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Extensions and the irreducibilities of induced characters of some 2-groups

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Abstract. Let Q_n and D_n denote the generalized quaternion group and the dihedral group of order 2^{n+1} $(n \geq 2)$, respectively. Let SD_n denote the semidihedral group of order 2^{n+1} $(n \geq 3)$. Let ϕ be a faithful irreducible character of H, where $H = Q_n$ or D_n or SD_n. The purpose of this paper is to determine all 2-groups G such that $H \subset G$ and the induced character ϕ^G is also irreducible.

Key words: 2-group, induced character, faithful irreducible character, group extension.

1. Introduction

Let Q_n and D_n denote the generalized quaternion group and the dihedral group of order 2^{n+1} $(n \geq 2)$, respectively. Let SD_n denote the semidihedral group of order 2^{n+1} $(n \geq 3)$.

As is stated in [4], these groups have remarkable properties among all 2-groups.

Moreover, Yamada and Iida [5] proved the following interesting result:

Let $\mathbf{Q}$ denote the rational field. Let G be a 2-group and χ a complex irreducible character of G. Then there exist subgroups $H \rhd N$ in G and the complex irreducible character ϕ of H such that $\chi = \phi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\phi)$, $N = \operatorname{Ker}\phi$ and

$$H/N \cong Q_n \ (n \geq 2), \quad \text{or} \quad D_n \ (n \geq 3), \quad \text{or} \quad SD_n \ (n \geq 3),$$
$$\text{or} \quad C_n \ (n \geq 0),$$

where C_n is the cyclic group of order 2^n, and $\mathbf{Q}(\chi) = \mathbf{Q}(\chi(g),\ g \in G)$.

In [4], Yamada and Iida considered the case when $N = 1$, or equivalently ϕ is faithful. They studied the following problem:

Problem *Let ϕ be a faithful irreducible character of H, where $H = Q_n$ or D_n or SD_n. Determine the 2-group G such that $H \subset G$ and the induced character ϕ^G is also irreducible.*

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It is well-known that the groups Q_n, D_n and SD_n have faithful irreducible characters. It is also known that they are algebraically conjugate to each other. Hence the irreducibility of ϕ^G, where ϕ is a faithful irreducible character of $H = Q_n$ or D_n or SD_n, does not depend on the particular choice of ϕ, but depends only on these groups.

This problem has been solved in each of the following cases:

(1) When $[G : H] = 2$ or 4 ([4]),

(2) When $[G : H] = 8$ ([6]),

(3) When H is a normal subgroup of G ([3]),

for all $H = Q_n$ or D_n or SD_n.

The purpose of this paper is to give a complete answer to this problem for all $H = Q_n$ or D_n or SD_n.

For other results concerning this problem, see [2].

In this paper, we will frequently use the word "respectively" so it is abbreviated to "resp.".

2. Statements of the results

We use the following notation throught this paper.

- The dihedral group $D_n = \langle a,\ b\rangle$ $(n \geq 2)$ with

$$a^{2^n} = 1, \quad b^2 = 1, \quad bab^{-1} = a^{-1}.$$

- The generalized quaternion group $Q_n = \langle a,\ b\rangle$ $(n \geq 2)$ with

$$a^{2^n} = 1, \quad b^2 = a^{2^{n-1}}, \quad bab^{-1} = a^{-1}.$$

- The semidihedral group $SD_n = \langle a,\ b\rangle$ $(n \geq 3)$ with

$$a^{2^n} = 1, \quad b^2 = 1, \quad bab^{-1} = a^{-1+2^{n-1}}.$$

To state our results, we have to introduce the following groups:

(1) $D(n,m) = \langle a,\ b,\ u_m\rangle$ $(\triangleright D_n = \langle a,\ b\rangle)$ $(1 \leq m \leq n-2)$ with

$$a^{2^n} = b^2 = u_m^{2^m} = 1, \quad bab^{-1} = a^{-1}, \quad u_m a u_m^{-1} = a^{1+2^{n-m}},$$
$$u_m b = b u_m.$$

(2) $Q(n,m) = \langle a,\ b,\ u_m\rangle$ $(\triangleright Q_n = \langle a,\ b\rangle)$ $(1 \leq m \leq n-2)$ with

$$a^{2^n} = u_m^{2^m} = 1, \quad b^2 = a^{2^{n-1}}, \quad bab^{-1} = a^{-1}, \quad u_m a u_m^{-1} = a^{1+2^{n-m}},$$
$$u_m b = b u_m.$$

(3) $D_0(n,1,1) = \langle a,\ b,\ u_1,\ x\rangle$ ($\triangleright D(n,1) = \langle a,\ b,\ u_1\rangle$) with

$$a^{2^n} = b^2 = u_1^2 = x^2 = 1,\quad bab^{-1} = a^{-1},\quad u_1au_1^{-1} = a^{1+2^{n-1}},$$
$$u_1b = bu_1,\quad xax^{-1} = au_1,\quad xbx^{-1} = bu_1,\quad u_1x = xu_1.$$

(4) $Q_0(n,1,1) = \langle a,\ b,\ u_1,\ x\rangle$ ($\triangleright Q(n,1) = \langle a,\ b,\ u_1\rangle$) with

$$a^{2^n} = u_1^2 = x^2 = 1,\ b^2 = a^{2^{n-1}},\quad bab^{-1} = a^{-1},\quad u_1au_1^{-1} = a^{1+2^{n-1}},$$
$$u_1b = bu_1,\quad xax^{-1} = au_1,\quad xbx^{-1} = a^{2^{n-1}}bu_1,\quad u_1x = xu_1.$$

