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Tortile Yang-Baxter operators for crossed group-categories

Takeo FUKUSHI

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Abstract. The notion of a tortile Yang-Baxter operator in a crossed group-category is introduced. It is shown that a tortile Yang-Baxter operator on an object X induces a unique braiding and a twist on the free crossed group-category generated by the objects X and X^{*}.

Key words: tortile Yang-Baxter operator, crossed group-category.

1. Introduction

The category of tangles in 3 dimension has a beautiful algebraic characterization in terms of a universal property. This was initially developed by Yetter [10], Turaev [8], Freyd-Yetter [1] and Joyal-Street [3], and has culminated in the work of Shum [7] asserting that the category of framed tangles $\mathcal{FT}$ is monoidally equivalent to the tortile category freely generated by a single object. Joyal and Street [2] gave another purely algebraic interpretation of this category as the free tensor category containing an object equipped with a tortile Yang-Baxter operator.

Recently, Turaev [9] introduced the notion of a modular crossed group-category, and used it to develop 3-dimensional homotopy quantum field theory (HQFT). He started with defining the notion of a tortile (ribbon) crossed π-category for a group π, and showed that modular crossed π-categories induce invariants of 3-dimensional π-manifolds.

The aim of this paper is to give the Joyal and Street's interpretation for a crossed group-category. To do this, we define a balanced Yang-Baxter operator and a tortile Yang-Baxter operator in a crossed group-category. Then we prove that the free crossed group-category $\mathcal{F}$ generated by a single object equipped with a tortile Yang-Baxter operator admits a unique braiding and a twist. Although our construction owes much to the paper [2], several new aspects appear. First, it turns out that one should define a twist

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before a Yang-Baxter operator. This statement means that in a general crossed group-category, it is not possible to define a Yang-Baxter operator without a twist. Thus one can define only balanced Yang-Baxter operators in a crossed group-category. Second, we use the fact that the category $\mathcal{F}$ admits a connectivity structure, which we feel non-trivial. In general, for an object U in a crossed π-category $\mathcal{C}$, the centralizer $\mathcal{L}_{\mathcal{C}}(U)$ does not admit a crossed π-category structure. However, if a crossed π-category $\mathcal{C}$ is connected, then the category $\mathcal{L}_{\mathcal{C}}(U)$ admits a crossed π-category structure, so that we can aplly this procedure to $\mathcal{F}$. Third, since we have to consider a Yang-Baxter operator with a twist, various identities which were simple in [2] become much more complicated. To overcome this difficulty, we use a diagrammatic notion. Then, we can check that each equality between diagrams corresponds to a certain equality between morphisms in $\mathcal{F}$. As a result, we see that the constructions above are all well done and the theorem holds.

2. Preliminaries

Definition 1 Let π be a group and let $\mathcal{C}$ be a strict monoidal category with a unit object I. Then the category $\mathcal{C}$ is called a π-category if it satisfies the following conditions:

(a) there are full subcategories $\mathcal{C}_\alpha$ ($\alpha \in \pi$) of $\mathcal{C}$ such that each object of $\mathcal{C}$ belongs to $\mathcal{C}_\alpha$ for a unique $\alpha \in \pi$;

(b) if $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$ with $\alpha \neq \beta$ then there is not any morphism from U to V;

(c) $I \in \mathcal{C}_1$, and if $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$ then $U \otimes V \in \mathcal{C}_{\alpha\beta}$.

In [9] a K-additivity and a left duality are assumed in the monoidal category $\mathcal{C}$. In this paper, we do not assume those structures in $\mathcal{C}$.

Definition 2 In the setting above, an automorphism of $\mathcal{C}$ is defined as a functor $\varphi : \mathcal{C} \to \mathcal{C}$ wich preserves the tensor product and the unit object. Thus,

$$\varphi(I) = I, \quad \varphi(U \otimes V) = \varphi(U) \otimes \varphi(V), \quad \varphi(f \otimes g) = \varphi(f) \otimes \varphi(g),$$

for any objects U, V and any morphisms f, g in $\mathcal{C}$. We denote by $\mathrm{Aut}(\mathcal{C})$ the group of automorphisms of $\mathcal{C}$. A crossed π-category is a π-category $\mathcal{C}$ endowed with a group homomorphism $\varphi : \pi \to \mathrm{Aut}(\mathcal{C})$ such that for all

$\alpha, \beta \in \pi$ the functor $\varphi_\alpha = \varphi(\alpha) : \mathcal{C} \to \mathcal{C}$ maps $\mathcal{C}_\beta$ to $\mathcal{C}_{\alpha\beta\alpha^{-1}}$. For objects $U \in \mathcal{C}_\alpha$, $V \in \mathcal{C}_\beta$, set ${}^UV = \varphi_\alpha(V)$.

For crossed π-categories $\mathcal{C}$, $\mathcal{C}'$, a tensor functor $\mathcal{C} \to \mathcal{C}'$ is called a crossed π-functor if it preserves the action of π.

Definition 3 Let $\mathcal{C}$ be a crossed π-category. A braiding in $\mathcal{C}$ is a system of invertible morphisms $c_{U,V} : U \otimes V \to {}^UV \otimes U$ satisfying the following conditions:

(a) for any morphisms $f : U \to U'$ and $g : V \to V'$ such that U, U' lie in the same component of $\mathcal{C}$, we have

$$c_{U',V'}(f \otimes g) = ({}^Ug \otimes f)c_{U,V};$$

(b) for any objects U, V, W in $\mathcal{C}$ we have

$$c_{U\otimes V,W} = (c_{U,{}^VW} \otimes 1)(1 \otimes c_{V,W});$$

(c) for any objects U, V, W in $\mathcal{C}$ we have

$$c_{U,V\otimes W} = (1 \otimes c_{U,W})(c_{U,V} \otimes 1);$$

(d) the action of π on $\mathcal{C}$ preserves the braiding, i.e., for any $\alpha \in \pi$ and any $V, W \in \mathcal{C}$ we have

$$\varphi_\alpha(c_{V,W}) = c_{\varphi_\alpha(V),\varphi_\alpha(W)}.$$

A crossed π-category equipped with a braiding is called a braided crossed π-category. A braided crossed π-category $\mathcal{C}$ is called *balanced* if it is equipped with a natural family of invertible morphisms $\theta_U : U \to {}^UU$ (called twist) satisfying the following conditions:

(1) $\theta_I = \mathrm{id}_I : I \to I$;

(2) for any object U, V in $\mathcal{C}$ we have

$$\theta_{U\otimes V} = c_{({}^UV)_V,{}^UU}\, c_{({}^UU),({}^VV)}(\theta_U \otimes \theta_V);$$

(3) the action of π on $\mathcal{C}$ preserves the twist, i.e., for any $\alpha \in \pi$ and any $U \in \mathcal{C}$ we have $\varphi_\alpha(\theta_U) = \theta_{\varphi_\alpha(U)}$.

A braided crossed π-category $\mathcal{C}$ is called *tortile* if it is balanced and each object U has a dual U^* such that $\theta_{{}^UU^*} = (\theta_U)^*$.

3. Tortile Yang-Baxter operators in crossed group-categories

In this section we consider Yang-Baxter operators and twists in a crossed π-category $\mathcal{C}$. When $\pi = 1$, one can define a Yang-Baxter operator on each object U in $\mathcal{C}$ without a twist. However, for a general crossed π-category $\mathcal{C}$, one must define a twist first, then proceed to define a balanced Yang-Baxter operator by using the twist.

Definition 4 A twist on an object U of a crossed π-category $\mathcal{C}$ is an invertible arrow $z : U \to {}^{U}U$. A balanced Yang-Baxter operator on an object U is an invertible arrow $y : U \otimes U \to {}^{U}U \otimes U$ satisfying the hexagonal condition

$$({}^{U}y \otimes 1)(1 \otimes y)(y \otimes 1) = (1 \otimes y)(y' \otimes 1)(1 \otimes y)$$

where $y' = ({}^{U}z \otimes 1)y(1 \otimes z^{-1})$.

A left dual for an object U of $\mathcal{C}_\alpha$ is an object U^* in $\mathcal{C}_{\alpha^{-1}}$ together with arrows

$$b_U : I \to U \otimes U^* \quad \text{and} \quad d_U : U^* \otimes U \to I$$

such that

$$(d_U \otimes 1)(1 \otimes b_U) = 1 \quad \text{and} \quad (1 \otimes d_U)(b_U \otimes 1) = 1.$$

If both U, V have duals, then each arrow $f : U \to V$ gives rise to an arrow

$$f^* : (d_V \otimes 1)(1 \otimes f \otimes 1)(1 \otimes b_U) : V^* \to U^*.$$

A balanced Yantg-Baxter operator on an object $U \in \mathcal{C}$ is called *dualizable* if U has a dual and, both the arrows $u : {}^{U}U^* \otimes U \to U \otimes U^*$ and $v : U^* \otimes U \to {}^{U^*}U \otimes U^*$, given by the equations

$$u = (d_{{}^{U}U^*} \otimes 1 \otimes 1)(1 \otimes y \otimes 1)(1 \otimes 1 \otimes b_U)$$

and

$$\begin{aligned} v = (d_U \otimes 1 \otimes 1)(1 \otimes 1 \otimes {}^{U^*}z^{-1} \otimes 1)(1 \otimes y^{-1} \otimes 1) \\ (1 \otimes z \otimes 1 \otimes 1)(1 \otimes 1 \otimes b_U) \end{aligned}$$

are invertible. A balanced Yang-Baxter operator on an object U is called *tortile* if it is dualizable and the following identity holds.

$${}^{U}zz = (1 \otimes d_{{}^{U}U})(1 \otimes {}^{U}v^{-1})(y' \otimes 1)(1 \otimes b_{{}^{U}U}) : U \to {}^{U}({}^{U}U).$$

In a balanced crossed π-category $\mathcal{C}$, we have a balanced YB-operator $(y = c_{U,U}, z = \theta_U)$ on each object U. If U has a dual, then we have the identities $u = c^{-1}_{U,U^*}$ and $v = c_{U^*,U}$ in $\mathcal{C}$. Hence $(y = c_{U,U}, z = \theta_U)$ is dualizable. The next proposition shows that a balanced crossed π-category $\mathcal{C}$ becomes a tortile crossed π-category iff the above balanced YB-operators $(y = c_{U,U}, z = \theta_U)$ become tortile for all objects U in $\mathcal{C}$.

