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Extension of holomorphic functions through a hypersurface by tangent analytic discs

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Abstract. We prove a new criterion of extendibility of holomorphic functions from a domain $\Omega \subset \mathbb{C}^n$ by means of analytic discs A tangent to $\partial\Omega$, and verifying $\partial A \subset \bar{\Omega}$ and $\partial A \cap \Omega \neq \emptyset$.

Key words: CR functions, analytic discs.

1. Introduction

Let Ω be a domain of $\mathbb{C}^n$ with boundary M, A an analytic disc *attached* to $\bar{\Omega}$ and not to M, that is verifying $\partial A \subset \bar{\Omega}$ but $\partial A \not\subset M$. We assume A to be tangent to M at some boundary point $z^o \in \partial A \cap M$, and let B be a ball with center z^o which contains $\bar{A}$. Then holomorphic functions f on $\Omega \cap B$ extend holomorphically to a fixed neighborhood of z^o (Theorem 2.1 herein). In spite of its appearence, this is in fact a result on propagation of holomorphic extendibility from Ω along boundaries of discs attached to M at points of tangency (Theorem 2.2 herein). The first result of this type is the HANGES-TREVES Theorem in [6] on propagation along discs which are entirely contained in M. Successively, in the paper [9], TUMANOV proved that *defective* discs attached to M are as well *propagators* of extendibility. Note that defective discs are all tangent but they do not exhaust the class of tangent discs. Also, if we assume that $\bar{A} \subset \bar{\Omega}$ but $\bar{A} \not\subset M$ in any neighborhood of z^o, then our result implies extension of germs at z^o of holomorphic functions on Ω. We recall that extendibility of germs (to either side of M) is equivalent to *minimality* of M in the sense of TREPREAU and TUMANOV. Hence, if M is not minimal, then one-sided discs tangent to M at z^o are in fact entirely contained in M (Proposition 4.1 herein). The method of the present paper consists in the analysis of the properties of superharmonicity of $\log r_f^\nu$ where r_f^ν denotes the radius of convergence of the Taylor series of a holomorphic function f on Ω in a direction $\nu \in \mathbb{C}^n \setminus \{0\}$. Another

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approach which uses, instead, the technique of the infinitesimal deformation of the discs, can be found in the successessive joint paper of the author with BARACCO [2].

2. Statement and proof of the main result

Let Ω be a domain of $\mathbb{C}^n$, M the boundary of Ω (Ω on one side of M), TM (resp. $T^{\mathbb{C}}M = TM \cap iTM$) the tangent (resp. complex tangent) bundle to M, z^o a point of M. Let $C^{k,\gamma}$ ($k \geq 0$ integer, $0 \leq \gamma \leq 1$ fractional) denote the class of functions whose derivatives up to order k are Lipschitz-continuous with exponent γ. Let Δ be the standard disc in $\mathbb{C}$, denote by $A = A(\tau)$, $\tau \in \Delta$, an analytic disc in $\mathbb{C}^n$, $C^{1,\gamma}$ up to the boundary and small in $C^{1,\gamma}$-norm, and suppose that $z^o = A(1)$.

Theorem 2.1 *Let M be $C^{2,\gamma}$ and A be $C^{1,\gamma}$. Assume*

$$\partial A \subset \bar{\Omega} \tag{2.1}$$

$$\partial_\tau A(1) \in T_{z^o} M \tag{2.2}$$

$$\partial A \cap \Omega \neq \emptyset. \tag{2.3}$$

Let B be a ball with center z^o which contains $\bar{A}$. Then all holomorphic functions on $\Omega \cap B$ extend holomorphically to a fixed neighborhood of z^o.

