0)$ has an isolated singularity. $\square$

Lemma 4.3 *Let $f=(f_1,\dots,f_k):(\mathbf{R}^n,0)\to(\mathbf{R}^k,0)$, with $n>k$, be an analytic germ of finite singularity type and let $g:(\mathbf{R}^n,0)\to(\mathbf{R},0)$ be an analytic germ such that $F_{\mathbf{C}}=(f_{\mathbf{C}},g_{\mathbf{C}}):(\mathbf{R}^n,0)\to(\mathbf{R}^{k+1},0)$ has an isolated singularity then*

$$\dim_{\mathbf{R}} \frac{\mathcal{O}_{\mathbf{R}^n,0}}{\left(f_1,\dots,f_k,\frac{\partial(g,f_1,\dots,f_k)}{\partial(x_{i_1},\dots,x_{i_{k+1}})}\right)}<+\infty.$$

Proof. Suppose that this vector space is not finite dimensional. Then $0\in\mathbf{C}^n$ is not isolated in

$$f_{\mathbf{C}}^{-1}(0)\cap\left(\cap_{1\le i_1<\dots<i_{k+1}\le n}\left\{\frac{\partial(g_{\mathbf{C}},f_{1\mathbf{C}},\dots,f_{k\mathbf{C}})}{\partial(x_{i_1},\dots,x_{i_{k+1}})}=0\right\}\right).$$

Hence, by the curve selection lemma ([Mi2]), there exists a real analytic curve $P(t):[0,\lambda)\to\mathbf{C}^n$ such that $P(0)=0$, $f_{\mathbf{C}}(P(t))=0$ and $\frac{\partial(g_{\mathbf{C}},f_{1\mathbf{C}},\dots,f_{k\mathbf{C}})}{\partial(x_{i_1},\dots,x_{i_{k+1}})}(P(t))=0$ for all $(k+1)$-tuple $(i_1,\dots,i_{k+1})$. This means that the vectors $\nabla f_{j\mathbf{C}}$, $j=1,\dots,k$ and $\nabla g_{\mathbf{C}}$ are linearly dependent over $\mathbf{C}$ at $P(t)$ and there exist $a_1(t),\dots,a_k(t)$ and $b(t)$ such that

$$\sum_{j=1}^{k} a_j(t)\cdot\nabla f_{j\mathbf{C}}(P(t))=b(t)\cdot\nabla g_{\mathbf{C}}(P(t)),$$

and $(a_1(t),\dots,a_n(t),b(t))\neq(0,\dots,0)$. Since f is of finite singularity type, $f_{\mathbf{C}}^{-1}(0)$ is a smooth $(n-k)$-dimensional complex manifold outside the origin and $\nabla f_{1\mathbf{C}}(P(t)),\dots,\nabla f_{k\mathbf{C}}(P(t))$ are linearly independent for $t\neq 0$. This implies that for all $t\neq0$, $b(t)\neq0$ and

$$\overline{\nabla(g_{\mathbf{C}})(P(t))}=\sum_{j=1}^{k}\overline{c_j(t)}\cdot\overline{\nabla f_{j\mathbf{C}}(P(t))},$$

where $c_j(t)=\frac{a_j(t)}{b(t)}$. Therefore

$$\frac{\partial}{\partial t}(g\circ P(t))=\left\langle\frac{\partial P(t)}{\partial t},\overline{\nabla(g_{\mathbf{C}})(P(t))}\right\rangle,$$

$$\frac{\partial}{\partial t}(g\circ P(t))=\left\langle\frac{\partial P(t)}{\partial t},\sum_{j=1}^{k}\overline{c_j(t)}\cdot\overline{\nabla f_{j\mathbf{C}}(P(t))}\right\rangle$$

$$= \sum_{j=1}^{k} c_j(t) \cdot \frac{\partial}{\partial t}(f_j \circ P(t)) = 0,$$

since $\{P(t)\} \subset f_{\mathbf{C}}^{-1}(0)$. Thus we have $\{P(t)\} \subset \{g = 0\}$ which contradicts the fact that $(f_{\mathbf{C}}, g_{\mathbf{C}})$ has an isolated singularity. □

Before proving our main result, we need to prove it in the case of curves.