Proposition 1 *In a balanced crossed π-category $\mathcal{C}$, if U is an object with a dual U^*, then the pair $(c_{U,U}, \theta_U)$ is a tortile YB-operator iff $\theta_{{}^U U^*} = (\theta_U)^*$.*

Proof. We first observe that if (U^*, d_U, b_U) is a dual for U, then $({}^U U, d_{{}^U U} c_{{}^U U, U^*}, c^{-1}_{{}^U U^*, {}^U U} b_U)$ is a dual for U^*. Then for an arrow $f : U^* \to V^*$ in $\mathcal{C}$, we obtain an arrow

$$f^\sharp = (1 \otimes d_{{}^V V})(1 \otimes {}^U f \otimes 1)(b_{{}^U U} \otimes 1) : {}^V V \to {}^U U.$$

Applying this construction to the arrow $\theta_{{}^U U^*} : {}^U U^* \to U^*$, we see that

$$\begin{aligned}
&(\theta_{{}^U U^*})^\sharp \theta_U \\
&= (d_{{}^U U} \otimes 1)(c_{{}^U U, {}^U U^*} \otimes 1)(1 \otimes \theta_{{}^U U^*} \otimes 1)(1 \otimes c^{-1}_{{}^U U^*, {}^U({}^U U)})(1 \otimes b_{{}^U U})\theta_U \\
&= (d_{{}^U U} \otimes 1)(c_{{}^U U, {}^U U^*} \otimes 1)(1 \otimes \theta_{{}^U U^*} \otimes 1)(\theta_U \otimes 1 \otimes 1) \\
&\quad (1 \otimes c^{-1}_{{}^U U^*, {}^U({}^U U)})(1 \otimes b_{{}^U U}) \\
&= (d_{{}^U U} \otimes 1)(c_{{}^U U, {}^U U^*} \otimes 1)(\theta_U \otimes \theta_{{}^U U^*} \otimes 1)(1 \otimes c^{-1}_{{}^U U^*, {}^U({}^U U)})(1 \otimes b_{{}^U U}) \\
&= (d_{{}^U U} \otimes 1)(\theta_{{}^U U^*} \otimes \theta_{{}^U U} \otimes 1)(c^{-1}_{{}^U U^*, {}^U U} \otimes 1)(1 \otimes c^{-1}_{{}^U U^*, {}^U({}^U U)})(1 \otimes b_{{}^U U}) \\
&= (d_{{}^U U} \otimes 1)(c^{-1}_{{}^U U^*, {}^U U} \otimes 1)(1 \otimes c^{-1}_{{}^U U^*, {}^U({}^U U)})(1 \otimes b_{{}^U U}) \\
&= (d_{{}^U U} \otimes 1)c^{-1}_{{}^U U^*, {}^U U \otimes {}^U({}^U U)}(1 \otimes b_{{}^U U}) \\
&= (1 \otimes d_{{}^U U})c_{{}^U U^* \otimes {}^U U, {}^U({}^U U)} c^{-1}_{{}^U U^*, {}^U U \otimes {}^U({}^U U)}(1 \otimes b_{{}^U U}) \\
&= (1 \otimes d_{{}^U U})(c_{{}^U U^*, {}^U({}^U({}^U U))} \otimes 1)(1 \otimes c_{{}^U U, {}^U({}^U U)})c^{-1}_{{}^U U^*, {}^U U \otimes {}^U({}^U U)}(1 \otimes b_{{}^U U}) \\
&= (1 \otimes d_{{}^U U})(c_{{}^U U^*, {}^U({}^U({}^U U))} \otimes 1)c^{-1}_{{}^U U^*, {}^U({}^U({}^U U)) \otimes {}^U U}(c_{U, {}^U U} \otimes 1)(1 \otimes b_{{}^U U}) \\
&= (1 \otimes d_{{}^U U})(1 \otimes c^{-1}_{{}^U U^*, {}^U U})(c_{U, {}^U U} \otimes 1)(1 \otimes b_{{}^U U}) \\
&= (1 \otimes d_{{}^U U})(1 \otimes {}^U v^{-1})(y' \otimes 1)(1 \otimes b_{{}^U U}).
\end{aligned}$$

Thus the balanced YB-operator (y, z) is tortile iff $(\theta_{{}^U U^*})^\sharp \theta_U = {}^U z z = {}^U \theta_U \theta_U$. That is, $(\theta_{{}^U U^*})^\sharp = {}^U \theta_U$. Then it is easy to see that the condition $(\theta_{{}^U U^*})^\sharp = {}^U \theta_U$ is equivalent to the relation $\theta_{{}^U U^*} = (\theta_U)^*$. $\square$

Definition 5 For a crossed π-category $\mathcal{C}$, the centralizer $\mathcal{L}_{\mathcal{C}}(U)$ of an object $U \in \mathcal{C}$ is the category whose objects are pairs (X, α) where X is an object in $\mathcal{C}$ and $\alpha : U \otimes X \to {}^{U}X \otimes U$ is an isomorphism in $\mathcal{C}$, and whose arrows $f : (X, \alpha) \to (Y, \beta)$ are the arrows $f : X \to Y$ in $\mathcal{C}$ such that $\beta(1 \otimes f) = ({}^{U}f \otimes 1)\alpha$.

Then $\mathcal{L}_{\mathcal{C}}(U)$ becomes a π-category by the rule $(X, \alpha) \in \mathcal{C}_x \Leftrightarrow X \in \mathcal{C}_x$ for $x \in \pi$ and the tensor product $(X, \alpha) \otimes (Y, \beta) = (X \otimes Y, (\alpha \otimes 1)(1 \otimes \beta))$. Similarly, one can define the centralizer $\mathcal{L}_{\mathcal{C}}(h)$ of an arrow $h : U \to V$ in $\mathcal{C}$ as follows. The objects of $\mathcal{L}_{\mathcal{C}}(h)$ are triples (X, α, β) where X is an object in $\mathcal{C}$ and $\alpha : U \otimes X \to {}^{U}X \otimes U$ and $\beta : V \otimes X \to {}^{V}X \otimes V$ are isomorphisms in $\mathcal{C}$ such that $\beta(h \otimes 1) = (1 \otimes h)\alpha$. The arrows $(X, \alpha, \beta) \to (Y, \gamma, \delta)$ in $\mathcal{L}_{\mathcal{C}}(h)$ are arrows $f : X \to Y$ in $\mathcal{C}$ such that $({}^{U}f \otimes 1)\alpha = \gamma(1 \otimes f)$ and $({}^{V}f \otimes 1)\beta = \delta(1 \otimes f)$. This category $\mathcal{L}_{\mathcal{C}}(h)$ also admits a π-category structure.

Definition 6 A crossed π-category $\mathcal{C}$ is called *connected* if for each pair (X, Y) of objects, there exists an invertible arrow $g(X, Y) : {}^{X}Y \to Y$ such that $g(I, Y) = \mathrm{id}_Y$, $g(X, I) = \mathrm{id}_I$, $g(X' \otimes X, Y) = g(X', Y)\ {}^{X'}g(X, Y)$ and $g(X, Y \otimes Y') = g(X, Y) \otimes g(X, Y')$.

When a crossed π-category $\mathcal{C}$ is connected, the categories $\mathcal{L}_{\mathcal{C}}(U)$ and $\mathcal{L}_{\mathcal{C}}(h)$ become crossed π-categories via ${}^{(X,\alpha)}(Y, \beta) = ({}^{X}Y,\ ({}^{U}g^{-1}(X, Y) \otimes 1)\beta(1 \otimes g(X, Y)))$ and ${}^{(X,\alpha,\beta)}(Y, \gamma, \delta) = ({}^{X}Y,\ ({}^{U}g^{-1}(X, Y) \otimes 1)\gamma(1 \otimes g(X, Y)),\ ({}^{U}g^{-1}(X, Y) \otimes 1)\delta(1 \otimes g(X, Y)))$.

Definition 7 For a crossed π-category $\mathcal{C}$, the center $\mathcal{L}_{\mathcal{C}}$ of $\mathcal{C}$ is the category whose objects are pairs (U, α) where $U \in \mathcal{C}$ and $\alpha : U \otimes - \to {}^{U} - \otimes U$ is a natural isomorphism obeying the following two conditions:

(1) $\alpha_I = 1$;

(2) $\alpha_{X \otimes Y} = (1 \otimes \alpha_Y)(\alpha_X \otimes 1)$ for all $X, Y \in \mathcal{C}$.

