Hokkaido Mathematical Journal Vol. 32 (2003) p. 551–560

Absolute continuity of analytic measures

Hiroshi YAMAGUCHI

(Received February 21, 2002)

Abstract. We give an extension of a result due to Asmar, Montgomery-Smith and Saeki, which is concerned with absolute continuity of analytic measures. We also discuss the relation between the space $N(\sigma)$ and absolute continuity of analytic measures.

Key words: LCA group, measure, Fourier transform, absolute continuity.

1. Introduction

Let G be a LCA group with dual group $\hat{G}$. Let $L^1(G)$ and $M(G)$ be the group algebra and the measure algebra, respectively. Let ψ be a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbb{R}$, and let $\phi : \mathbb{R} \to G$ be the dual homomorphism of ψ. Defining an action of $\mathbb{R}$ on G by $t\cdot x = \phi(t)+x$ ($t \in \mathbb{R}$, $x \in G$), we get a transformation group $(\mathbb{R}, G)$. Let σ be a quasi-invariant, (positive) Radon measure on G, and set $N(\sigma) = \{\mu \in M(G) : \phi(h) * \mu \ll \sigma$ $\forall h \in L^1(\mathbb{R})\}$. Then $N(\sigma)$ is an $L^1(\mathbb{R})$-module and an L-subspace of $M(G)$. In general, we have

$$L^1(\sigma) \subset N(\sigma) \subset M(G).$$

According to choice of G and σ, it may happen that $N(\sigma) = M(G)$ and $L^1(\sigma) \subsetneq N(\sigma) \subsetneq M(G)$ (cf. [6] and [14]). Any analytic measure in $N(\sigma)$ is absolutely continuous with respect to σ (Corollary 2.1 or [14, Corollary 2.1]). We show that $N(\sigma)$ is the largest $L^1(\mathbb{R})$-module, L-subspace of $M(G)$ such that any its analytic measure is necessarily absolutely continuous with respect to σ (Corollary 2.3). Recently, Asmar, Montgomery-Smith and Saeki obtained a new version of Bochner's generalization of the F. and M. Riesz theorem ([3, Theorem 4.5]). We also give another proof of it (Theorem 2.2).

2. Notation and results

Let G be a LCA group with dual group $\hat{G}$. We denote by $\mathfrak{B}(G)$ the σ-algebra of Borel sets in G. For $x \in G$, δ_x denotes the point mass at x. We

---

2000 Mathematics Subject Classification : 43A05, 43A10, 43A25.

denote by Trig(G) the set of trigonometric polynomials on G. Let $C_o(G)$ be the Banach space of continuous functions on G which vanish at infinity. Then $M(G)$ is identified with the dual space of $C_o(G)$. Let $M^+(G)$ be the set of nonnegative measures in $M(G)$. For $\mu \in M(G)$ and $f \in L^1(|\mu|)$, we often use the notation $\mu(f)$ as $\int_G f(x)d\mu(x)$. For $\lambda \in M(G)$, $\hat{\lambda}$ denotes the Fourier-Stieltjes transform of λ, i.e., $\hat{\lambda}(\gamma) = \int_G (-x, \gamma) d\lambda(x)$ for $\gamma \in \hat{G}$. For a closed subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transform vanish off E, and E is called a Riesz set if $M_E(G) \subset L^1(G)$. Obviously, compact subsets of $\hat{G}$ are Riesz sets.

Let ψ be a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbb{R}$ (the reals). We may assume that there exists $\chi_o \in \hat{G}$ such that $\psi(\chi_o) = 1$ by considering a multiplication of ψ if necessary. Let $\phi : \mathbb{R} \to G$ be the dual homomorphism of ψ, i.e., $(\phi(t), \gamma) = \exp(i\psi(\gamma)t)$ for $t \in \mathbb{R}$ and $\gamma \in \hat{G}$.