(5) $D(n,m,1) = \langle a,\ b,\ u_m,\ x\rangle$ ($\triangleright D(n,m) = \langle a,\ b,\ u_m\rangle$) $(2 \leq m \leq n-3)$ with

$$a^{2^n} = b^2 = u_m^{2^m} = 1,\quad bab^{-1} = a^{-1},\quad u_mau_m^{-1} = a^{1+2^{n-m}},$$
$$u_mb = bu_m,\quad xax^{-1} = a^{1+2^{n-m-1}}u_m^{2^{m-1}},\quad xbx^{-1} = bu_m^{2^{m-1}},$$
$$xu_mx^{-1} = u_m,\ x^2 = u_m^{e_m},$$

where e_m is an odd integer defined by the relation,

$$(1+2^{n-m})^{e_m} \equiv (1+2^{n-m-1})^2 \pmod{2^n}.$$

(6) $Q(n,m,1) = \langle a,\ b,\ u_m,\ x\rangle$ ($\triangleright Q(n,m) = \langle a,\ b,\ u_m\rangle$) $(2 \leq m \leq n-3)$ with

$$a^{2^n} = u_m^{2^m} = 1,\ b^2 = a^{2^{n-1}},\quad bab^{-1} = a^{-1},\quad u_mau_m^{-1} = a^{1+2^{n-m}},$$
$$u_mb = bu_m,\quad xax^{-1} = a^{1+2^{n-m-1}}u_m^{2^{m-1}},\quad xbx^{-1} = bu_m^{2^{m-1}},$$
$$xu_mx^{-1} = u_m,\quad x^2 = u_m^{e_m},$$

where e_m is an odd integer defined by the relation,

$$(1+2^{n-m})^{e_m} \equiv (1+2^{n-m-1})^2 \pmod{2^n}.$$

Remark (1) Later, in the proof of Theorem 1, Case II, we will note that the elements $u_m^{e_m}$ defined in (5) and (6) are uniquely determined, so the groups $D(n,m,1)$ and $Q(n,m,1)$ are uniquely determined for each integers n and m

(2) Note that some of the notations used in this paper are different from those used in [4] and [6]. For example, we use the notation $D_0(n,1,1)$ and $Q_0(n,1,1)$ instead of $G_2^{(2)}(D_n)$ and $G_2^{(2)}(Q_n)$ in [4].

For a finite group G, we denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G and by $\mathrm{FIrr}(G)$ ($\subset \mathrm{Irr}(G)$) the set of faithful irreducible characters of G.

Yamada and Iida ([4]) proved the following

Theorem 0.1 ([4, Theorems 5 and 6]) (1) *Let* $n \geq 4$ *and* $\phi \in \mathrm{FIrr}(D_n)$. *Let* G *be a 2-group such that* $D_n \subset G$ *and* $[G : D_n] = 2^2$. *Suppose that* $\phi^G \in \mathrm{Irr}(G)$, *then* $G \cong D(n, 2)$ *or* $D_0(n, 1, 1)$.

(2) *Let* $n \geq 4$ *and* $\phi \in \mathrm{FIrr}(Q_n)$. *Let* G *be a 2-group such that* $Q_n \subset G$ *and* $[G : Q_n] = 2^2$. *Suppose that* $\phi^G \in \mathrm{Irr}(G)$, *then* $G \cong Q(n, 2)$ *or* $Q_0(n, 1, 1)$.

(3) *Let* $n \geq 4$ *and* $\phi \in \mathrm{FIrr}(SD_n)$. *Let* G *be a 2-group such that* $SD_n \subset G$ *and* $[G : SD_n] = 2^2$. *Suppose that* $\phi^G \in \mathrm{Irr}(G)$, *then* $G \cong Q(n, 2)$ *or* $Q_0(n, 1, 1)$ *or* $D(n, 2)$ *or* $D_0(n, 1, 1)$.

Further, Iida ([3]) proved the following

Theorem 0.2 ([3, Theorem 7]) (1) *Let* $\phi \in \mathrm{FIrr}(D_n)$. *Let* G *be a 2-group such that* $D_n \subsetneqq G$ *and* $D_n \lhd G$. *Suppose that* $\phi^G \in \mathrm{Irr}(G)$, *then* $G \cong D(n, m)$ *for some integer* m, $1 \leq m \leq n - 2$.

(2) *Let* $\phi \in \mathrm{FIrr}(Q_n)$. *Let* G *be a 2-group such that* $Q_n \subsetneqq G$ *and* $Q_n \lhd G$. *Suppose that* $\phi^G \in \mathrm{Irr}(G)$, *then* $G \cong Q(n, m)$ *for some integer* m, $1 \leq m \leq n - 2$.

(3) *Let* $\phi \in \mathrm{FIrr}(SD_n)$. *Let* G *be a 2-group such that* $SD_n \subsetneqq G$ *and* $SD_n \lhd G$. *Suppose that* $\phi^G \in \mathrm{Irr}(G)$, *then* $G \cong Q(n, m)$ *or* $D(n, m)$ *for some integer* m, $1 \leq m \leq n - 2$.

On the other hand, we have shown the following

Proposition 0.3 ([6, Theorems 1 and 2, Case II])

(1) *Let* $\phi \in \mathrm{FIrr}(D_n)$, *and let* G *be a 2-group such that* $D_0(n, 1, 1) \subsetneqq G$. *Then* $\phi^G \notin \mathrm{Irr}(G)$.

(2) *Let* $\phi \in \mathrm{FIrr}(Q_n)$, *and let* G *be a 2-group such that* $Q_0(n, 1, 1) \subsetneqq G$. *Then* $\phi^G \notin \mathrm{Irr}(G)$.