An arrow $f : (U, \alpha) \to (V, \beta)$ in $\mathcal{L}_{\mathcal{C}}$ is an arrow $f : U \to V$ in $\mathcal{C}$ such that $\beta_X(f \otimes 1) = (1 \otimes f)\alpha_X$ for all $X \in \mathcal{C}$.

Then $\mathcal{L}_{\mathcal{C}}$ becomes a crossed π-category with $(U, \alpha) \otimes (V, \beta) = (U \otimes V, (\alpha \otimes 1)(1 \otimes \beta))$ and ${}^{(U,\alpha)}(V, \beta) = ({}^{U}V, {}^{U}\beta_{U^*X})$.

Proposition 2 (a) *For a crossed π-category $\mathcal{C}$, the crossed π-category $\mathcal{L}_{\mathcal{C}}$ is braided via $\alpha_V : (U, \alpha) \otimes (V, \beta) \to {}^{(U,\alpha)}(V, \beta) \otimes (U, \alpha)$.*

(b) *Let $\mathcal{C}$ be a crossed π-category. Then for each object $U \in \mathcal{C}$, the equation $F(X) = (X, \alpha_X)$ determines a bijection between objects $(U, \alpha) \in \mathcal{L}_\mathcal{C}$ and tensor functors $F : \mathcal{C} \to \mathcal{L}_\mathcal{C}(U)$. Similarly, for each arrow $h : U \to V$ in $\mathcal{C}$, the equation $F'(X) = (X, \alpha_X, \beta_X)$ determines a bijection between arrows $h : (U, \alpha) \to (V, \beta) \in \mathcal{L}_\mathcal{C}$ and tensor functors $F' : \mathcal{C} \to \mathcal{L}_\mathcal{C}(h)$.*

(c) *For a crossed π-category $\mathcal{C}$, the equation $G(U) = (U, c_{U,-})$ determines a bijection between braidings c on $\mathcal{C}$ and crossed π-functors $G : \mathcal{C} \to \mathcal{L}_\mathcal{C}$.*

Proof. Straightforward. □

For a connected crossed π-category $\mathcal{C}$, let (y, z) be a balanced YB-operator on an object U such that $g(U, U) = z^{-1}$. Then we have the following lemma:

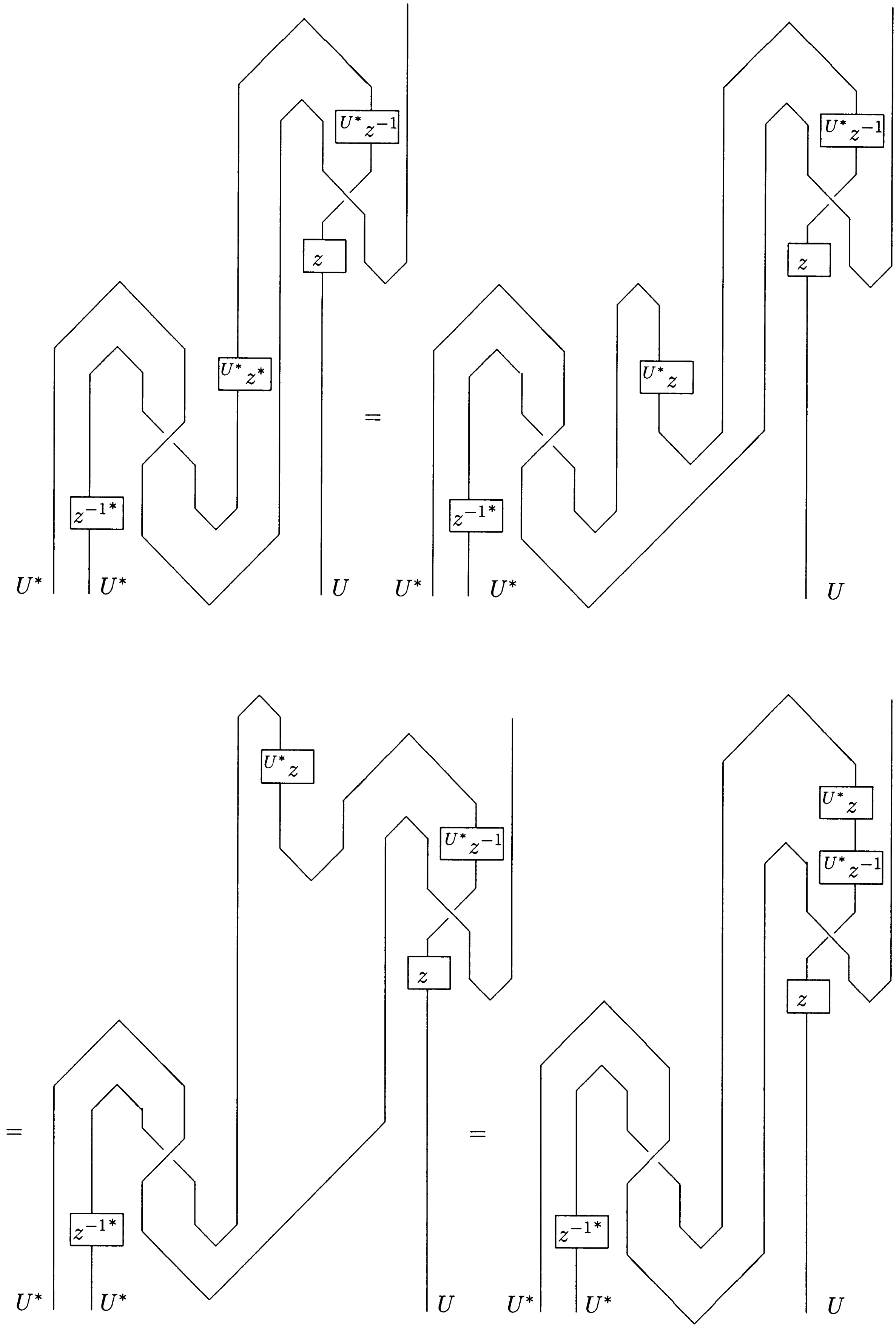
Lemma 1 (a) *The balanced YB-operator (y, z) defines a balanced YB-operator on the object $(U, y) \in \mathcal{L}_\mathcal{C}(U)$. If (y, z) is dualizable, then $(U^*, u^{-1}) \in \mathcal{L}_\mathcal{C}(U)$ is a left dual for the object $(U, y) \in \mathcal{L}_\mathcal{C}(U)$. Moreover, (y, z) defines a dualizable balanced YB-operator on (U, y).*

(b) *The centralizer $\mathcal{L}_\mathcal{C}(U^*)$ contains (U, v) and (U^*, w) where $w = (d_U \otimes 1 \otimes 1)(1 \otimes 1 \otimes {}^{U^*}z^* \otimes 1)(1 \otimes u \otimes 1)(1 \otimes {z^*}^{-1} \otimes 1 \otimes 1)(1 \otimes 1 \otimes b_U)$. The object (U^*, w) is dual to (U, v) and (y, z) defines a balanced YB-operator on (U, v).*

(c) *If (y, z) is a tortile Yang-Baxter operator, then it is also a tortile Yang-Baxter operator on $(U, y) \in \mathcal{L}_\mathcal{C}(U)$ and $(U, v) \in \mathcal{L}_\mathcal{C}(U^*)$.*

Proof. (a) For arrows $(U, y) \to {}^{(U,y)}(U, y)$ and $(U, y) \otimes (U, y) \to {}^{(U,y)}(U, y) \otimes (U, y)$ in $\mathcal{L}_\mathcal{C}(U)$, we take the arrows $z : U \to {}^U U$ and $y : U \otimes U \to {}^U U \otimes U$ in $\mathcal{C}$. Then by the hexagnal condition on (y, z) and the assumption $g(U, U) = z^{-1}$, we see that these arrows are indeed arrows in $\mathcal{L}_\mathcal{C}(U)$.