Let Λ be a discrete subgroup of $\hat{G}$ generated by χ_o, and let $K = \Lambda^\perp$, the annihilator of Λ. We define a continuous homomorphism $\alpha : \mathbb{R} \oplus K \to G$ by

$$\alpha(t, u) = \phi(t) + u. \tag{2.1}$$

Then $\ker(\alpha) = \{(2\pi n, -\phi(2\pi n)) : n \in \mathbb{Z}\}$ and $\ker(\alpha)^\perp = \{(\psi(\gamma), \gamma|_K) : \gamma \in \hat{G}\} \cong \hat{G}$. For $0 < \epsilon < \frac{1}{6}$, we define a function $\Delta_\epsilon(t, \omega)$ on $\mathbb{R} \oplus \hat{K}$ by

$$\Delta_\epsilon(t, \omega) = \begin{cases} \max\Big(1 - \dfrac{1}{\epsilon}|t|, 0\Big) & (\omega = 0), \\ 0 & (\omega \neq 0). \end{cases}$$

For $\mu \in M(G)$, define a function $\Phi_\mu^\epsilon(t, \omega)$ on $\mathbb{R} \oplus \hat{K}$ by

$$\Phi_\mu^\epsilon(t, \omega) = \sum_{\gamma \in \hat{G}} \hat{\mu}(\gamma) \Delta_\epsilon((t, \omega) - (\psi(\gamma), \gamma|_K)).$$

Then $\Phi_\mu^\epsilon \in M(\mathbb{R} \oplus K)^\wedge$, $\|(\Phi_\mu^\epsilon)^\vee\| = \|\mu\|$ and $\alpha((\Phi_\mu^\epsilon)^\vee) = \mu$ for $\mu \in M(G)$ (cf. [14]), where "$\vee$" denotes the inverse Fourier transform. We define an isometry $T_\psi^\epsilon : M(G) \to M(\mathbb{R} \oplus K)$ by

$$T_\psi^\epsilon(\mu) = (\Phi_\mu^\epsilon)^\vee. \tag{2.2}$$

Defining an action of $\mathbb{R}$ on G by $t \cdot x = \phi(t) + x$, we get a transformation group $(\mathbb{R}, G)$. For $\lambda \in M(\mathbb{R})$ and $\mu \in M(G)$, we define $\lambda * \mu \in M(G)$ by

$$\lambda * \mu(f) = \int_G \int_{\mathbb{R}} f(t \cdot x) d\lambda(t) d\mu(x)$$

for $f \in C_o(G)$. When there is a possibility of confusion, we may use the the notation $\lambda *_{\mathbb{R}} \mu$ instead of $\lambda * \mu$. We note that $\lambda *_{\mathbb{R}} \mu = \phi(\lambda) * \mu$ (cf. [14, Proposition 4.1]), where $\phi(\lambda) * \mu$ is the usual convolution in $M(G)$. For $\mu \in M(G)$, its spectrum $\mathrm{sp}(\mu)$ is defined by $\mathrm{sp}(\mu) = \bigcap_{h \in J(\mu)} \hat{h}^{-1}(0)$, where $J(\mu) = \{h \in L^1(\mathbb{R}) : h *_{\mathbb{R}} \mu = 0\}$. For $\mu \in M(G)$ and a closed set E in $\mathbb{R}$, we note that $\mathrm{sp}(\mu) \subset E$ if and only if $\mathrm{supp}(\hat{\mu}) \subset \psi^{-1}(E)$ (cf. [14, Remark 4.1]).

A (positive) Radon measure σ on G is said to be quasi-invariant if $\sigma(F) = 0$ implies $\sigma(t \cdot F)\,(= \sigma(\phi(t) + F)) = 0$ for all $t \in \mathbb{R}$. For a quasi-invariant Radon measure σ on G, let $N(\sigma) = \{\mu \in M(G) : h * \mu \ll \sigma \ \forall h \in L^1(\mathbb{R})\}$. $N(\sigma)$ is a closed subspace of $M(G)$, and $L^1(\sigma) \subset N(\sigma) \subset M(G)$ (cf. [9]). Moreover, $N(\sigma)$ is an $L^1(\mathbb{R})$-module and an L-subspace of $M(G)$ (cf. [10, Corollary 5]). For $\epsilon > 0$, $\bar{V}_\epsilon$ and V_ϵ denote a closed interval $[-\epsilon, \epsilon]$ and an open interval $(-\epsilon, \epsilon)$, respectively. We state our first result.