Our main theorems are the following

Theorem 1 *Let* $\phi \in \mathrm{FIrr}(D_n)$. *Suppose that* G *is a 2-group such that* $D_n \subset G$, $\phi^G \in \mathrm{Irr}(G)$ *and* $[G : D_n] = 2^m$. *Then*

(1) $m \leq n - 2$,

(2) $G \cong D(n, 1)$ *if* $m = 1$.

(3) $G \cong D(n, 2)$ *or* $D_0(n, 1, 1)$ *if* $m = 2$.

(4) $G \cong D(n, m)$ *or* $D(n, m - 1, 1)$ *if* $3 \leq m \leq n - 2$.

Theorem 2 *Let $\phi \in \mathrm{FIrr}(Q_n)$. Suppose that G is a 2-group such that $Q_n \subset G$, $\phi^G \in \mathrm{Irr}(G)$ and $[G : Q_n] = 2^m$. Then*
(1) $m \leq n-2$,
(2) $G \cong Q(n,1)$ *if* $m = 1$.
(3) $G \cong Q(n,2)$ *or* $Q_0(n,1,1)$ *if* $m = 2$.
(4) $G \cong Q(n,m)$ *or* $Q(n,m-1,1)$ *if* $3 \leq m \leq n-2$.

Theorem 3 *Let $\phi \in \mathrm{FIrr}(SD_n)$. Suppose that G is a 2-group such that $SD_n \subset G$, $\phi^G \in \mathrm{Irr}(G)$ and $[G : SD_n] = 2^m$. Then*
(1) $m \leq n-2$,
(2) $G \cong D(n,1)$ *or* $Q(n,1)$ *if* $m = 1$.
(3) $G \cong D(n,2)$ *or* $Q(n,2)$ *or* $D_0(n,1,1)$ *or* $Q_0(n,1,1)$ *if* $m = 2$.
(4) $G \cong D(n,m)$ *or* $Q(n,m)$ *or* $D(n,m-1,1)$ *or* $Q(n,m-1,1)$ *if* $3 \leq m \leq n-2$.

To prove the theorems, we need some results concerning the criterion of the irreducibility of induced characters.

We denote by $\zeta = \zeta_{2^n}$ a primitive 2^nth root of unity. It is known that, for $H = Q_n$ or D_n, there are $2^{n-1} - 1$ irreducible characters ϕ_ν $(1 \leq \nu < 2^{n-1})$ of H, which are not linear:

$$\phi_\nu(a^i) = \zeta^{\nu i} + \zeta^{-\nu i}, \quad \phi_\nu(a^i b) = 0 \quad (1 \leq i \leq 2^n).$$

For $H = SD_n$, there are $2^{n-1} - 1$ irreducible characters ϕ_ν $(-2^{n-2} \leq \nu \leq 2^{n-2}$ for odd ν, $1 \leq \nu < 2^{n-1}$ for even $\nu)$ of H, which are not linear:

$$\phi_\nu(a^i) = \zeta^{\nu i} + \zeta^{\nu i(-1+2^{n-1})}, \quad \phi_\nu(a^i b) = 0 \quad (1 \leq i \leq 2^n).$$

Each irreducible character ϕ_ν of Q_n or D_n or SD_n is induced from a linear character η_ν of the maximal normal cyclic subgroup $\langle a \rangle$:

$$\eta_\nu(a^i) = \zeta^{\nu i} \quad (1 \leq i \leq 2^n).$$

Therefore, for a group $G \supset H = D_n$, or Q_n or SD_n ϕ_ν^G is irreducible if and only if $\eta_\nu^G = (\eta_\nu^H)^G$ is irreducible. For $H = Q_n$ or D_n or SD_n, an irreducible character ϕ_ν of H is faithful if and only if ν is odd. The faithful irreducible characters ϕ_ν of H are algebraically conjugate to each other.

We need the following result of Shoda (cf. [1, p.329]):

Proposition 0.4 *Let G be a group and H be a subgroup of G. Let ϕ be a linear character of H. Then the induced character ϕ^G of G is irreducible* *if and only if, for each* $x \in G - H = \{g \in G \mid g \notin H\}$, *there exists* $h \in xHx^{-1} \cap H$ *such that* $\phi(h) \neq \phi(x^{-1}hx)$. *In particular, when* ϕ *is faithful, the condition* $\phi(h) \neq \phi(x^{-1}hx)$ *is equivalent to that of* $h \neq x^{-1}hx$.

Using this result, we have the following:

Proposition 0.5 *Let* $\langle a \rangle \subset H \subset G$, *where* $H = D_n$ *or* Q_n *or* SD_n *and* $\langle a \rangle$ *is a maximal normal cyclic subgroup of* H. *Let* ϕ *be a faithful irreducible character of* H. *Then the following conditions are equivalent*

(1) ϕ^G *is irreducible.*

(2) *For each* $x \in G - \langle a \rangle$, *there exists* $y \in \langle a \rangle \cap x\langle a \rangle x^{-1}$ *such that* $xyx^{-1} \neq y$.

Definition When the condition (2) of Proposition 0.5 holds, we say that G satisfies (EX, H), where $H = D_n$ or Q_n or SD_n.

Remark It is easy to see that the groups $D_0(n,1,1)$, $D(n,m)$ and $D(n,m,1)$ (resp. $Q_0(n,1,1)$, $Q(n,m)$ and $Q(n,m,1)$) satisfy (EX, D_n) (resp. (EX, Q_n)). It is also easy to see that $D_0(n,1,1)$, $D(n,m)$, $D(n,m,1)$, $Q_0(n,1,1)$, $Q(n,m)$ and $Q(n,m,1)$ satisfy (EX, SD_n).