Lemma 4.4 *Let $F : (\mathbf{R}^n, 0) \to (\mathbf{R}^{n-1}, 0)$ be a complete intersection of finite singularity type. Then*

$$\psi(F) \equiv 1 + \mu(F) \mod 2.$$

Proof. We have

$$\mu = 2\delta - r + 1,$$

where r is the number of branches of the complex curve $F_{\mathbf{C}}^{-1}(0)$ (see [BuG, Mi2]). The complex conjugation acts on the set of branches: those interchanged in pairs do not yield a real branch, whereas those left invariant yield a single real branch. Thus $r \equiv s \bmod 2$ where s is the number of real branches of $f^{-1}(0)$. Now we just have to use the fact that

$$s \equiv \frac{1}{2}\chi(F^{-1}(0) \cap S_\varepsilon) \equiv \psi(F) \mod 2.$$

□

Now we are ready to state:

Theorem 4.5 *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^k, 0)$ be an analytic germ of finite singularity type then*

$$\psi(f) \equiv 1 + \mu(f) \mod 2.$$

Proof. With Lemma 4.2 we construct $n - k - 1$ functions $g_1, \ldots, g_{n-k-1}$ such that the $n - k - 1$ mappings

$$\begin{array}{rcccl} F_1 = (f, g_1) & : & (\mathbf{R}^n, 0) & \to & (\mathbf{R}^{k+1}, 0), \\ F_2 = (f, g_1, g_2) & : & (\mathbf{R}^n, 0) & \to & (\mathbf{R}^{k+2}, 0), \\ \vdots & & & \vdots & \\ F_{n-k-1} = (f, g_1, \ldots, g_{n-k-1}) & : & (\mathbf{R}^n, 0) & \to & (\mathbf{R}^{n-1}, 0), \end{array}$$

are of finite singularity type. Applying Corollary 3.10, Lemma 4.3 and the Lê-Greuel formula, we find

$$\begin{array}{rcl} \psi(f)+\psi(F_1) & \equiv & \mu(f)+\mu(F_1) \mod 2, \\ \psi(F_1)+\psi(F_2) & \equiv & \mu(F_1)+\mu(F_2) \mod 2, \\ \vdots & & \vdots \\ \psi(F_{n-k-2})+\psi(F_{n-k-1}) & \equiv & \mu(F_{n-k-2})+\mu(F_{n-k-1}) \mod 2. \end{array}$$

Adding them together leads to

$$\psi(f)+\psi(F_{n-k-1}) \equiv \mu(f)+\mu(F_{n-k-1}) \mod 2.$$

By the previous lemma, we know that $\psi(F_{n-k-1}) \equiv 1+\mu(F_{n-k-1}) \mod 2$ and it is easy to see that

$$\psi(f) \equiv 1+\mu(f) \mod 2.$$

□

5. Appendix: a direct proof of Lemma 3.3

We give here a direct, but rather technical, proof of Lemma 3.3. We are in the following situation:

$$\begin{array}{rcl} f:(\mathbf{K}^n,p) & \to & (\mathbf{K}^k,\delta) \\ x & \mapsto & (f_1(x),\ldots,f_k(x)), \end{array}$$

where δ is a regular value of f so that $f^{-1}(\delta)$ is an analytic manifold of dimension $n-k$. Let

$$M(x)=\det\left[\frac{\partial f_i}{\partial x_j}(x)\right]_{1\le i,j\le k}.$$

We adopt the notation f_{ix_j} for $\frac{\partial f_i}{\partial x_j}$. We assume that $M(p)\neq 0$ which implies that $\frac{1}{M}$ is analytic at p and that for all x near p, $M(x)\neq 0$. We have for all $j\in\{1,\ldots,n\}$