(b) We have to show that the object (U^*, w) is dual to (U, v). For this it is convenient to use a diagrammatic notion as used in [3], [5]. For example, the following equalities show that the arrow $d_U : U^* \otimes U \to I$ becomes an arrow $(U^*, w) \otimes (U, v) \to (I, \mathrm{id}_{U^*})$ in $\mathcal{L}_\mathcal{C}(U^*)$.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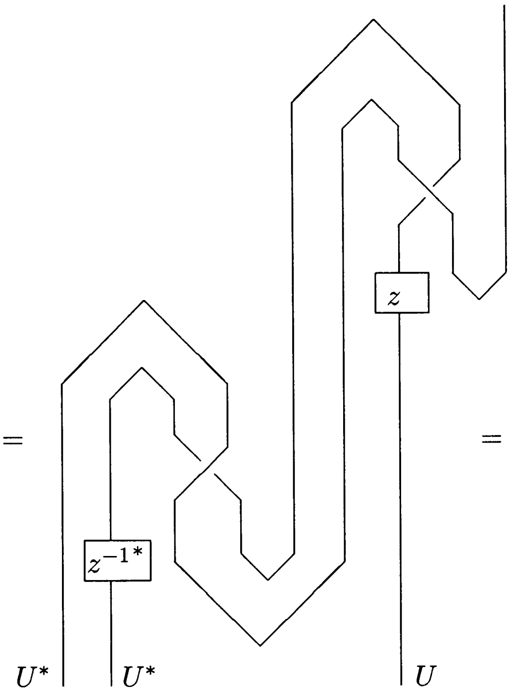
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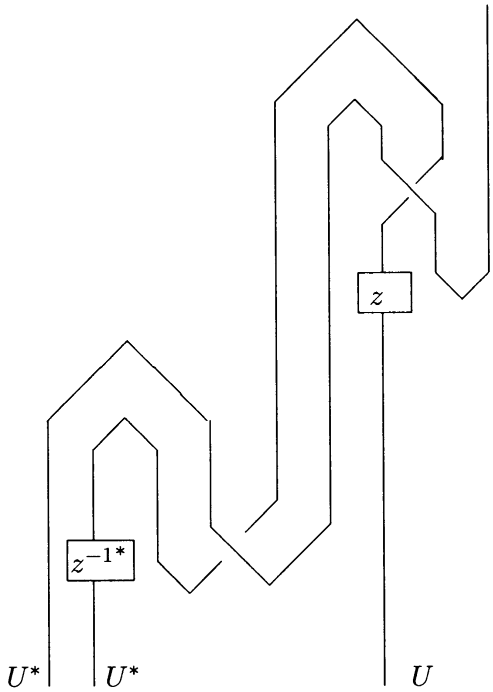
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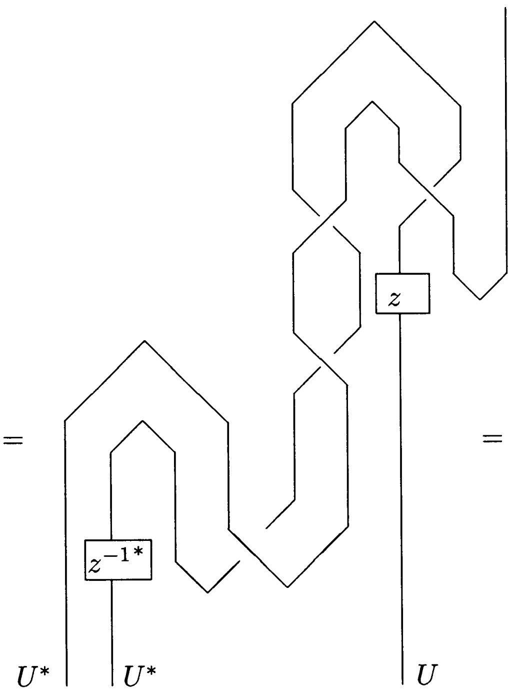
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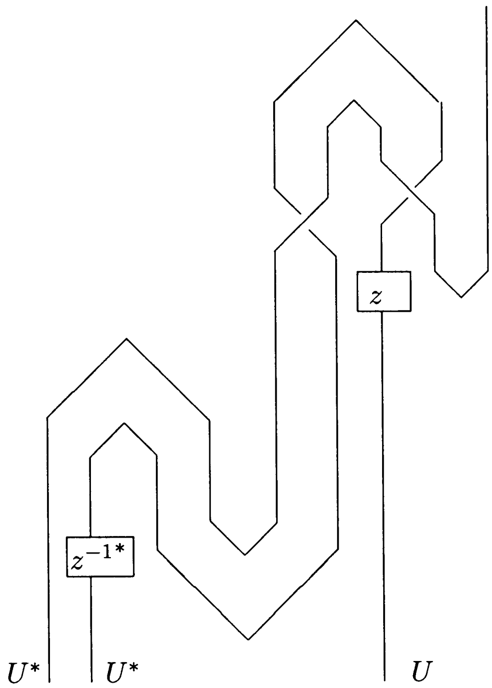
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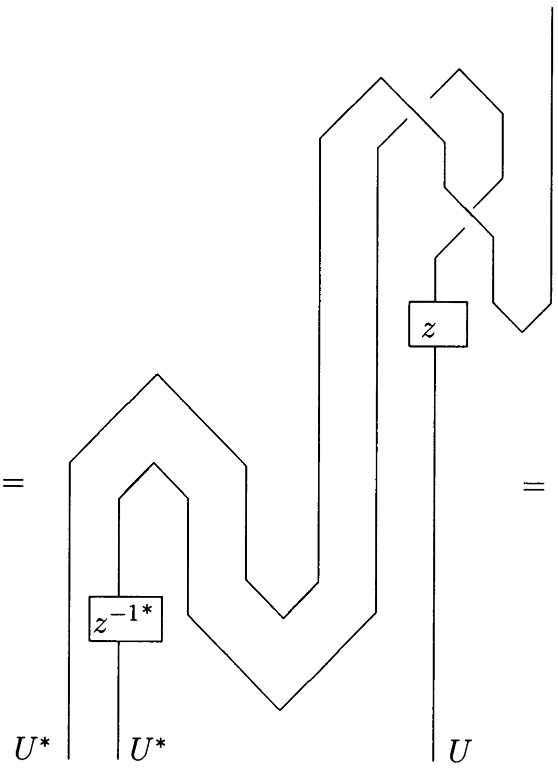
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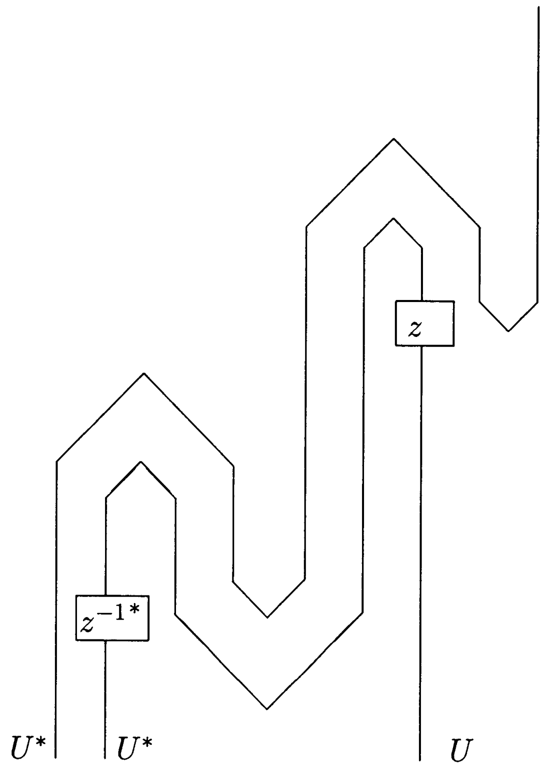
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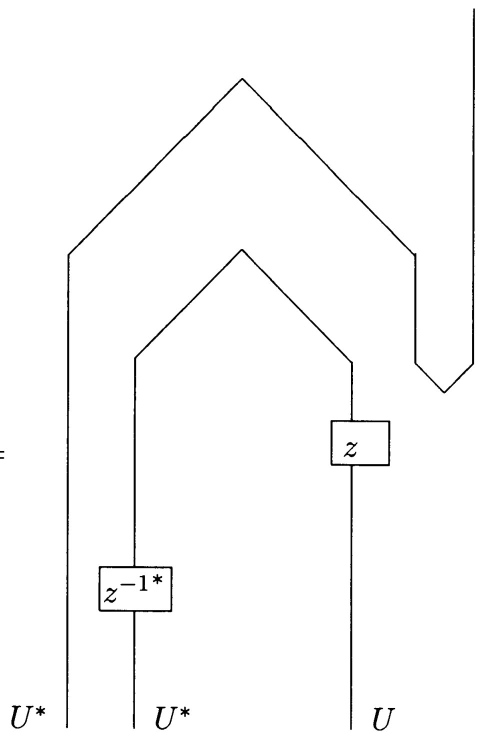
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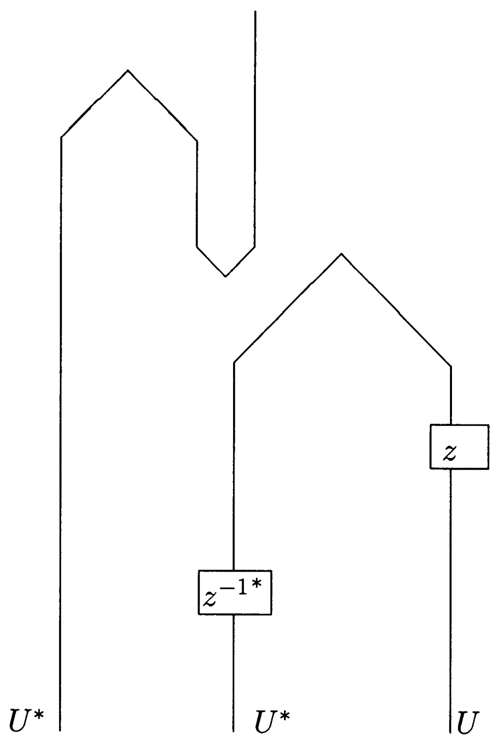
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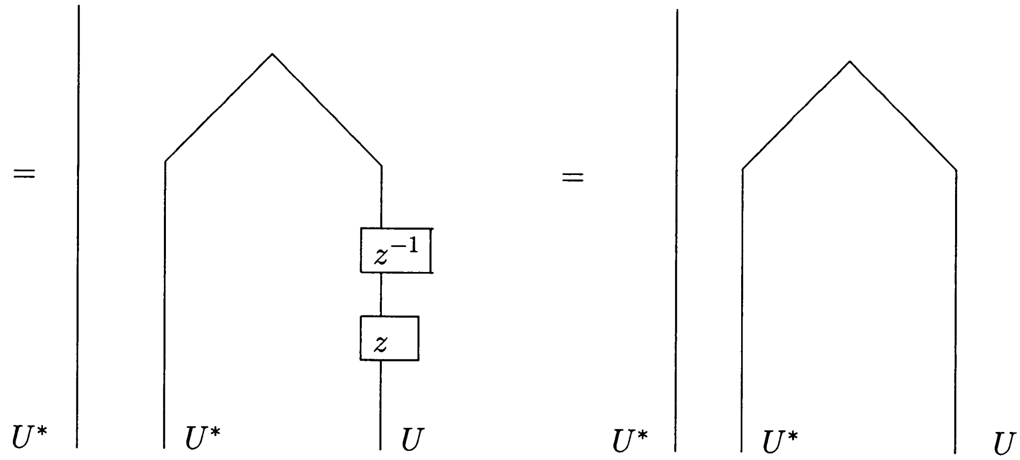
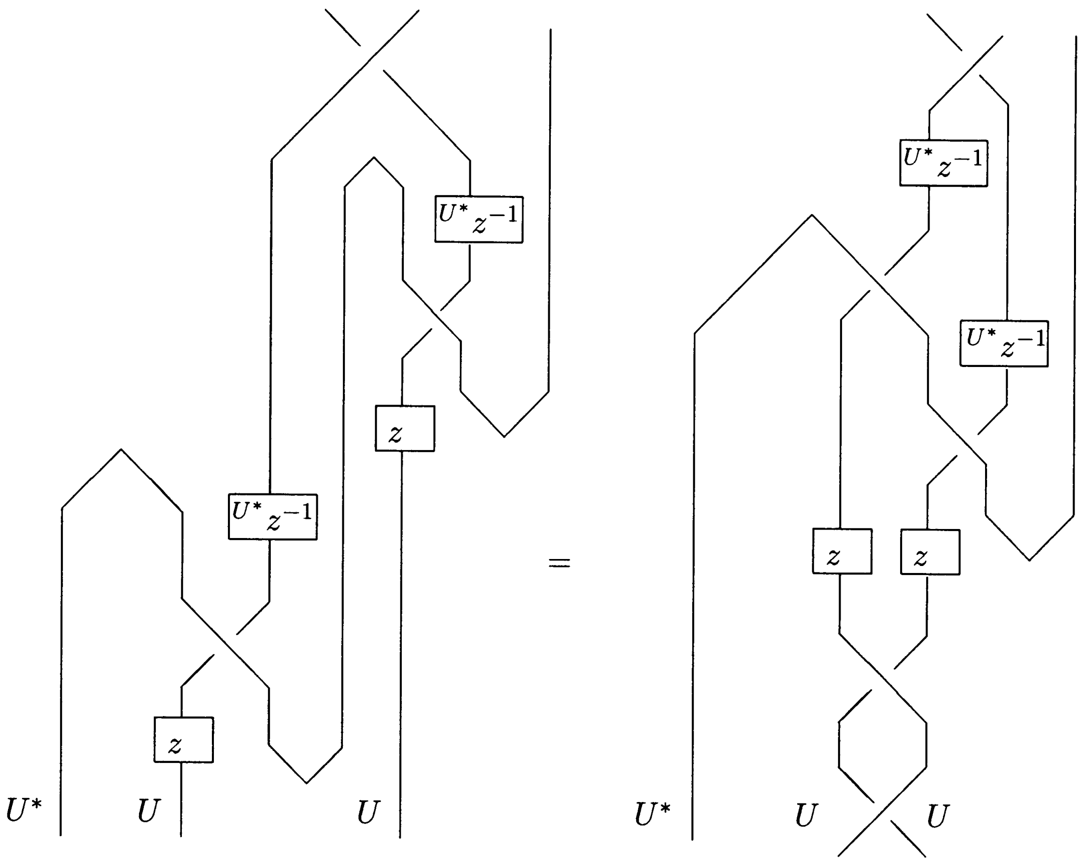