3. Proof of Theorem 1

Let G be a 2-group, satisfying the conditions of Theorem 1. As usual, we denote by $N_G(H)$, the normalizer of H in G for a subgroup H of G. We define the subgroups N_i, $i = 1, 2$, of G as follows:

$$N_1 = N_G(D_n), \quad N_2 = N_G(N_1).$$

By Theorem 0.2, we have

$$N_1 = D(n,z) = \langle a,\ b,\ u_z \rangle,$$

for some integer z, $1 \leq z \leq n-2$, and it is easy to see that $D(n,z)/D_n \cong C_z$. Hence we have only to consider the case where $N_1 \subsetneqq G$. In this case, we have $N_1 \subsetneqq N_G(N_1) = N_2$, since G is a 2-group.

First, we show the following

Claim I Suppose that $N_1 = D(n,z) \subsetneqq G$, then $z \leq n-3$.

Proof of Claim I. Suppose that $N_1 \subsetneqq N_2$ and $z = n-2$.
Let $x \in N_2 - N_1$. Then, by the condition (EX, D_n), there exist an integer

t, $0 \le t \le n-1$, and $y \in \langle a^{2^t} \rangle$, such that the following conditions hold:

$$\langle a \rangle \cap x\langle a \rangle x^{-1} = \langle a^{2^t} \rangle \quad \text{and} \quad xyx^{-1} \ne y.$$

It is well-known that

$$\mathrm{Aut}\langle a \rangle \cong (\mathbf{Z}/2^n\mathbf{Z})^* = \langle -1 \rangle \times \langle 5 \rangle \cong C_1 \times C_{n-2}$$

where $(\mathbf{Z}/2^n\mathbf{Z})^*$ is the unit group of the factor ring $\mathbf{Z}/2^n\mathbf{Z}$ and $\langle -1 \rangle$ and $\langle 5 \rangle$ are the cyclic subgroups of $(\mathbf{Z}/2^n\mathbf{Z})^*$ generated by -1 and 5 respectively. Notice that C_1 is the cyclic group of order 2. Hence, when $z = n-2$, we have

$$\mathrm{Aut}\langle a \rangle \cong D(n, n-2)/\langle a \rangle \cong N_1/\langle a \rangle$$

Therefore there exists the element $v \in N_1$, such that

$$\langle a \rangle \cap (vx)\langle a \rangle (vx)^{-1} = \langle a^{2^t} \rangle$$

and vx acts trivially on $\langle a^{2^t} \rangle$ by conjugation. This contradicts the condition (EX, D_n). Hence the proof of Claim I is completed. □

Hereafter we may assume that $D(n,z) = N_1 \subsetneqq N_2$ and $z \le n-3$.

Let H be a group. For a normal subgroup N of H, and any g, $h \in H$, we write

$$g \equiv h \pmod{N}$$

when $g^{-1}h \in N$.

For an element $g \in H$ we denote by $|\, g \,|$ the order of g.

Now, we show the following

Claim II $N_2/N_1 = N_2/D(n,z) \cong C_1$.

Proof of Claim II. For the sake of simplicity, we write u instead of u_z. Note that any element in $D(n,z)$ is represented as $a^i u^j b^k$ where i, j, $k \in \mathbf{Z}$, $0 \le i \le 2^n - 1$, $0 \le j \le 2^z - 1$, $0 \le k \le 1$.

We need the following

Lemma 1 *For integers i, j and a positive integer s, the following equalities and inequality hold.*

(1) $(a^i u^j)^{2^s} = a^{i \cdot 2^s \cdot t_s} u^{2^s j}$, *for some odd integer t_s.*

(2) $(a^i u^{2^{z-1}})^2 = a^{2i(1+2^{n-2})}$.

(3) $(a^i u^{2^{z-1}})^{2^s} = a^{2^s i}$ *for* $2 \leq s$.
(4) $|\, a^i u^j b \,| \leq 2^{z+1}$.
(5) $a^i u^{2^{z-1}} \equiv u^{2^{z-1}} a^i \pmod{\langle a^{2^{n-1}} \rangle}$.

Proof of Lemma 1. (1) can be shown by induction on s.
(2), (3) and (5) can be shown by direct calculations. So we omit the proof.
(4) Since $(a^i u^j b)^2 = a^{-i2^{n-z}j_1} u^{2j}$ for some $j_1 \in \mathbf{Z}$, we have $(a^i u^j b)^{2^{z+1}} = 1$ by (1). □

Let $x \in N_2 - N_1$.

First, we consider the element xax^{-1}. Since $z \leq n-3$, by Claim I, and $|\, a^i u^j b \,| \leq 2^{z+1}$, by Lemma 1 (4), we must have

$$xax^{-1} = a^i u^j,$$

for some integers i, j. Further, since

$$(xax^{-1})^{2^z} = (a^i u^j)^{2^z} = a^{i \cdot 2^z \cdot t_z},$$

where t_z is an odd integer defined in Lemma 1 (1), i must be an odd integer. If $i \in \langle -1 \rangle \times \langle 5 \rangle - \langle 5 \rangle$, then $(bx)a(bx)^{-1} = a^{-i}u^j$ and $-i \in \langle 5 \rangle$. Hence we may assume that,

$$i \in \langle 5 \rangle. \tag{1}$$

Write $a_0 = xax^{-1}$ and $b_0 = xbx^{-1}$. Taking the conjugate of both sides of the equality, $bab^{-1} = a^{-1}$, by x, we get