Before giving the proof of Theorem 2.1, we discuss some variants of it. First we notice that we can weaken the third hypothesis (2.3) by asking $\bar{A} \cap \Omega \neq \emptyset$, if we correspondingly stengthen (2.1) to $\bar{A} \subset \bar{\Omega}$. In fact, if $z^1 = A(\tau^1)$ is a point in $A \cap \Omega$, we take a (smooth and small) subdisc $\Delta_1 \subset \Delta$ with $\tau^1 \in \partial\Delta_1$, and which coincides with Δ in a neighborhood of $\tau = 1$. Define A_1 as A restricted to Δ_1 (in particular we still have $\bar{A}_1 \subset \bar{\Omega}$); then Theorem 2.1 applies to this new A_1 and yields holomorphic extension from $B \cap \Omega$ to a full neighborhood of z^o provided that $B \supset \bar{A}_1$. Also, if we still assume $\bar{A} \subset \bar{\Omega}$ and suppose that there is a sequence of points $z_k \to z^o$ which belong to A but not to M, then germs of holomorphic functions on Ω at z^o extend to a full neighborhood of z^o. In fact in this situation we can find a sequence of discs A_k which shrink to z^o and satisfy the assumptions of Theorem 2.1.

We notice now that Theorem 2.1 is in fact a result of propagation according to the following restatement

Theorem 2.2 *Let M and A be $C^{2,\gamma}$ and $C^{1,\gamma}$ respectively, still assume* (2.2) *at z^o, and suppose, instead of* (2.1):

$$\partial A \subset M. \tag{2.4}$$

Let z^1 be another point of ∂A, and let f be holomorphic in $B \cap \Omega$ for a ball B with center z^o such that $B \supset \bar{A}$. If f extends holomorphically to a neighborhood of z^1, then it also extends holomorphically to a neighborhood of z^o.

We show how Theorem 2.2 is an immediate consequence of Theorem 2.1. We let $A(1) = z^o$ and $A(-1) = z^1$. By a small perturbation $\tilde{\Omega}$ of Ω, such that $\bar{\Omega} \subset \tilde{\Omega}$ in a neighborhood of z^1, and which leaves Ω unchanged at z^o, we keep the hypothesis $\partial A \subset \tilde{\Omega} \cup \partial\tilde{\Omega}$ while achieving $z^1 \in \tilde{\Omega}$. Then Theorem 2.1 applies.

We recall again that this result is contained in [6] when $A \subset M$ and z^o and z^1 are points in the interior of A. In our more general setting, the result follows from [9] under the additional assumption that A is *defective* (whence automatically tangent).

3. Proof of Theorem 2.1

By any choice of complex coordinates z in $\mathbb{C}^n$, we denote by $\delta(z)$ the Euclidean distance of z to M. We consider the foliation $\{M_\alpha\}_{\alpha\in\mathbb{R}^+}$ of Ω by the hypersurfaces $M_\alpha = \{z \in \Omega : \delta(z) = \alpha\}$. We make our choice of coordinates $z = (z_1, z')$, $z = x + iy$ such that each M_α is defined by the equation

$$y_1 = h(x_1, z', \alpha) \quad \text{with} \quad h(0,0,\alpha) = -\alpha, \ \partial_{x_1 z'} h(0,0,\alpha) = 0. \tag{3.1}$$

We shall also write $h_\alpha(x_1, z')$ instead of $h(x_1, z', \alpha)$, and use the notation $r_\alpha = y_1 - h_\alpha$. We define, for $\tau \in \Delta$

$$w(\tau) = z' \circ A(\tau),$$
$$\eta(\tau) = \delta \circ A(\tau),$$

and call them the z' and δ components of A respectively. For $\alpha \in \mathbb{R}^+$ small, and for $\beta \in [1-\epsilon, 1]$, we wish to get a family of discs $A_{\alpha\beta} = (u_{\alpha\beta} + iv_{\alpha\beta}, w - w(\beta))$, which coincide with the initial disc A for $\alpha = 0$, $\beta = 1$, and whose boundaries satisfy the conditions

$$\begin{cases} v_{\alpha\beta}(\tau) = h(u_{\alpha\beta}(\tau), w(\tau) - w(\beta), \alpha + \eta(\tau)), & \tau \in \partial\Delta \\ u_{\alpha\beta}(\beta) = 0. \end{cases} \tag{3.2}$$