The next equalities show that the arrow $y : U \otimes U \to {}^{U}U \otimes U$ defines an arrow $(U, v) \otimes (U, v) \to {}^{(U,v)}(U, v) \otimes (U, v)$ in $\mathcal{L}_{\mathcal{C}}(U^*)$.



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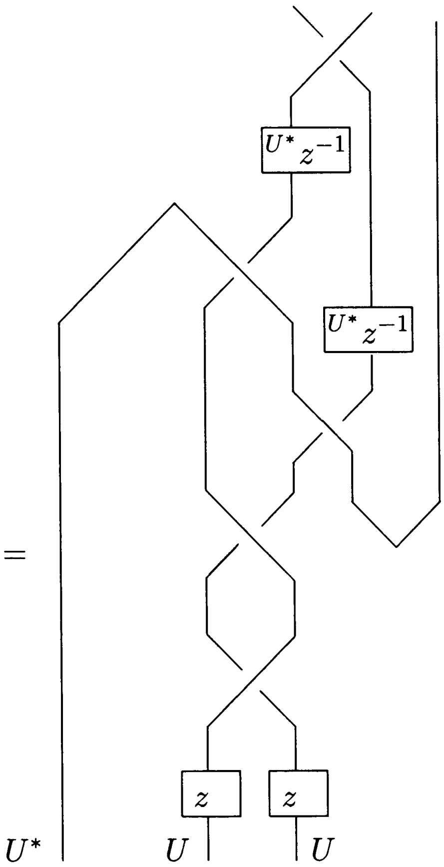
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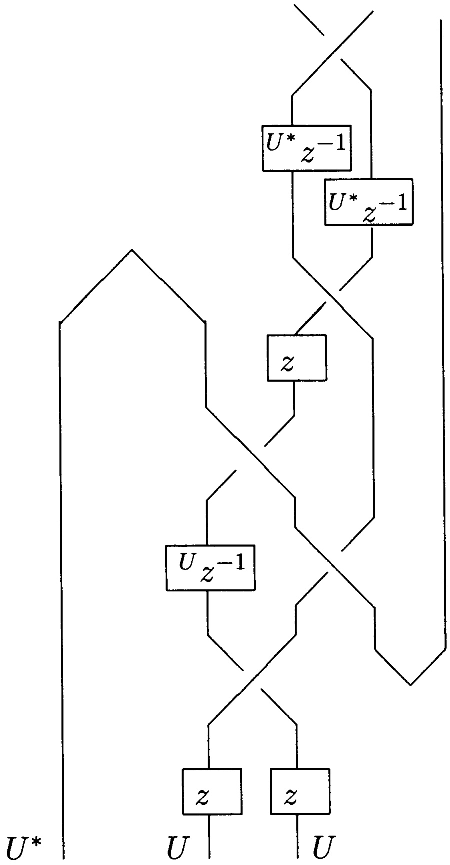
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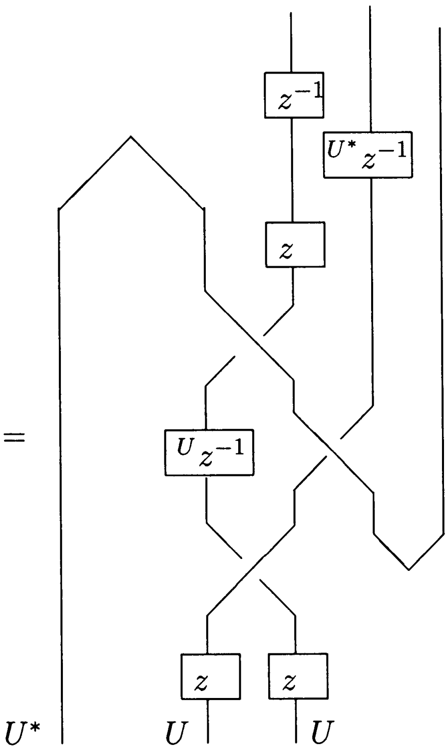
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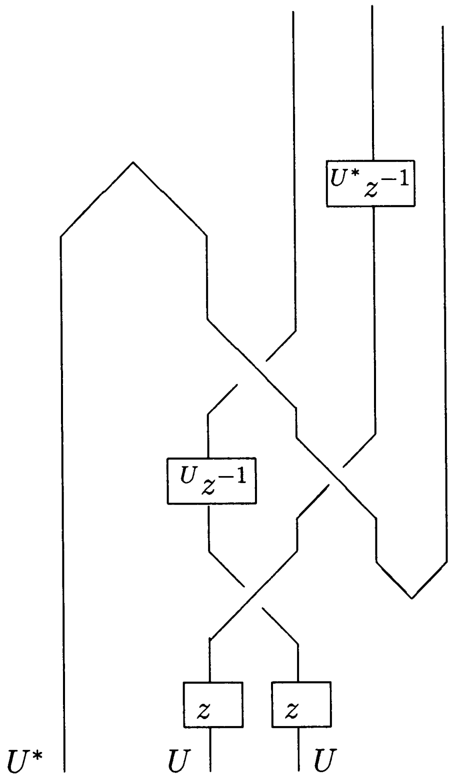
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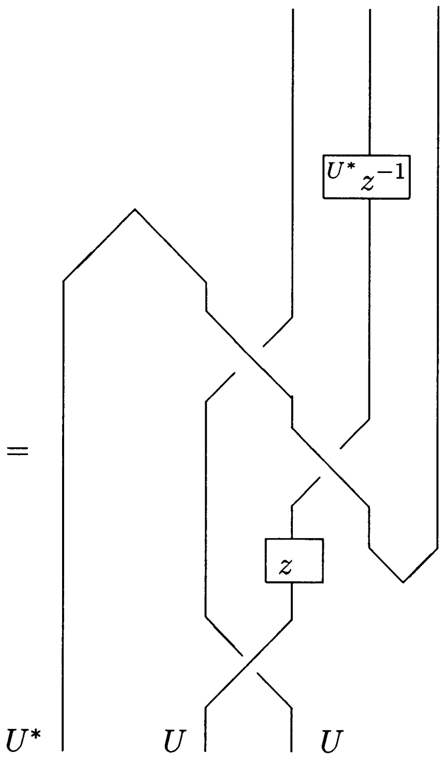
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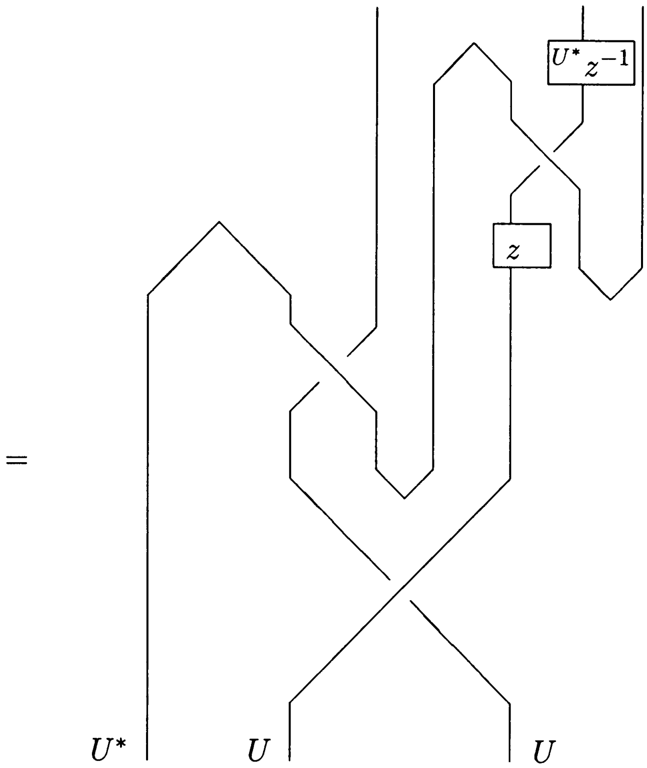
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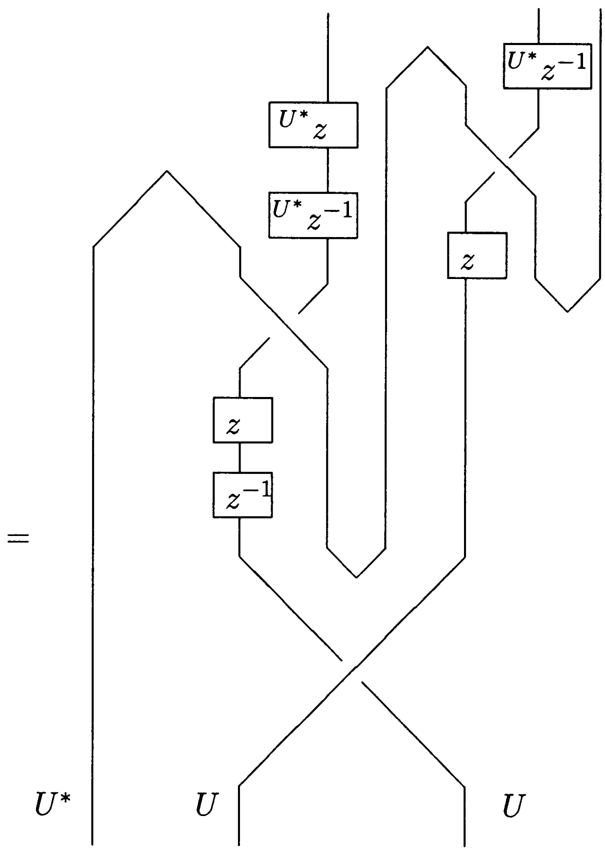
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(c) Straightforward. $\square$