$$b_0(a^i u^j)b_0^{-1} = u^{-j}a^{-i}.$$

Since

$$N_1/\langle a \rangle = D(n,z)/\langle a \rangle \cong C_1 \times C_z$$

is the abelian group, we have

$$b_0(a^i u^j)b_0^{-1} \equiv a^i u^j \pmod{\langle a \rangle}.$$

Hence we have

$$a^i u^j \equiv u^{-j}a^{-i} \pmod{\langle a \rangle}.$$

Therefore $u^{2j} = 1$. This means that we can write $j = 2^{z-1}j_0$, and

$$xax^{-1} = a^i u^{2^{z-1}j_0},$$

where $j_0 = 0$ or 1. If $j_0 = 0$, then

$$xax^{-1} = a^i$$

and

$$b_0 a^i b_0^{-1} = a^{-i}.$$

Since i is odd, we get

$$b_0 a b_0^{-1} = a^{-1}.$$

So, b_0 must be written as

$$b_0 = a^t b, \tag{2}$$

for some $t \in \mathbf{Z}$. Thus

$$xD_n x^{-1} = x\langle a,\ b\rangle x^{-1} = \langle a,\ b\rangle = D_n.$$

This contradicts the hypothesis that $x \in N_2 - N_1$. Hence we must have

$$xax^{-1} = a^i u^{2^{z-1}}.$$

Next, consider the element xux^{-1}. Write $u_0 = xux^{-1}$. Taking the conjugate of both sides of the equality, $ua^{2^z}u^{-1} = a^{2^z}$, by x, we get

$$u_0 a^{i\cdot 2^z\cdot t_z} u_0^{-1} = a^{i\cdot 2^z\cdot t_z},$$

where t_z is the odd integer defined in Lemma 1 (1). Since i is also odd, we can see that

$$u_0 a^{2^z} u_0^{-1} = a^{2^z}.$$

Suppose that $u_0 = a^{d_0}u^t b$ for some $d_0,\ t \in \mathbf{Z}$. Then

$$a^{2^z} = (a^{d_0}u^t b)(a^{2^z})(a^{d_0}u^t b)^{-1} = a^{-2^z}$$

So, $a^{2^{z+1}} = 1$, which contradicts the fact that $z + 3 \leq n$. Thus we must have $u_0 = a^{d_0}u^t$ for some $d_0,\ t \in \mathbf{Z}$.

But again by Lemma 1 (1),

$$1 = u_0^{2^z} = (a^{d_0}u^t)^{2^z} = a^{d_0 2^z t_z}.$$

So we have $d_0 \equiv 0 \pmod{2^{n-z}}$. Therefore we may write $d_0 = 2^{n-z}d$, and

$$xux^{-1} = a^{2^{n-z}d}u^t,$$

for some $d \in \mathbf{Z}$. Taking the conjugate of both sides of the equality, $uau^{-1} = a^{1+2^{n-z}}$, by x, we get

$$(a^{2^{n-z}d}u^t)(a^iu^{2^{z-1}})(a^{2^{n-z}d}u^t)^{-1}$$
$$= (a^iu^{2^{z-1}})^{1+2^{n-z}} = (a^iu^{2^{z-1}})(a^iu^{2^{z-1}})^{2^{n-z}} = a^{i(1+2^{n-z})}u^{2^{z-1}}.$$

Hence, we have

$$a^{i(1+2^{n-z})^t}u^{2^{z-1}} = a^{i(1+2^{n-z})}u^{2^{z-1}}.$$

Therefore,

$$i(1+2^{n-z})^t \equiv i(1+2^{n-z}) \pmod{2^n}.$$

Since i is odd, we get $t \equiv 1 \pmod{2^z}$, and hence

$$xux^{-1} = a^{2^{n-z}d}u.$$

Therefore, for any x_1, $x_2 \in N_2 - N_1$, we can write as follows:

$$x_1ax_1^{-1} = a^{i_1}u^{2^{z-1}} \quad \text{and} \quad x_1ux_1^{-1} = a^{2^{n-z}d_1}u,$$
$$x_2^{-1}ax_2 = a^{i_2}u^{2^{z-1}} \quad \text{and} \quad x_2^{-1}ux_2 = a^{2^{n-z}d_2}u,$$

where i_1, i_2, d_1, $d_2 \in \mathbf{Z}$ and i_1 and i_2 are odd. Using these relations, we have

$$(x_1x_2^{-1})a(x_1x_2^{-1})^{-1} = x_1(a^{i_2}u^{2^{z-1}})x_1^{-1}$$
$$= (a^{i_1}u^{2^{z-1}})^{i_2}(a^{2^{n-z}d_1}u)^{2^{z-1}} = (a^{i_1}u^{2^{z-1}})^{i_2}(a^{2^{n-1}d_1t_{z-1}}u^{2^{z-1}})$$

for some t_{z-1}, by Lemma 1 (1).

Therefore

$$(x_1x_2^{-1})a(x_1x_2^{-1})^{-1} \equiv 1 \pmod{\langle a \rangle}.$$

This means that

$$(x_1x_2^{-1})a(x_1x_2^{-1})^{-1} \in \langle a \rangle.$$

But, in this case, we also have

$$(x_1x_2^{-1})b(x_1x_2^{-1})^{-1} \in \langle a,\ b \rangle = D_n,$$

by the same argument as in (2). Hence, we have shown that

$$x_1x_2^{-1} \in N_1,$$

for any x_1, $x_2 \in N_2 - N_1$. Thus the proof of Claim II is completed. $\square$

Now, we will determine the group structure of N_2 ($\supsetneqq N_1 = D(n, z)$). We show the following

Claim III (1) $N_2 \cong D_0(n, 1, 1)$ ($\supsetneqq D(n, 1)$) if $z = 1$.
(2) $N_2 \cong D(n, z, 1)$ ($\supsetneqq D(n, z)$) if $2 \leq z \leq n - 3$.

Proof of Claim III. (1) When $z = 1$, the group N_2 has been considerd in [4], and the isomorphism of (1) follows from Theorem 0.1.