Theorem 2.1 *Let $0 < \epsilon < \frac{1}{6}$, and let E be a closed set in $\mathbb{R}$ such that $E + \bar{V}_\epsilon$ is a Riesz set in $\mathbb{R}$. Let μ be a measure in $M(G)$ with $\mathrm{sp}(\mu) \subset E$. Then $\lim_{t \to 0} \|\mu - \delta_{\phi(t)} * \mu\| = 0$.*

Proof. Since $\mathrm{supp}(\hat{\mu}) \subset \psi^{-1}(E)$ and $T_\psi^\epsilon(\mu)^\wedge = \Psi_\mu^\epsilon$, we have

(1) $\qquad \mathrm{supp}(T_\psi^\epsilon(\mu)^\wedge) \subset (E + \bar{V}_\epsilon) \times \hat{K}.$

Let $\pi_K : \mathbb{R} \oplus K \to K$ be the projection, and put $\eta = \pi_K(|T_\psi^\epsilon(\mu)|)$. It follows from [13, Corollary 1.6] that there exists a family $\{\lambda_u\}_{u \in K} \subset M(\mathbb{R})$ with the following properties:

(2) $\qquad u \to (\lambda_u \times \delta_u)(f)$ is η-measurable for each bounded Borel function f on $\mathbb{R} \oplus K$,

(3) $\qquad \|\lambda_u\| = 1, \quad$ and

(4) $\qquad T_\psi^\epsilon(\mu)(f) = \int_K (\lambda_u \times \delta_u)(f) d\eta(u)$ for each bounded Borel function f on $\mathbb{R} \oplus K$.

Then, by (1) and [13, Lemma 2.1], we have

$$\lambda_u \in M_{E + \bar{V}_\epsilon}(\mathbb{R}) \quad \eta\text{-a.a. } u \in K,$$

which, together with the fact that $E + \bar{V}_\epsilon$ is a Riesz set, yields

(5) $\lambda_u \in L^1(\mathbb{R}) \quad \eta\text{-a.a.} \;\; u \in K.$

We note that

$$\alpha((\delta_t \times \delta_0) * T^\epsilon_\psi(\mu)) = \delta_{\phi(t)} * \mu.$$

Hence

(6) $$\begin{aligned}\|\delta_{\phi(t)} * \mu - \mu\| &= \|\alpha((\delta_t \times \delta_0) * T^\epsilon_\psi(\mu) - T^\epsilon_\psi(\mu))\| \\ &\leq \|(\delta_t \times \delta_0) * T^\epsilon_\psi(\mu) - T^\epsilon_\psi(\mu)\|.\end{aligned}$$

By (4), we have

(7) $$(\delta_t \times \delta_0) * T^\epsilon_\psi(\mu)(f) = \int_K \{(\delta_t * \lambda_u) \times \delta_u\}(f) d\eta(u)$$

for each bounded Borel function f on $\mathbb{R} \oplus K$. For $f \in C_o(\mathbb{R})$, we have $\lambda_u(f) = (\lambda_u \times \delta_u)(f)$, where, on the right hand side, f is considered as a bounded continuous function on $\mathbb{R} \oplus K$. Let $\mathcal{A}$ be a countable dense set in $C_o(\mathbb{R})$. Then

$$\|\delta_t * \lambda_u - \lambda_u\| = \sup_{\substack{f \in \mathcal{A} \\ \|f\|_\infty \leq 1}} |(\delta_t * \lambda_u - \lambda_u)(f)|,$$

which, together with (2), yields that $u \to \|\delta_t * \lambda_u - \lambda_u\|$ is η-measurable. It follows from (4) and (7) that

$$\|(\delta_t \times \delta_0) * T^\epsilon_\psi(\mu) - T^\epsilon_\psi(\mu)\| \leq \int_K \|\delta_t * \lambda_u - \lambda_u\| \, d\eta(u).$$

On the other hand, (5) implies

$$\lim_{t \to 0} \|\delta_t * \lambda_u - \lambda_u\| = 0 \quad \eta\text{-a.a.} \;\; u \in K.$$

Thus, by the Lebesgue convergence theorem, we have

$$\lim_{t \to 0} \|(\delta_t \times \delta_0) * T^\epsilon_\psi(\mu) - T^\epsilon_\psi(\mu)\| = 0,$$

which, combined with (6), yields

$$\lim_{t \to 0} \|\delta_{\phi(t)} * \mu - \mu\| = 0.$$

This completes the proof. □

When $E = [0, \infty)$, Theorem 2.1 is already known (cf. [7] or [2]). We give another example.

Definition 2.1 Let $0 < p < \infty$. A subset E of $\mathbb{Z}$ is called a $\Lambda(p)$-set if for some $0 < q < p$, there exists a constant $C > 0$ such that

$$\| f \|_p \le C \| f \|_q$$

for all $f \in \mathrm{Trig}_E(\mathbb{T})$, where $\mathrm{Trig}_E(\mathbb{T})$ is the set of trigonometric polynomials on $\mathbb{T}$ whose Fourier transforms vanish off E.