The first of (3.2) means that the "δ"-component of $\partial A_{\alpha\beta}$ equals $\alpha + \eta(\tau)$. To find the family of discs $A_{\alpha\beta}$, is equivalent as to find a solution $u = u_{\alpha\beta}$ in $\partial\Delta$ of *Bishop's equation*

$$u = T_\beta h\left(u, w - w(\beta), \alpha + \eta\right), \tag{3.3}$$

and to take the holomorphic extension of $u + iv = u + ih(u, w - w(\beta), \alpha + \eta)$ from $\partial\Delta$ to Δ. Here T_β is the Hilbert transform normalized by the condition $T_\beta u(\beta) = 0$. As for (3.3) we have the following

Lemma 3.1 *Let h_α be $C^{2,\gamma}$ with respect to x_1, z', α. Then for small w, η in $C^{j,\gamma}(\partial\Delta)$, $j = 0, 1$, and for α and β in $[0,1]$ close to 0 and 1 respectively, there is a unique solution $u = u_{\alpha\beta}$ to (3.3) in $\partial\Delta$. Moreover, for the holomorphic extension of $u_{\alpha\beta} + iv_{\alpha\beta}$ from $\partial\Delta$ to Δ, there exist continuous $\partial^i_\alpha \partial^l_\beta \partial^j_\tau (u + iv)$ for any $i + l + j \leq 2$ and $j = 0, 1$, and the derivation order in α, β and τ can be interchanged.*

Proof. We refer to [4, Th. 1 p. 220] as a general reference and, more closely, to [10, Th. 1.2]. We give herein a sketch of the proof. We fix $j = 0$ or $j = 1$ and write

$$F : C^{j,\gamma}(\partial\Delta, \mathbb{R}) \times C^{j,\gamma}(\partial\Delta, \mathbb{R}) \times C^{j,\gamma}(\partial\Delta, \mathbb{C}^{n-1}) \times \mathbb{R}^+ \times \mathbb{C} \to C^{j,\gamma}(\partial\Delta, \mathbb{R})$$
$$F : (u, \eta, w, \alpha, \beta) \mapsto u + T_\beta h(u, w - w(\beta), \alpha + \eta).$$

We have to solve the equation $F = 0$ in the unknown u. F is C^1 as an application between functional spaces, $F(0) = 0$, and $F'_u(0) = \mathrm{id}_u$. Hence by the implicit function theorem, there exists a unique solution $u = u_{\alpha\beta}$ for small w, η, α and for β close to 1. It is also clear that the mapping $\mathbb{R}^2_{\alpha,\beta} \to C^{j,\gamma}(\partial\Delta, \mathbb{R})$, $(\alpha, \beta) \mapsto u_{\alpha,\beta} + iv_{\alpha,\beta}$ is C^{2-j}. Let us choose first $j = 0$. Then there exists continuous $\partial_\alpha \partial_\beta (u + iv)$ and derivation in α and β can be interchanged. This completes the proof in case $j = 0$. Let now $j = 1$ and first choose $i = 1$. If $\tau = e^{i\theta}$ is the variable in $\partial\Delta$, we know that there exists continuous $\partial_\alpha \partial_\theta (u + iv)$. To discuss further about $\partial_\alpha u_{\alpha\beta}$, we look to the linearized equation

$$\dot{u} + T_\beta(\partial_{x_1} h \dot{u} + \partial_\alpha h) = 0. \tag{3.4}$$