(2) Let $x \in N_2 - N_1$. Then

$$xax^{-1} = a^i u^{2^{z-1}} \notin \langle a \rangle,$$

and

$$xa^2x^{-1} = (a^i u^{2^{z-1}})^2 = a^{2i(1+2^{n-2})} \in \langle a \rangle.$$

Recall that we may assume

$$i \in \langle 5 \rangle,$$

by (1). Suppose that $i \in \langle 1 + 2^{n-z} \rangle$, then there exists $v \in N_1$, such that

$$(vx)a(vx)^{-1} \notin \langle a \rangle,$$

and

$$(vx)a^2(vx)^{-1} = a^2.$$

This contradicts the condition (EX, D_n).

Hence we must have

$$i \notin \langle 1 + 2^{n-z} \rangle. \tag{3}$$

On the other hand, we have

$$x^2ax^{-2} = x(a^i u^{2^{z-1}})x^{-1} = (a^i u^{2^{z-1}})^i (a^{2^{n-1} \cdot d \cdot t_{z-1}} u^{2^{z-1}})$$

for some t_{z-1} by Lemma 1 (1). Hence we have

$$x^2ax^{-2} \equiv a^{i^2} u^{2^{z-1} i} u^{2^{z-1}} = a^{i^2} \pmod{\langle a^{2^{n-1}} \rangle},$$

by using Lemma 1 (5). So we can write

$$x^2ax^{-2} = a^{i^2 + \beta \cdot 2^{n-1}}$$

where $\beta = 0$ or 1.

Since $x^2 \in D(n,z) = N_1$, we have

$$i^2 \in \langle 1 + 2^{n-z} \rangle, \tag{4}$$

where $\langle 1+2^{n-z}\rangle$ is the cyclic subgroup of $(\mathbf{Z}/2^n\mathbf{Z})^*$ generated by $1+2^{n-z}$. By (1), (3) and (4), we may write as

$$i = 1 + k \cdot 2^{n-z-1},$$

and

$$xax^{-1} = a^{1+k\cdot 2^{n-z-1}} u^{2^{z-1}},$$

for some odd integer k. In this case, we have

$$\begin{aligned} i^2 + \beta \cdot 2^{n-1} &= (1 + k \cdot 2^{n-z-1})^2 + \beta \cdot 2^{n-1} \\ &= 1 + (k + k^2 \cdot 2^{n-z-2} + \beta \cdot 2^{z-1}) \cdot 2^{n-z}. \end{aligned}$$

If we set $k_1 = k + k^2 \cdot 2^{n-z-2} + \beta \cdot 2^{z-1}$, then k_1 is the odd integer, since $3 \le n - z$. And we have

$$x^2 a x^{-2} = a^{1+k_1 2^{n-z}}. \tag{5}$$

Now, we consider the element x^2 ($\in D(n,z)$). By (5), x^2 must be written as

$$x^2 = a^{t_1} u^{l_1}$$

for some t_1, $l_1 \in \mathbf{Z}$. Since

$$a^{1+k_1 2^{n-z}} = x^2 a x^{-2} = (a^{t_1} u^{l_1}) a (a^{t_1} u^{l_1})^{-1} = a^{(1+2^{n-z})^{l_1}},$$

l_1 must be odd. On the other hand, since

$$\begin{aligned} a^{t_1} u^{l_1} &= x^2 = x x^2 x^{-1} = x (a^{t_1} u^{l_1}) x^{-1} \\ &= (a^{1+k\cdot 2^{n-z-1}} u^{2^{z-1}})^{t_1} (a^{2^{n-z} d} u)^{l_1}, \end{aligned}$$

we have

$$u^{l_1} \equiv u^{2^{z-1} t_1} u^{l_1} \pmod{\langle a \rangle}.$$

Therefore we can write as $t_1 = 2t_2$, and

$$x^2 = a^{2t_2} u^{l_1}$$

for some integer t_2. For any integer s, we have

$$(a^s x)^2 = a^s (a^{1+k\cdot 2^{n-z-1}} u^{2^{z-1}})^s x^2 = a^s (a u^{2^{z-1}})^s a^{k\cdot s\cdot 2^{n-z-1}} a^{2t_2} u^{l_1},$$

since $a^{2^{n-z-1}} u^{2^{z-1}} = u^{2^{z-1}} a^{2^{n-z-1}}$. But

$$(a u^{2^{z-1}})^s = a^{s(1+2^{n-2})}, \quad \left(\text{resp. } (a u^{2^{z-1}})^s = a^{s(1+2^{n-2})-2^{n-2}} u^{2^{z-1}}\right),$$

when s is even (resp. s is odd), by direct calculations. Therefore

$$\begin{aligned}
&(a^s x)^2 = a^s a^{s(1+2^{n-2})} a^{k\cdot s\cdot 2^{n-z-1}} a^{2t_2} u^{l_1} = a^{2s(1+2^{n-3}+k\cdot 2^{n-z-2})+2t_2} u^{l_1} \\
&(\text{resp. } (a^s x)^2 = a^s a^{s(1+2^{n-2})-2^{n-2}} a^{k\cdot s\cdot 2^{n-z-1}} a^{2t_2} u^{l_1+2^{z-1}} \\
&\qquad\qquad = a^{2s(1+2^{n-3}+k\cdot 2^{n-z-2})+2t_2-2^{n-2}} u^{l_1+2^{z-1}})
\end{aligned}$$

when s is even (when s is odd). Take the integer s_1 which satisfies the following equality

$$\begin{aligned}
&s_1(1+2^{n-3}+k\cdot 2^{n-z-2}) + t_2 \equiv 0 \pmod{2^{n-1}}, \\
&\left(\text{resp. } s_1(1+2^{n-3}+k\cdot 2^{n-z-2}) + t_2 - 2^{n-3} \equiv 0 \pmod{2^{n-1}}\right),
\end{aligned}$$

when t_2 is even (resp. when t_2 is odd).