Example 2.1 Let $\mathbb{Z}_-$ and $\mathbb{R}^+$ be the set of nonpositive integers and the set of nonnegative real numbers, respectively. Let $F \subset \mathbb{Z}_- \backslash \{0\}$ be a $\Lambda(2)$-set in $\mathbb{Z}$, and put $E = \left(F + \bar{V}_{\frac{1}{6}}\right) \cup \mathbb{R}^+$. Then, for $0 < \epsilon < \frac{1}{6}$, $E + \bar{V}_\epsilon$ is a Riesz set in $\mathbb{R}$.

In fact, let $\mu \in M_{E+\bar{V}_\epsilon}(\mathbb{R})$. Let $\pi : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{T}$ be the natural homomorphism. Put $F_n(x) = \frac{1}{\pi} \cdot \frac{1-\cos nx}{nx^2}$ $(n \in \mathbb{N})$. Then $\pi(F_n * \mu) \in \mathrm{Trig}(\mathbb{T})$. We note that $\hat{F}_n(s) = \int_{\mathbb{R}} F_n(x) e^{-isx} dx = \max\left(1 - \frac{1}{n}|s|, 0\right)$. Let $P_- : \mathrm{Trig}(\mathbb{T}) \to \mathrm{Trig}(\mathbb{T})$ be a projection defined by

$$P_-(u)(x) = \sum_{k<0} \hat{u}(k) e^{ikx}.$$

Let $0 < p < 1$. It follows from [12, Theorem 8.7.6] that there exists a constant $A_p > 0$ such that

$$\|P_-(\pi(F_n * \mu))\|_p \le A_p \|\pi(F_n * \mu)\|_1.$$

Since $P_-(\pi(F_n * \mu)) \in \mathrm{Trig}_F(\mathbb{T})$ and F is a $\Lambda(2)$ set, there exists a constant $C_F > 0$ such that

$$\|P_-(\pi(F_n * \mu))\|_2 \le C_F \|P_-(\pi(F_n * \mu))\|_p.$$

Hence

$$\begin{aligned}\|P_-(\pi(F_n * \mu))\|_2 &\le A_p C_F \|\pi(F_n * \mu)\|_1 \\ &\le A_p C_F \| \mu \|,\end{aligned}$$

which yields

$$\sum_{k \in F} |(F_n * \mu)^\wedge(k)|^2 \le A_p^2 C_F^2 \| \mu \|^2.$$

Letting $n \to \infty$, we have

$$\sum_{k\in F} |\hat{\mu}(k)|^2 \leq A_p^2 C_F^2 \| \mu \|^2.$$

Let $x \in \bar{V}_{\frac{1}{6}} + \bar{V}_\epsilon = \bar{V}_{\frac{1}{6}+\epsilon}$. Considering $e^{-ix\cdot} F_n * \mu$, we similarly get

$$\sum_{k\in F} |\hat{\mu}(x+k)|^2 \leq A_p^2 C_F^2 \| \mu \|^2.$$

Thus

$$\begin{aligned} \int_{(-\infty,0]} |\hat{\mu}(x)|^2 dx &= \sum_{k\in F} \int_{\bar{V}_{\frac{1}{6}+\epsilon}} |\hat{\mu}(k+x)|^2 dx \\ &\leq \Big(\frac{1}{3} + 2\epsilon\Big) A_p^2 C_F^2 \| \mu \|^2 < \infty, \end{aligned}$$

which, together with [5, Main Theorem], yields $\mu \in L^1(\mathbb{R})$. This shows that $E + \bar{V}_\epsilon$ is a Riesz set in $\mathbb{R}$.

Set $A = \{\mu \in M(G) : \lim_{t\to 0} \|\delta_{\phi(t)} * \mu - \mu\| = 0\}$. The following theorem is due to Liu and van Rooij.

Theorem A (cf. [9, Therem 9]) *Let σ be a quasi-invariant Radon measure on G. Then $A \cap N(\sigma) = L^1(\sigma)$.*

By Theorem 2.1 and Theorem A, we get the following corollary, which was obtained in [14], by a different method.