$$m_j(x)=\begin{vmatrix} g_{x_1}(x) & \cdots & g_{x_k}(x) & g_{x_j}(x) \\ f_{1x_1}(x) & \cdots & f_{1x_k}(x) & f_{1x_j}(x) \\ \vdots & \ddots & \vdots & \vdots \\ f_{kx_1}(x) & \cdots & f_{kx_k}(x) & f_{kx_j}(x) \end{vmatrix},$$

hence, developing m_j along the first line and since $M \neq 0$, and omitting the x

$$g_{x_j} = \frac{(-1)^k}{M}\left(m_j + \sum_{i=1}^{k}(-1)^i g_{x_i}\frac{\partial(f_1,\ldots,f_k)}{\partial(x_1,\ldots,\hat{x}_i,\ldots,x_k;x_j)}\right). \tag{1}$$

Remark For all $1 \leq j \leq k \quad m_j = 0$.

We have for all $(i_1,\ldots,i_{k+1})$

$$\frac{\partial(g,f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_{k+1}})} = \begin{vmatrix} g_{x_{i_1}} & \cdots & g_{x_{i_{k+1}}} \\ f_{1x_{i_1}} & \cdots & f_{1x_{i_{k+1}}} \\ \vdots & \ddots & \vdots \\ f_{kx_{i_1}} & \cdots & f_{kx_{i_{k+1}}} \end{vmatrix}.$$

We develop along the first line and get

$$\frac{\partial(g,f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_{k+1}})} = \sum_{l=1}^{k+1}(-1)^{l+1} g_{x_{i_l}}\frac{\partial(f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,\hat{x_{i_l}},\ldots,x_{i_{k+1}})}.$$

Replacing $g_{x_{i_l}}$ by the expression (1), we obtain

$$\begin{aligned}&\frac{\partial(g,f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_{k+1}})}\\ &= \sum_{l=1}^{k+1}\frac{(-1)^{k+l+1}}{M}\frac{\partial(f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,\hat{x_{i_l}},\ldots,x_{i_{k+1}})}m_{i_l} + \sum_{i=1}^{k}\frac{(-1)^{k+1}g_{x_{i_l}}}{M}\\ &\left(\sum_{l=1}^{k+1}(-1)^{l+1}\frac{\partial(f_1,\ldots,f_k)}{\partial(x_1,\ldots,\hat{x}_i,\ldots,x_k;x_{i_l})}\frac{\partial(f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,\hat{x_{i_l}},\ldots,x_{i_{k+1}})}\right).\end{aligned}$$

It is enough to prove that each term between the big parenthesis is zero. But one sees clearly that is the determinant of the following matrix

$$B = \begin{bmatrix} \frac{\partial(f_1,\ldots,f_k)}{\partial(x_1,\ldots,\hat{x}_i,\ldots,x_k;x_{i_1})} & \cdots & \frac{\partial(f_1,\ldots,f_k)}{\partial(x_1,\ldots,\hat{x}_i,\ldots,x_k;x_{i_{k+1}})} \\ \frac{\partial f_1}{\partial x_{i_1}} & \cdots & \frac{\partial f_1}{\partial x_{i_{k+1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_{i_1}} & \cdots & \frac{\partial f_k}{\partial x_{i_{k+1}}} \end{bmatrix}.$$

But we have for all $l \in \{1, \ldots, k+1\}$,

$$\frac{\partial(f_1, \ldots, f_k)}{\partial(x_1, \ldots, \hat{x}_i, \ldots, x_k; x_{i_l})} = \sum_{j=1}^{k} (-1)^{j+k} \frac{\partial f_j}{\partial x_{i_l}} A_j,$$

where

$$A_j = \frac{\partial(f_1, \ldots, \hat{f}_j, \ldots, f_k)}{\partial(x_1, \ldots, \hat{x}_i, \ldots, x_k)},$$

and then

$$L_0 = \sum_{j=1}^{k} (-1)^{j+k} A_j L_j,$$

where $L_0, L_1, \ldots, L_k$ are the lines of the matrix B, which allows us to conclude that $\det B = 0$ and $\frac{\partial(g, f_1, \ldots, f_k)}{\partial(x_{i_1}, \ldots, x_{i_k})}$ belongs to the ideal generated by the m_{i_l}, which is included in J_p.

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