Set $x_1 = a^{s_1} x$. Then

$$x_1^2 = (a^{s_1} x)^2 = u^{l_1} \quad (\text{resp. } x_1^2 = u^{l_1+2^{z-1}}),$$

when t_2 is even (resp. t_2 is odd). For any cases, we can write

$$x_1^2 = u^{l_2}$$

for some odd integer l_2. Since u is a power of x_1^2, we have $x_1 u x_1^{-1} = u$. On the other hand, we can write as

$$x_1 a x_1^{-1} = a^{1+k_2\cdot 2^{n-z-1}} u^{2^{z-1}},$$

for some odd integer k_2. Since $2 \le n-z-1$, we have

$$x_1 a^{2^{n-z-1}} x_1^{-1} = (a^{1+k_2 2^{n-z-1}} u^{2^{z-1}})^{2^{n-z-1}} = a^{(1+k_2 2^{n-z-1})2^{n-z-1}}$$

by Lemma 1 (3). Hence

$$\begin{aligned}
x_1^2 a x_1^{-2} &= x_1 (a^{1+k_2 2^{n-z-1}} u^{2^{z-1}}) x_1^{-1} \\
&= (a^{1+k_2 2^{n-z-1}} u^{2^{z-1}}) a^{(1+k_2 2^{n-z-1})2^{n-z-1} k_2} u^{2^{z-1}} \\
&= a^{(1+k_2 2^{n-z-1})^2} \qquad (6)
\end{aligned}$$

Therefore, for any integer s, we have

$$x_1^{2s} a x_1^{-2s} = a^{(1+k_2 2^{n-z-1})^{2s}}$$

and

$$x_1^{2s+1} a x_1^{-2s-1} = x_1^{2s}(a^{1+k_2 2^{n-z-1}} u^{2^{z-1}}) x_1^{-2s} = a^{(1+k_2 2^{n-z-1})^{2s+1}} u^{2^{z-1}}$$

Take the integer s_2 which satisfies the following equality

$$(1 + k_2 \cdot 2^{n-z-1})^{2s_2+1} \equiv 1 + 2^{n-z-1} \pmod{2^n},$$

and set $x_2 = x_1^{2s_2+1}$. Then

$$x_2 a x_2^{-1} = a^{1+2^{n-z-1}} u^{2^{z-1}}.$$

Further we have $x_2 u x_2^{-1} = u$, and

$$x_2^2 = x_1^{2(2s_2+1)} = u^{l_2(2s_2+1)} = u^{l_3},$$

where we set $l_3 = l_2(2s_2 + 1)$, which is an odd integer. By the same way as in (6), we have

$$x_2^2 a x_2^{-2} = a^{(1+2^{n-z-1})^2} = u^{l_3} a u^{-l_3} = a^{(1+2^{n-z})^{l_3}}.$$

Hence

$$(1 + 2^{n-z})^{l_3} \equiv (1 + 2^{n-z-1})^2 \pmod{2^n}.$$

It is easy to see that such an integer l_3 is uniquely determined mod 2^z. Hence the element u^{l_3} is uniquely determined by this relation. In the definition of $D(n, m, 1)$ in Section 2, (5), we write $l_3 = e_m$, when $m = z$. So we may write as

$$x_2^2 = u^{e_z}.$$

In particular, when $2z + 2 \leq n$, we have $e_z \equiv 1 \pmod{2^z}$. Hence $x_2^2 = u$, in this case.

Finally, we consider the element $b_0 = x_2 b x_2^{-1}$. Taking the conjugate of both sides of the equality, $ba^2b^{-1} = a^{-2}$, by x_2, we get

$$b_0 a^{2(1+2^{n-z-1})(1+2^{n-2})} b_0^{-1} = a^{-2(1+2^{n-z-1})(1+2^{n-2})}.$$

Hence

$$b_0 a^2 b_0^{-1} = a^{-2},$$

and

$$bb_0a^2b_0^{-1}b^{-1} = a^2.$$

Therefore we may write as

$$bb_0 = a^{-t}u^{2^{z-1}r},$$

and

$$b_0 = a^t u^{2^{z-1}r}b,$$

for some $t \in \mathbf{Z}$, and $r = 0$ or 1. Since $x_2^2 = u^{e_z}$,

$$\begin{aligned} b = u^{e_z}bu^{-e_z} &= x_2^2 b x_2^{-2} = x_2(a^t u^{2^{z-1}r}b)x_2^{-1} \\ &= (a^{1+2^{n-z-1}}u^{2^{z-1}})^t(u^{2^{z-1}r})(a^t u^{2^{z-1}r}b). \end{aligned} \tag{7}$$

Therefore we have

$$b \equiv u^{2^{z-1}t}b \pmod{\langle a\rangle}.$$

So,

$$t \equiv 0 \pmod 2.$$

Write $t = 2t_3$, where $t_3 \in \mathbf{Z}$. Substituting $t = 2t_3$ to (7), we get

$$b = (a^{1+2^{n-z-1}}u^{2^{z-1}})^{2t_3}a^{2t_3}b = a^{2(1+2^{n-z-1})(1+2^{n-2})t_3+2t_3}b.$$