Corollary 2.1 (cf. [14, Corollary 2.1]) *Let σ be a quasi-invariant Radon measure on G. Let E be as in Theorem 2.1, and let μ be a measure in $N(\sigma)$ with $\mathrm{sp}(\mu) \subset E$. Then $\mu \ll \sigma$.*

Recently, Asmar, Montgomery-Smith and Saeki ([3]) got significant results concerned with analytic measures, and they gave the following theorem as an application.

Theorem B ([3, Theorem 4.5]) *Let $\mu \in M(G)$, and suppose that, for every $s \in \mathbb{R}$, $\psi^{-1}((-\infty, s]) \cap \mathrm{supp}(\hat{\mu})$ is compact. Then $\mu \ll m_G$.*

Next we show that Theorem B follows from Corollary 2.1. It is easy to verify that Therem B and the following Theorem B′ are equivalent.

Theorem B′ *Let $\mu \in M(G)$ be of analytic type, i.e., $\hat{\mu}(\gamma) = 0$ for $\gamma \in \hat{G}$ with $\psi(\gamma) < 0$. Suppose that, for every $s \in \mathbb{R}$, $\psi^{-1}([s-1, s+1]) \cap \mathrm{supp}(\hat{\mu})$ is compact. Then $\mu \ll m_G$.*

The following is a slight extension of Theorem B′.

Theorem 2.2 *Let E be as in Theorem 2.1. Let μ be a measure in $M(G)$ with $\mathrm{sp}(\mu) \subset E$. Suppose that, for every $s \in \mathbb{R}$, $\psi^{-1}([s-1, s+1]) \cap \mathrm{supp}(\hat{\mu})$ is a Riesz set in $\hat{G}$. Then $\mu \ll m_G$.*

Proof. By Corollary 2.1, it is sufficient to prove that $\mu \in N(m_G)$. Let $h \in L^1(\mathbb{R})$. For any $\epsilon > 0$, it follows from [12, Theorem 2.6.6] that there exists $\nu_\epsilon \in L^1(\mathbb{R})$ such that $\hat{\nu}_\epsilon$ has a compact support and $\|h - h * \nu_\epsilon\|_1 < \epsilon$. Since $(h * \nu_\epsilon)^\wedge$ has a compact support, there exist $g_1, \ldots, g_m \in L^1(\mathbb{R})$ such that

(1) $$\mathrm{supp}(\hat{g}_i) \subset [s_i - 1, s_i + 1] \text{ for some } s_i \in \mathbb{R} \ (i = 1, 2, \ldots, m), \text{ and}$$

(2) $$h * \nu_\epsilon = \sum_{i=1}^{m} h * \nu_\epsilon * g_i.$$

Since $\mathrm{supp}((\phi(h * \nu_\epsilon * g_i) * \mu)^\wedge) \subset \psi^{-1}([s_i - 1, s_i + 1]) \cap \mathrm{supp}(\hat{\mu})$, we have

(3) $$\phi(h * \nu_\epsilon) * \mu = \sum_{i=1}^{m} \phi(h * \nu_\epsilon * g_i) * \mu \in L^1(G).$$

On the other hand,

$$\begin{aligned} \|\phi(h) * \mu - \phi(h * \nu_\epsilon) * \mu\| &\le \|\phi(h - h * \nu_\epsilon)\| \, \|\mu\| \\ &\le \|h - h * \nu_\epsilon\|_1 \, \|\mu\| \le \epsilon \|\mu\|. \end{aligned}$$

Since ϵ is any positive real number, we have, by (3),

$$\phi(h) * \mu \in L^1(G),$$

which shows that $\mu \in N(m_G)$. This completes the proof. □

As we pointed out before, $N(\sigma)$ is an $L^1(\mathbb{R})$-module and an L-subspace of $M(G)$. Moreover, by Corollary 2.1, every analytic measure in $N(\sigma)$ is absolutely continuous with respect to σ. Finally, we show that $N(\sigma)$ is the largest $L^1(\mathbb{R})$-module, L-subspace of $M(G)$ with this property.

Theorem 2.3 *Let σ be a quasi-invariant, Radon measure on G. Let V be an open set in $\mathbb{R}$ with $V \cap \psi(\hat{G}) \neq \emptyset$. Let $\mathcal{L}(G)$ be an $L^1(\mathbb{R})$-module, L-subspace of $M(G)$ such that $\mathcal{L}(G) \not\subset N(\sigma)$. Then there exists $\mu \in \mathcal{L}(G) \backslash N(\sigma)$ with* $\mathrm{sp}(\mu) \subset V$.