Let $\mathcal{F}$ be the crossed $\mathbf{Z}$-category freely generated by objects X and X^* and arrows $d_X : X^* \otimes X \to I$, $b_X : I \to X \otimes X^*$, $y : X \otimes X \to {}^X X \otimes X$, $z : X \to {}^X X$, subject to the conditions that $X \in \mathcal{F}_1$, $X^* \in \mathcal{F}_{-1}$, X^* is left dual to X via d_X and b_X, and (y, z) is a tortile YB-operator on X.

Now we state our main theorem in this paper.

Theorem *The category $\mathcal{F}$ admits a unique braiding c and a twist θ such that $c_{X,X} = y$, $\theta_X = z$ and $\theta_{X^*} = {}^{X^*}(\theta_X)^*$.*

To prove the existence of such a braiding, we begin with the following proposition.

Proposition 3 *There is a natural isomorphisms*

$$c_{X,-} : X \otimes - \to {}^X - \otimes X \quad \text{and} \quad c_{X^*,-} : X^* \otimes - \to {}^{X^*} - \otimes X^*$$

such that $(X, c_{X,-})$ and $(X^*, c_{X^*,-}) \in \mathcal{L}_{\mathcal{F}}$ where $c_{X,X} = y$ and $c_{X^*,X} = v$.

Proof. We first show that the category $\mathcal{F}$ is a connected crossed $\mathbf{Z}$-category. For this, observe that the category $\mathcal{F}$ is generated as a monoidal category by the objects ${}^{X^n}X$, ${}^{X^{*n}}X$, ${}^{X^n}X^*$, ${}^{X^{*n}}X^*$, where X^n and X^{*n} mean $X \otimes X \otimes \cdots \otimes X$ (n times) and $X^* \otimes X^* \otimes \cdots \otimes X^*$ (n times) respectively, and n runs through the set of non-negative integers. Besides, the objects ${}^{X^n}X$ and ${}^{X^{*n}}X$ belong to the same class of X, and the objects ${}^{X^n}X^*$ and ${}^{X^{*n}}X^*$ belong to the same class of X^*. Using this property we can prove that for any object Y there exists an isomorphism ${}^Y X \to X$. Indeed, for any objects Y and Z we can define an isomophism ${}^{Y \otimes Z}X \to X$ once we obtain isomorphisms $f : {}^Y X \to X$ and $g : {}^Z X \to X$ since we can use the composition $f\,{}^Y g$. Thus it is enough to show that there are isomorphisms ${}^X X \to X$ and ${}^{X^*}X \to X$, and we can take the arrows z^{-1} and ${}^{X^*}z$. The same argument applies to show that for any object Y there exists an isomorphism ${}^Y({}^{X^n}X) \to {}^{X^n}X$. Actually, we only need isomorphisms ${}^X({}^{X^n}X) \to {}^{X^n}X$ and ${}^{X^*}({}^{X^n}X) \to {}^{X^n}X$, and we can take the arrows ${}^{X^n}z^{-1}$ and ${}^{X^n}({}^{X^*}z)$. Similarly, one can prove that for any object Y there are isomorphisms ${}^Y({}^{X^{*n}}X) \to {}^{X^{*n}}X$, ${}^Y({}^{X^n}X^*) \to {}^{X^n}X^*$ and ${}^Y({}^{X^{*n}}X^*) \to {}^{X^{*n}}X^*$. Finally, we observe that for any object Y, Z and W, an isomorphism ${}^Y(Z \otimes W) = {}^Y Z \otimes {}^Y W \to Z \otimes W$ is obtained from ${}^Y Z \to Z$ amd ${}^Y W \to W$. Hence the category $\mathcal{F}$ is connected, and the categories $\mathcal{L}_{\mathcal{F}}(X)$ and $\mathcal{L}_{\mathcal{F}}(X^*)$ become crossed **Z**-categories. Moreover, the condition $g(X,X) = z^{-1}$ is satisfied. Thus we can use the universality of the category $\mathcal{F}$ and Lemma 1 to get crossed **Z**-functors $\mathcal{F} \to \mathcal{L}_{\mathcal{F}}(X)$ and $\mathcal{F} \to \mathcal{L}_{\mathcal{F}}(X^*)$. In particular, these functors are tensor functors, hence by Proposition 2 (b) we obtain the natural isomorphisms $c_{X,-} : X \otimes - \to {}^{X} - \otimes X$ and $c_{X^*,-} : X^* \otimes - \to {}^{X^*} - \otimes X^*$. □

Lemma 2 *The pair (y,z) becomes a tortile YB-operator on the object $(X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$. Also, the object $(X^*, c_{X^*,-})$ is left dual to the object $(X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$.*

Proof. To obtain a tortile YB-operator on the object $(X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$, we first consider the centralizer $\mathcal{L}_{\mathcal{F}}(y)$ of $y : X \otimes X \to {}^{X}X \otimes X$. Then we see that the category $\mathcal{L}_{\mathcal{F}}(y)$ contains the objects (X, α, β) and (X^*, γ, δ) where $\alpha = (y' \otimes 1)(1 \otimes y)$, $\beta = ({}^{X}y \otimes 1)(1 \otimes y)$, $\gamma = ({}^{X}z^{*-1} \otimes 1)(u^{-1} \otimes 1)(1 \otimes z^* \otimes 1)(1 \otimes u^{-1})$, $\delta = ({}^{X}u^{-1} \otimes 1)(1 \otimes u^{-1})$. Moreover, the object (X^*, γ, δ) is dual to (X, α, β), and we obtain a tortile YB-operator (y,z) on (X, α, β) in $\mathcal{L}_{\mathcal{F}}(y)$. Since $\mathcal{L}_{\mathcal{F}}(y)$ is a crossed **Z**-category, the universal property of $\mathcal{F}$ induces a crossed tensor functor $\mathcal{F} \to \mathcal{L}_{\mathcal{F}}(y)$, which is a section of the projection $\mathcal{L}_{\mathcal{F}}(y) \to \mathcal{F}$. In particular, this functor is a tensor functor, hence by Proposition 2 (b), we obtain an arrow $y : (X, c_{X,-}) \otimes (X, c_{X,-}) \to {}^{(X,c_{X,-})}(X, c_{X,-}) \otimes (X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$.