Here both $\partial_{x_1} h$ and $\partial_\alpha h$ are $C^{1,\gamma}$ in $\theta \in \partial\Delta$ and therefore (3.4) has a unique solution $\dot{u} = \partial_\alpha u$ which is $C^{1,\gamma}$ with respect to $\theta \in \partial\Delta$ (and C^1 with respect to β). Hence there exists continuous $\partial_\theta \partial_\alpha u$ and moreover, taking into account the first part of the proof, we have commutation of derivation order in α and θ. The same also holds for v due to $v|_{\partial\Delta} = h(u, w - w(\beta), \alpha + \eta)|_{\partial\Delta}$. Finally, from

$$\partial_\theta = i\tau\partial_\tau - i\bar{\tau}\partial_{\bar{\tau}},$$

we get for the holomorphic extension of $u + iv$

$$\partial_\alpha \partial_\tau (u + iv) = \partial_\tau \partial_\alpha (u + iv).$$

We pass to $\partial_\beta (u + iv)$. For this we consider the equation

$$\dot{u} + \partial_\beta T_\beta h + T_\beta(\partial_{x_1} h \dot{u} - \partial_w h \dot{w}) = 0. \tag{3.5}$$

This equation has a unique solution $\dot{u} = \partial_\beta u$ which is $C^{1,\gamma}$ in θ. Hence there exists continuous $\partial_\theta \partial_\beta u$ and we have commutation of derivation order in β and θ. In the same way as above we finally conclude that we have commuatation of derivation in β and τ for $u + iv$. □

By Lemma 3.1 we are allowed to take the Taylor expansions of $\partial_\tau A_{\alpha\beta}$ in α, β up to order 1. On the other hand, since the initial disc A is tangent to M at $z^o = A(1)$, we have

$$\mathrm{Re}\langle \partial r, \partial_\tau A \rangle|_{\alpha=0,\beta=1,\tau=1} = 0. \tag{3.6}$$

It follows

$$\mathrm{Re}\langle \partial r, \partial_\tau A_{\alpha\beta} \rangle|_{\tau=1} = O((1-\beta) + \alpha). \tag{3.7}$$

In particular

$$v_{\alpha\beta}(\beta) \geq -\alpha - c(1-\beta)((1-\beta) + \alpha). \tag{3.8}$$

Let ν be a unit vector in $\mathbb{C}^n$, f a holomorphic function, ξ a point in the domain of f, $\{V\}$ the system of open neighborhoods of ξ, Δ_s the disc $\{\tau \in \mathbb{C} : |\tau| < s\}$. We define

$$r_f^{\nu,V} = \sup\{s : f \text{ is holomorphic in a neighborhood of } z + \nu\Delta_s \text{ for any } z \in V\},$$

and

$$r_f^\nu(\xi) = \sup_V r_f^{\nu,V}.$$

Let f be holomorphic on Ω, let $z \in \Omega$ and $z^o \in \partial\Omega$. Suppose that the vector $z^o - z$ is normal to $\partial\Omega$ in z^o, and write $\nu = \frac{z^o - z}{|z^o - z|}$ and $\delta(z) = |z^o - z|$; thus $\delta(z)$ is the distance from z to $\partial\Omega$. In this situation it is clear, by the definition of r_f^ν itself, that if

$$r_f^\nu(z) > \delta(z),$$

then f extends holomorphically to a full neighborhood of z^o. This is the crucial point in the proof of Theorem 2.1. We discuss further properties of r_f^ν. We first show that it describes the convergence radius of the Taylor expansion of f in the ν-direction. In other terms we claim that

$$r_f^\nu(\xi) = \sup\{s : |\partial_\nu^k f(z)| \le ck!s^{-k} \text{ for some } V, \ \forall z \in V, \ \forall k \in \mathbb{N}\}, \tag{3.9}$$