Proof. We choose $\gamma_o \in \hat{G}$ and a symmetric open neighborhood U of 0 in $\mathbb{R}$, with compact closure, so that $\psi(\gamma_o) + \bar{U} \subset V$. Since $\mathcal{L}(G)$ is an L-subspace of $M(G)$, there exists a nonzero measure $\nu \in (\mathcal{L}(G) \cap M^+(G)) \backslash N(\sigma)$. Then there exists a quasi-invariant measure σ_ν in $M^+(G)$ and a σ-compact subset X_ν of G such that

(1) $\sigma_\nu \ll \sigma,$

(2) $\phi(\mathbb{R}) + X_\nu = X_\nu \quad (\text{i.e., } \mathbb{R} \cdot X_\nu = X_\nu),$

(3) $\nu(X_\nu^c) = \sigma_\nu(X_\nu^c) = 0, \text{ and}$

(4) $\sigma_\nu|_{X_\nu}$ and $\sigma|_{X_\nu}$ are mutually absolutely continuous.

(cf. [14, Proposition 4.2]). Then

(5) $\nu \notin N(\sigma_\nu).$

It follows from [14, Theorem 5.1] that $\pi_K(T_\psi^\epsilon(\nu))$ is not absolutely continuous with respect to $\pi_K(T_\psi^\epsilon(\sigma_\nu))$, where $\pi_K : \mathbb{R} \oplus K \to K$ is the projection. Let $\pi_K(T_\psi^\epsilon(\nu)) = \eta_a + \eta_s$ be the Lebesgue decomposition of $\pi_K(T_\psi^\epsilon(\nu))$ with respect to $\pi_K(T_\psi^\epsilon(\sigma_\nu))$. Then $\eta_s \neq 0$. There exists a Borel set $K_s \subset K$ such that $\pi_K(T_\psi^\epsilon(\sigma_\nu))(K_s) = 0$ and $\eta_s(K_s^c) = 0$. Put $B = \mathbb{R} \times K_s$, and define a measure $\omega_B \in M^+(\mathbb{R} \oplus K)$ by

$$\omega_B(F) = T_\psi^\epsilon(\nu)(B \cap F)$$

for $F \in \mathfrak{B}(\mathbb{R} \oplus K)$. Then $\pi_K(\omega_B) = \eta_s \neq 0$. In particular, $\omega_B \neq 0$. Since $\omega_B \ll T_\psi^\epsilon(\nu)$,

$$\alpha(\omega_B) \ll \alpha(T_\psi^\epsilon(\nu)) = \nu.$$

It follows from the facts that $\nu \in \mathcal{L}(G)$ and $\mathcal{L}(G)$ is an L-subspace of $M(G)$ that $\alpha(\omega_B) \in \mathcal{L}(G)$. Let $h \neq 0$ be a nonngative function in $L^1(\mathbb{R})$ such that $\mathrm{supp}(\hat{h}) \subset \bar{U}$. Then $\phi(h) * \alpha(\omega_B) \neq 0$, and

(6) $\mathrm{sp}(\phi(h) * \alpha(\omega_B)) = \mathrm{sp}(h *_{\mathbb{R}} \alpha(\omega_B)) \subset \bar{U}.$

Since $\mathcal{L}(G)$ is an $L^1(\mathbb{R})$-module and $\alpha(\omega_B) \in \mathcal{L}(G)$, we have

(7) $\phi(h) * \alpha(\omega_B) \in \mathcal{L}(G)$.

For any nononegative, nonzero $g \in L^1(\mathbb{R})$,

$$\begin{aligned} g *_{\mathbb{R}} (\phi(h) * \alpha(\omega_B)) &= \phi(g * h) * \alpha(\omega_B) \\ &= \alpha((g * h) \times \delta_0) * \alpha(\omega_B). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \pi_K\big(((g * h) \times \delta_0) * \omega_B\big) &= \pi_K((g * h) \times \delta_0) * \pi_K(\omega_B) \\ &= \|g * h\|_1 \pi_K(\omega_B) = \|g * h\|_1 \eta_s, \end{aligned}$$

hence $\pi_K\big(((g * h) \times \delta_0) * \omega_B\big) \perp \pi_K(T_\psi^\epsilon(\sigma_\nu))$. Thus

$$((g * h) \times \delta_0) * \omega_B \perp T_\psi^\epsilon(\sigma_\nu),$$

which, combined with [14, Lemma 4.1], yields

$$0 \neq g *_{\mathbb{R}} (\phi(h) * \alpha(\omega_B)) = \alpha\big(((g * h) \times \delta_0) * \omega_B\big) \perp \sigma_\nu.$$

This shows that $\phi(h) * \alpha(\omega_B) \notin N(\sigma_\nu)$. Since $\phi(h) * \alpha(\omega_B)$ is concentrated on X_ν and $\sigma_\nu|_{X_\nu}$ and $\sigma|_{X_\nu}$ are mutually absolutely continuous, we have

(8) $\phi(h) * \alpha(\omega_B) \notin N(\sigma)$.