Next consider the category $\mathcal{L}_{\mathcal{F}}(z)$. Then we see that the category $\mathcal{L}_{\mathcal{F}}(z)$ contains the objects (X, y, η) and (X^*, u^{-1}, ζ) where $\eta = (1 \otimes z)y(z^{-1} \otimes 1)$ and $\zeta = (1 \otimes z)u^{-1}(z^{-1} \otimes 1)$. In addition, (X^*, u^{-1}, ζ) is dual to (X, y, η), and we obtain a tortile YB-operator (y,z) on the objects (X, y, η) in $\mathcal{L}_{\mathcal{F}}(z)$. Thus we have a crossed tensor functor $\mathcal{F} \to \mathcal{L}_{\mathcal{F}}(z)$, which corresponds to an arrow $z : (X, c_{X,-}) \to {}^{(X,c_{X,-})}(X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$.

Then it is easy to see that the arrows $y : (X, c_{X,-}) \otimes (X, c_{X,-}) \to {}^{(X,c_{X,-})}(X, c_{X,-}) \otimes (X, c_{X,-})$ and $z : (X, c_{X,-}) \to {}^{(X,c_{X,-})}(X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$ define a tortile YB-operator on the object $(X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$.

Finally, we check that the object $(X^*, c_{X^*,-})$ is left dual to the object $(X, c_{X,-})$ in $\mathcal{L}_{\mathcal{F}}$. For this we consider the centralizers $\mathcal{L}_{\mathcal{F}}(d_X)$ and $\mathcal{L}_{\mathcal{F}}(b_X)$. We see that the category $\mathcal{L}_{\mathcal{F}}(d_X)$ contains the objects (X, τ, id_X) and $(X^*, \sigma, \mathrm{id}_{X^*})$ where $\tau = ({}^{X^*}z \otimes 1 \otimes 1)(v \otimes 1)(1 \otimes z^{-1} \otimes 1)(1 \otimes y)$ and $\sigma = ({}^{X^*}z^{*-1} \otimes 1 \otimes 1)(w \otimes 1)(1 \otimes z^* \otimes 1)(1 \otimes u^{-1})$. Moreover, the object $(X^*, \sigma, \mathrm{id}_{X^*})$ is dual to (X, τ, id_X), and (y,z) defines a tortile YB-operator on the object (X, τ, id_X) in $\mathcal{L}_{\mathcal{F}}(d_X)$. Thus we obtain a crossed tensor functor $\mathcal{F} \to$

$\mathcal{L}_{\mathcal{F}}(d_X)$, which corresponds to an arrow $(X^*, c_{X^*,-}) \otimes (X, c_{X,-}) \to (I, \mathrm{id})$ in $\mathcal{L}_{\mathcal{F}}$. Similarly, the category $\mathcal{L}_{\mathcal{F}}(b_X)$ contains the objects (X, μ, id_X) and $(X^*, \nu, \mathrm{id}_{X^*})$ where $\mu = (z^{-1} \otimes 1 \otimes 1)(y \otimes 1)(1 \otimes {}^{X^*}z \otimes 1)(1 \otimes v)$ and $\nu = (z^* \otimes 1 \otimes 1)(u^{-1} \otimes 1)(1 \otimes {}^{X^*}z^{*-1} \otimes 1)(1 \otimes w)$, which induces a crossed tensor functor $\mathcal{F} \to \mathcal{L}_{\mathcal{F}}(b_X)$ and hence an arrow $(I, \mathrm{id}) \to (X, c_{X,-}) \otimes (X^*, c_{X^*,-})$ in $\mathcal{L}_{\mathcal{F}}$. □

Using the lemma above, we can prove the existence of the braiding in Theorem. In fact, since (y, z) is a tortile YB-operator on the object $(X, c_{X,-})$, and $(X^*, c_{X^*,-})$ is dual to the object $(X, c_{X,-})$, we obtain a crossed tensor functor $\mathcal{F} \to \mathcal{L}_{\mathcal{F}}$. Thus by Proposition 2 (c), we see that $\mathcal{F}$ has a braiding c with $c_{X,X} = y$.

Next we consider the existence of a twist θ in $\mathcal{F}$. For any braided crossed π-category $\mathcal{C}$, let $\mathcal{C}'$ be the category whose objects are pairs (U, ξ) where $\xi : U \to {}^U U$ is an isomorphism in $\mathcal{C}$, and whose arrows $f : (U, \xi) \to (V, \zeta)$ are arrows $f : U \to V$ in $\mathcal{C}$ such that ${}^U f \xi = \zeta f$. We can define a tensor product and a cross action on $\mathcal{C}'$ by putting

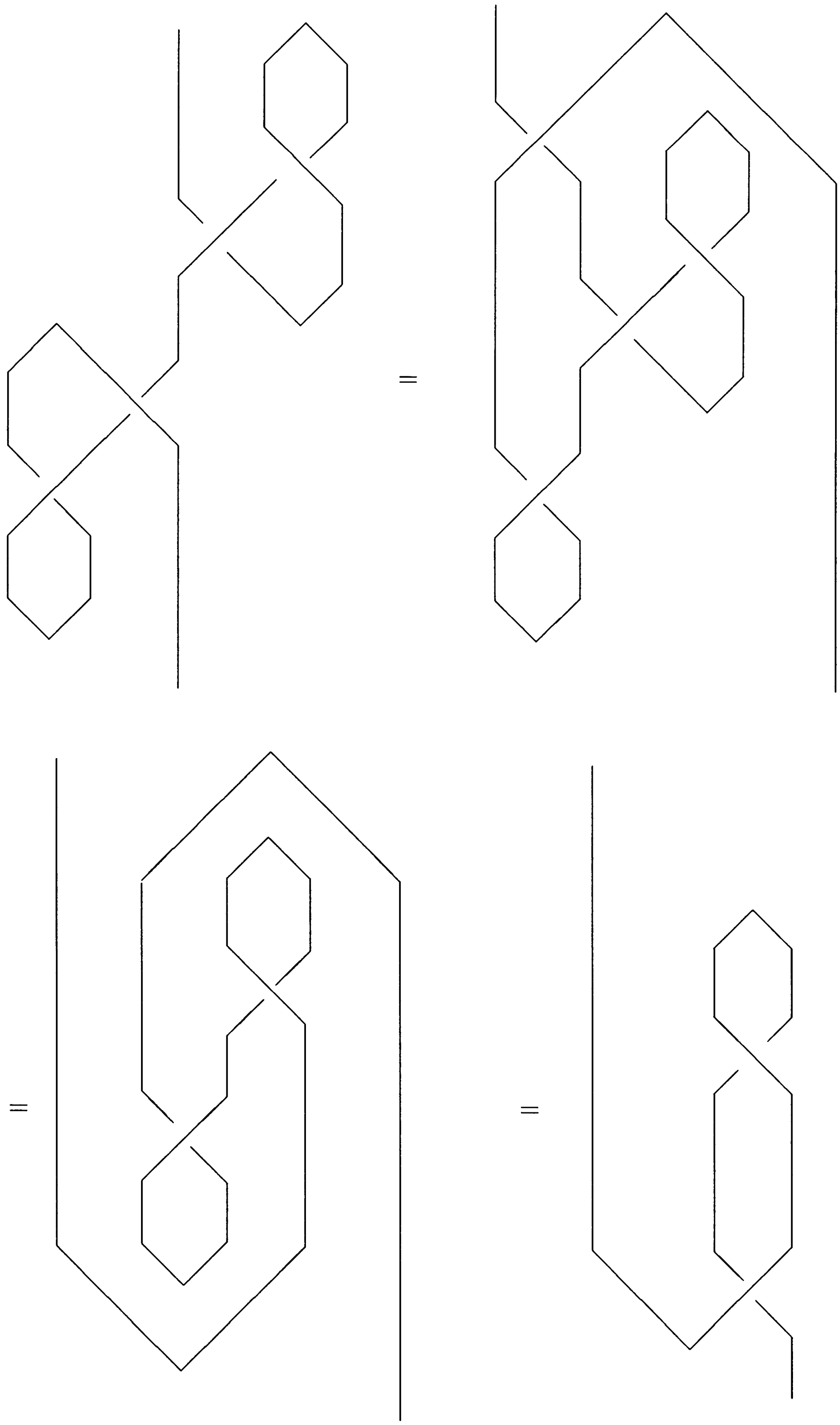
$$(U, \xi) \otimes (V, \zeta) = (U \otimes V, \chi) \quad \text{and} \quad {}^{(U,\xi)}(V, \zeta) = ({}^U V, {}^U \zeta)$$