where ∂_ν denotes the holomorphic derivative along the ν-direction. In fact "$\le$" is clear by Cauchy's inequalities. As for "$\ge$", we denote by (z_1, z') the variable in $\mathbb{C}^n$, and suppose that the direction of ν is that of the z_1-axis. In a polydisc $\xi + (\Delta_\epsilon \times \Delta_\epsilon \times \dots)$, f is the sum of a "double" series in $z_1 - \xi_1$ and $z' - \xi'$ that we may rearrange as $\sum_k a_k(z')(z_1 - \xi_1)^k$ the coefficients $a_k(z')$ being thus holomorphic. If s is a number as in the right of (3.9), then the above series converges for $z_1 \in \xi_1 + \Delta_s$ and therefore defines a holomorphic function on $\xi + (\Delta_s \times \Delta_\epsilon \times \dots)$. This proves that f is holomorphic in a neighborhood of $z + (\Delta_s \times \{0\} \times \{0\} \times \dots)$ $\forall z \in V$ for a neighborhood V of ξ; in particular $r_f^\nu(\xi) > s$ which proves our claim. Next we have that

$$\log r_f^\nu \text{ is plurisuperharmonic,} \tag{3.10}$$

that is, over any 1-dimensional disc contained in Ω, it stands above the harmonic extension from the boundary. To see this, we consider points ξ close to a fixed ξ_o, and denote by S_ξ a family of discs with center ξ which are contained in Ω and cluster to a disc S_{ξ_o}. We use the notation "m.v.$_{\partial S_\xi}$" to denote the mean value along ∂S_ξ. For any (continuous) function s which verifies $s < r_f^\nu$ on ∂S_ξ, we have

$$\log|\partial_\nu^k f(\xi)| \leq \text{m.v.}_{\partial S_\xi} \log|\partial_\nu^k f| \\ \leq \log ck! - k(\text{m.v.}_{\partial S_\xi} \log s), \tag{3.11}$$

where the first inequality is clear because $\log|\partial_\nu^k f|_{S_\xi}$ is subharmonic, and the second is a consequence of (3.9). It follows

$$\begin{aligned} \log r_f^{\nu,V} &= \sup\{t : \log|\partial_\nu^k f(\xi)| < \log ck! - kt \ \forall \xi \in V\} \\ &\geq \sup\{t : t < \text{m.v.}_{\partial S_\xi} \log r_f^\nu \ \forall \xi \in V\} \\ &= \inf_{\forall \xi \in V} \text{m.v.}_{\partial S_\xi} \log r_f^\nu, \end{aligned} \tag{3.12}$$

where the first equality follows from (3.9) and the central inequality from (3.11). By the lower semi-continuity of $\log r_f^\nu$, and by Fatou's Theorem, we have

$$\begin{aligned} \text{m.v.}_{\partial S_{\xi_o}} \log r_f^\nu &\leq \liminf_{\xi} \text{m.v.}_{\partial S_\xi} \log r_f^\nu \\ &= \sup_V \inf_\xi \text{m.v.}_{\partial S_\xi} \log r_f^\nu \\ &\leq \sup_V \log r_f^{\nu,B} = \log r_f^\nu(\xi_o). \end{aligned} \tag{3.13}$$

We notice now that, though f is a priori only defined in Ω, and $A_{\alpha\beta}$ needs not to be contained in Ω, nevertheless f extends automatically to a neighborhood of $A_{\alpha\beta}$ for $\alpha > 0$. To see this, we choose a projection $z \mapsto z'$ transveral to $A_{\alpha\beta}$ and write points near $A_{\alpha\beta}$ in the form $z = z' + A_{\alpha\beta}(\tau)$. We use (z', τ) as local holomorphic coordinates, note that $z' + A_{\alpha\beta}(\tau) \subset \Omega$ $\forall |\tau| = 1$, and define

$$\tilde{f}(z) = (2\pi i)^{-1} \int_{|\zeta|=1} \frac{f(z' + A_{\alpha\beta}(\zeta))}{\zeta - \tau} d\zeta.$$

If we move z' inward Ω, it is clear that we achieve $z = z' + A_{\alpha\beta}(\tau) \in \Omega$ $\forall \tau \in \Delta$; but then, for these values of z, we have $\tilde{f}(z) = f(z)$ by Cauchy's formula; hence $\tilde{f}$ coincides with f over Ω.