Put $\mu = \gamma_o \phi(h) * \alpha(\omega_B)$. Since $\psi(\gamma_o) + \bar{U} \subset V$, it follows from (6)–(8) that $\mathrm{sp}(\mu) \subset V$ and $\mu \in \mathcal{L}(G) \backslash N(\sigma)$. This completes the proof. □

For a quasi-invariant Radon measure σ on G and a closed subset E of $\mathbb{R}$, let $\mathfrak{F}_E(\sigma)$ be a family of $L^1(\mathbb{R})$-modules $\mathcal{L}(G)$, which are L-subspaces of $M(G)$, satisfying the following condition:

$$\mu \in \mathcal{L}(G), \ \ \mathrm{sp}(\mu) \subset E \implies \mu \ll \sigma. \tag{2.3}$$

By Corollary 2.1 and Theorem 2.3, we have the following corollaries.

Corollary 2.2 *Let E be as in Theorem* 2.1, *and suppose that $\psi(\hat{G}) \cap \overset{\circ}{E} \neq \emptyset$, where $\overset{\circ}{E}$ denotes the interior of E. Then $N(\sigma)$ is the largest $L^1(\mathbb{R})$-module, L-subspace of $M(G)$ in $\mathfrak{F}_E(\sigma)$. Namely, $\mathcal{L}(G) \subset N(\sigma)$ for every $\mathcal{L}(G) \in \mathfrak{F}_E(\sigma)$.*

Corollary 2.3 *$N(\sigma)$ is the largest $L^1(\mathbb{R})$-module, L-subspace of $M(G)$ in $\mathfrak{F}_{[0,\infty)}(\sigma)$*

References

[1] Asmar N.H., *A note on analytic measures*. Internat. J. Math. and Math. Sci. **13** (1990), 187–188.

[2] Asmar N.H. and Montgomery-Smith S.J., *Analytic measures and Bochner measurability*. Bull. Sci. Math. **122** (1998), 39–66.

[3] Asmar N.H., Montgomery-Smith S.J. and Saeki S., *Transference in spaces of measures*. J. Funct. Anal. **165** (1999), 1–23.

[4] DeLeeuw K. and Glicksberg I., *Quasi-invariance and analyticity of measures on compact groups*. Acta Math. **109** (1963), 179–205.

[5] Doss R., *On measures with small transforms*. Pacific J. Math. **26** (1968), 257–263.

[6] Finet C. and Tardivel-Nachet V., *Lacunary sets on transformation groups*. Hokkaido Math. J. **23** (1994), 1–19.

[7] Forelli F., *Analytic and quasi-invariant measures*. Acta Math. **118** (1967), 33–57.

[8] Gulick S.L., Liu T.S. and van Rooij A.C.M., *Group algebra modules I and II*. Canad. J. Math. **19** (1967), 133–150 and 151–173.

[9] Liu T.S. and van Rooij A.C.M., *Transformation groups and absolutely continuous measures*. Indag. Math. **30** (1969), 225–231.

[10] Liu, T.S., van Rooij A.C.M. and Wang J.K., *Transformation groups and absolutely continuos measures II*. Indag. Math. **32** (1970), 57–61.

[11] Rudin W., *Trigonometric series with gaps*. J. Math. and Mech. **9** (1960), 203–228.

[12] Rudin W., *Fourier Analysis on Groups*. New York, Interscience, 1962.

[13] Yamaguchi H., *Idempotent multipliers on the space of analytic singular measures*. Hokkaido Math. J. **14** (1985), 49–73.

[14] Yamaguchi H., *The space $N(\sigma)$ and the F. and M. Riesz theorem*. Hokkaido Math. J. **25** (1996), 567–589.

Department of mathematics
Josai University
Sakado, Saitama 350-0295
Japan
E-mail: hyama@math.josai.ac.jp