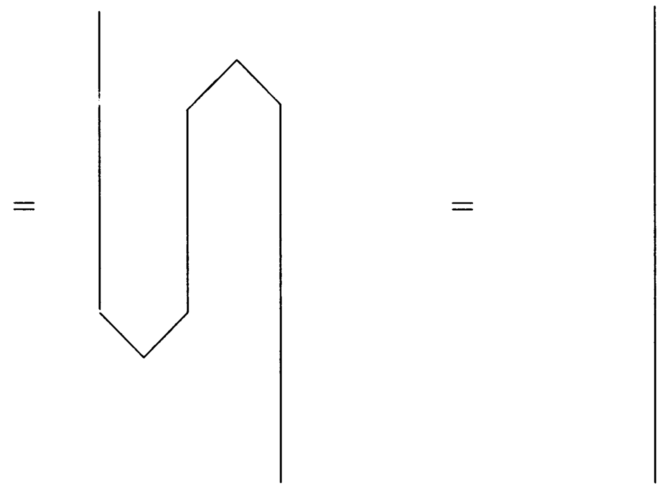
where $\chi = c_{U({}^V V), {}^U U} c_{U_U, {}^V V}(\xi \otimes \zeta)$. This tensor product makes $\mathcal{C}'$ into a crossed π-category. Applying this procedure to the braided crossed $\mathbf{Z}$-category $\mathcal{F}$, we obtain a crossed $\mathbf{Z}$-category $\mathcal{F}'$. To get a left dual for the object (X, z) in $\mathcal{F}'$, we use the following lemma:

Lemma 3 *The following identity holds in $\mathcal{F}$.*

$$\begin{aligned} z\, {}^{X^*}z\, (d_{X_X} \otimes 1)(1 \otimes c^{-1}_{X_X, X^*_X})(c_{X,X^*} \otimes 1)(b_X \otimes 1) \\ = 1 : {}^X X \to {}^X X. \end{aligned}$$

Proof. Since z is a tortile YB-operator the term $z\, {}^{X^*}z$ coincides with the composition $(1 \otimes d_X)(1 \otimes c^{-1}_{X^*,X})(c_{X^*_X, X} \otimes 1) b_X$. Then the following equalities show the identity in Lemma 3.





□

Using the lemma above, we can prove the following one:

Lemma 4 *The object* $(X^*, {}^{X^*}z^*)$ *is dual to the object* (X, z) *in* $\mathcal{F}'$*, and* (y, z) *defines a tortile* YB*-operator on* (X, z)*.*

Proof. We have to show that the arrows $d_X : X^* \otimes X \to I$ and $b_X : I \to X \otimes X^*$ define arrows $(X^*, {}^{X^*}z^*) \otimes (X, z) \to (I, \mathrm{id})$ and $(I, \mathrm{id}) \to (X, z) \otimes (X^*, {}^{X^*}z^*)$ in $\mathcal{F}'$. The next caluculations show that the arrow $d_X : X^* \otimes X \to I$ is an arrow $(X^*, {}^{X^*}z^*) \otimes (X, z) \to (I, \mathrm{id})$ in $\mathcal{F}'$.

$$
\begin{aligned}
& d_X(c_{X,X^*X^*})(c_{X^*X^*,XX})({}^{X^*}z^* \otimes z) \\
& = d_X c_{X,X^*X^*}(d_{X^*X} \otimes 1 \otimes 1)(1 \otimes c^{-1}_{X^*X,X} \otimes 1)(1 \otimes 1 \otimes b_{X^*X})({}^{X^*}z^* \otimes z) \\
& = d_{X^*X}(1 \otimes 1 \otimes d_X)(1 \otimes 1 \otimes c_{X,X^*X^*})(1 \otimes c^{-1}_{X^*X,X} \otimes 1) \\
& \quad (1 \otimes 1 \otimes b_{X^*X})({}^{X^*}z^* \otimes z) \\
& = d_{X^*X}(1 \otimes 1 \otimes d_X)(1 \otimes c^{-1}_{X^*X,X^*} \otimes 1)(1 \otimes c_{X^*X,X^*} \otimes 1) \\
& \quad (1 \otimes 1 \otimes c_{X,X^*X^*})(1 \otimes c^{-1}_{X^*X,X} \otimes 1)(1 \otimes 1 \otimes b_{X^*X})({}^{X^*}z^* \otimes z) \\
& = d_{X^*X}(1 \otimes d_{XX} \otimes 1)(1 \otimes 1 \otimes c_{X^*X,X})(1 \otimes c_{X^*X,X^*} \otimes 1) \\
& \quad (1 \otimes 1 \otimes c_{X,X^*X^*})(1 \otimes c^{-1}_{X^*X,X} \otimes 1)(1 \otimes 1 \otimes b_{X^*X})({}^{X^*}z^* \otimes z) \\
& = d_{X^*X}(1 \otimes d_{XX} \otimes 1)(1 \otimes c_{XX,X^*} \otimes 1)(1 \otimes 1 \otimes c_{X^*X,X^*X^*}) \\
& \quad (1 \otimes c_{X^*X,X} \otimes 1)(1 \otimes c^{-1}_{X^*X,X} \otimes 1)(1 \otimes 1 \otimes b_{X^*X})({}^{X^*}z^* \otimes z) \\
& = d_{X^*X}(1 \otimes d_{XX} \otimes 1)(1 \otimes c_{XX,X^*} \otimes 1)(1 \otimes 1 \otimes c_{X^*X,X^*X^*}) \\
& \quad (1 \otimes 1 \otimes b_{X^*X})({}^{X^*}z^* \otimes z)
\end{aligned}
$$

$$= d_{X^*X}(1 \otimes d_{X_X} \otimes 1)(1 \otimes 1 \otimes c^{-1}_{X_{X,X^*X}})(1 \otimes c_{X,X^*} \otimes 1)$$
$$(1 \otimes b_X \otimes 1)({}^{X^*}z^* \otimes z)$$
$$= d_{X^*X}(1 \otimes {}^{X^*}z^{-1})({}^{X^*}z^* \otimes 1)$$
$$= d_{X^*X}(1 \otimes {}^{X^*}z^{-1})(d_X \otimes 1 \otimes 1)(1 \otimes {}^{X^*}z \otimes 1 \otimes 1)(1 \otimes b_{X^*X} \otimes 1)$$
$$= d_{X^*X}(d_X \otimes 1 \otimes 1)(1 \otimes {}^{X^*}z \otimes 1 \otimes 1)(1 \otimes b_{X^*X} \otimes 1)(1 \otimes {}^{X^*}z^{-1})$$
$$= d_X(1 \otimes 1 \otimes d_{X^*X})(1 \otimes {}^{X^*}z \otimes 1 \otimes 1)(1 \otimes b_{X^*X} \otimes 1)(1 \otimes {}^{X^*}z^{-1})$$
$$= d_X(1 \otimes {}^{X^*}z)(1 \otimes 1 \otimes d_{X^*X})(1 \otimes b_{X^*X} \otimes 1)(1 \otimes {}^{X^*}z^{-1})$$
$$= d_X(1 \otimes {}^{X^*}z)(1 \otimes {}^{X^*}z^{-1})$$
$$= d_X.$$

Similarly, one can prove that the arrow $b_X : I \to X \otimes X^*$ defines an arrow $(I, \mathrm{id}) \to (X, z) \otimes (X^*, {}^{X^*}z^*)$ in $\mathcal{F}'$. □

Using the lemma above, we obtain a crossed tensor functor $\mathcal{F} \to \mathcal{F}'$ which takes X to (X, z) and takes X^* to $(X^*, {}^{X^*}z^*)$. Then for any object U in $\mathcal{F}$, the value (U, θ_U) of this tensor functor at U gives the twist $\theta : U \to {}^UU$. The uniqueness of the braiding and the twist follows from the same argument in [2]. Let c, c' be braidings on $\mathcal{F}$ such that $c_{X,X} = y = c'_{X,X}$. For any object U of $\mathcal{F}$, let $\mathcal{E}(U)$ be the set of objects Z for which $c_{U,Z} = c'_{U,Z}$, and let $\mathcal{E}$ be the set of objects U for which $\mathcal{E}(U) = \mathrm{obj}\,\mathcal{F}$. Then using the connectivity structure on $\mathcal{F}$, we see that both $\mathcal{E}(U)$ and $\mathcal{E}$ are closed under tensor product and crossed action. Moreover, using the fact that $u = c^{-1}_{X,X^*}$, $v = c_{X^*,X}$ and $w = c_{X^*,X^*}$, we see that $X, X^* \in \mathcal{E}(X)$ and $X, X^* \in \mathcal{E}(X^*)$. Thus $X, X^* \in \mathcal{E}$, and since X, X^* generate $\mathrm{obj}\,\mathcal{F}$, we have $\mathcal{E} = \mathrm{obj}\,\mathcal{F}$. Similarly, one can prove the uniqueness of twist. This completes our proof of Theorem.

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10-14 Yanagida, Kashiwagimachi
Hiraka-machi, Minamitsugaru-gun
Aomori 036-0104, Japan
E-mail: colour@mrh.biglobe.ne.jp