Therefore

$$2t_3\{(1+2^{n-z-1})(1+2^{n-2})+1\} \equiv 0 \pmod{2^n}.$$

So

$$4t_3(1+2^{n-z-2}+2^{n-3}) \equiv 0 \pmod{2^n}.$$

Hence

$$t = 2t_3 \equiv 0 \pmod{2^{n-1}}.$$

Thus we may write as $t = 2^{n-1}t_4$, and

$$x_2bx_2^{-1} = a^{2^{n-1}t_4}u^{2^{z-1}r}b,$$

where $t_4 = 0$ or 1. Taking the conjugate of both sides of the equality,

$bab^{-1} = a^{-1}$, by x_2, we get

$$\begin{aligned}&(a^{2^{n-1}t_4}u^{2^{z-1}r}b)(a^{1+2^{n-z-1}}u^{2^{z-1}})(a^{2^{n-1}t_4}u^{2^{z-1}r}b)^{-1}\\ &\qquad = a^{-(1+2^{n-z-1})(1+2^{n-1})}u^{2^{z-1}}.\end{aligned}$$

Therefore

$$a^{-(1+2^{n-z-1})(1+r2^{n-1})} = a^{-(1+2^{n-z-1})(1+2^{n-1})}.$$

Hence $r = 1$, so $u^{2^{z-1}r} = u^{2^{z-1}}$.

Summarizing the results, we get

$$\begin{aligned}x_2ax_2^{-1} &= a^{1+2^{n-z-1}}u^{2^{z-1}},\\ x_2bx_2^{-1} &= a^{2^{n-1}t_4}u^{2^{z-1}}b,\\ x_2ux_2^{-1} &= u,\\ x_2^2 &= u^{e_z}.\end{aligned}$$

If $t_4 = 0$, these relations are the same as that of $D(n,z,1)$. So, the group $N_2 = \langle a,\ b,\ u,\ x_2\rangle$ is clearly isomorphic to $D(n,z,1)$.

If $t_4 = 1$, we set $u_1 = a^{2^{n-1}}u$ and $x_3 = a^{2^{n-2}}x_2$. Then we have $u_1^{2^z} = 1$, $u_1b = bu_1$ and $u_1au_1^{-1} = a^{1+2^{n-z}}$. So,

$$\langle a,\ b,\ u_1\rangle = \langle a,\ b,\ u\rangle \cong D(n,z).$$

Further, we have

$$\begin{aligned}x_3ax_3^{-1} &= a^{1+2^{n-z-1}}u_1^{2^{z-1}},\\ x_3u_1x_3^{-1} &= u_1,\\ x_3bx_3^{-1} &= u_1^{2^{z-1}}b,\\ x_3^2 &= u_1^{e_z}.\end{aligned}$$

Thus, in this case also, the group $N_2 = \langle a,\ b,\ u_1,\ x_3\rangle$ is isomorphic to $D(n,z,1)$. Hence the proof of Claim III is completed. □

Finally, we show the following

Claim IV $N_G(N_2) = N_2$.

Proof of Claim IV. When $N_2 = D_0(n,1,1)$, we can show Claim IV, by using Proposition 0.3.

So, we have only to consider the case where $N_2 = D(n,z,1) =$

$\langle a,\ b,\ u,\ x\rangle$, $2 \le z \le n-3$. Assume that $N_2 \subsetneqq N_G(N_2)$. Let $y \in N_G(N_2) - N_2$.

First, consider the elements yby^{-1} and yuy^{-1}. Note that any element in $D(n,z,1)$ is represented as $a^i u^j b^k x^t$ where $i,\ j,\ k,\ t \in \mathbf{Z}$, $0 \le i \le 2^n - 1$, $0 \le j \le 2^z - 1$, $0 \le k \le 1$, $0 \le t \le 1$. Define the normal subgroup H_0 of N_2 as

$$H_0 = \langle a,\ u^{2^{z-1}}\rangle.$$

It is easy to see that $N_2/H_0 = D(n,z,1)/H_0$ is an abelian group. Hence

$$(a^i u^j b^k x)^2 \equiv u^{2j} x^2 = u^{2j+e_z} \pmod{H_0}.$$

So, we can write as

$$(a^i u^j b^k x)^2 = a^r u^{2^{z-1}s+2j+e_z}, \tag{8}$$

for some integers r, s. Since e_z is odd,

$$(a^i u^j b^k x)^{2^z} \ne 1,$$

by Lemma 1 (1). Since $|\,b\,| = 2$ and $|\,u\,| = 2^z$, we must have

$$yby^{-1} \in \langle a,\ b,\ u\rangle = N_1,$$

and

$$yuy^{-1} \in \langle a,\ b,\ u\rangle = N_1.$$

Next, consider the element yay^{-1}. Taking the conjugate of both sides of the equality, $a^{-1} = bab^{-1}$, by y, we get

$$a_0^{-1} = b_0 a_0 b_0^{-1} \equiv a_0 \pmod{H_0}.$$

So,

$$(yay^{-1})^2 = a_0^2 \in H_0.$$

On the other hand, by (8),

$$(a^i u^j b^k x)^2 \notin H_0.$$

Hence we must have

$$yay^{-1} \in \langle a,\ b,\ u\rangle = N_1.$$

Thus we have shown that

$$y \in N_G(N_1) = N_2.$$

This contradicts the assumption that $N_2 \subsetneqq N_G(N_2)$ and $y \in N_G(N_2) - N_2$. Therefore the proof of Claim IV is completed. □

Proof of Theorem 1. Since G is a 2-group, Claim IV means that $G = N_2$. Therefore we have $G = N_1$ or N_2. Hence we can get Theorem 1, by using Theorem 0.2, Proposition 0.3 and Claim III. □

4. Proof of Theorems 2 and 3

Proof of Theorems 2 is essentially the same as that of Theorem 1. So we omit the proof.

Theorem 3 follows from Theorem 0.2, Theorem 1 and Theorem 2.

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