End of Proof of Theorem 2.1 We pick up a point $z^1 \in \partial A \cap \Omega$ in view of (2.3), and fix our notations with $z^o = A(1)$ and $z^1 = A(-1)$. We observe that

$$\begin{cases} r_f^\nu|_{\partial_{\alpha\beta}} \geq \alpha, \\ r_f^\nu|_{\partial A_{\alpha\beta} \cap V} \geq \sigma, \end{cases} \tag{3.14}$$

for a suitable constant σ and a suitable neighborhood V of z^1 which are independent of α because $A_{\alpha\beta}(-1)$ converges to $A(-1) = z^1$, and f is holomorphic in a fixed neighborhood of z^1. We write $\tau = \rho e^{i\theta}$; the superharmonicity of $\log(r_f^\nu \circ A_{\alpha\beta})$ yields

$$\log(r_f^\nu \circ A_{\alpha\beta})(\tau) \geq (2\pi)^{-1} \int_0^{2\pi} \frac{(1-\rho^2)\log r_f^\nu(A_{\alpha\beta}(e^{it}))}{1+\rho^2-2\rho\cos(\theta-t)} dt$$

$$\geq \log\alpha + c\log\left(\frac{\sigma}{\alpha}\right)(1-\rho), \tag{3.15}$$

where $c = \frac{\text{diam}(V)}{2\,\text{diam}(A_{\alpha\beta})}$. It follows

$$r_f^\nu \circ A_{\alpha\beta}(\tau) \geq \alpha\left(1 + c'\log\left(\frac{\sigma}{\alpha}\right)(1-\rho)\right), \tag{3.16}$$

provided that $\log\left(\frac{\sigma}{\alpha}\right)(1-\rho)$ is small. In particular, evaluation of (3.16) for $\tau = \beta \in [-1, 1]$ approaching 1 yields

$$r_f^\nu \circ A_{\alpha\beta}(\beta) \geq \alpha + C\alpha(1-\beta), \text{ for any large } C. \tag{3.17}$$

If we plug together (3.8) and (3.17) with α and $1-\beta$ infinitesimal of the same order (and hence in particular with $\log\left(\frac{\sigma}{\alpha}\right)(1-\beta)$ small), we get

$$v_{\alpha\beta}(\beta) + r_f^\nu \circ A_{\alpha\beta}(\beta) > 0.$$

By recalling that $u_{\alpha\beta}(\beta) = 0$ and $w_{\alpha\beta}(\beta) = 0$, we conclude that z^o $(= 0)$ belongs to the domain where f is holomorphic. □

4. Corollaries and complements

Let Ω be a domain of $\mathbb{C}^n$ with $C^{2,\gamma}$ boundary $M = \partial\Omega$, z^o a point of M. We recall from [7] that M has the *one-sided extension property* from Ω at z^o when for any ball B with center z^o

any f holomorphic in $\Omega \cap B$ extends holomorphically to a fixed neighborhood of z^o.

Let A be a small disc, $C^{1,\gamma}$ up to the boundary, with $z^o \in \partial A$. We assume that A is tangent to M at z^o, and that it is contained in $\bar{\Omega}$. We also assume that A is not contained in M in any neighborhood of z^o; in other term we suppose that there is a sequence $z_k \to z^o$ with $z_k \in A \setminus M$. By the remark which follows the statement of Theorem 2.1, we have the one-sided extension property at z^o (from Ω). On the other hand the extension property (from either side of M) is characterized in [7] anf [8] as *minimality* that is the absence of germs of complex hypersurfaces belonging to M. By putting together these two facts we get

Proposition 4.1 *Let M contain a germ of a complex analytic hypersurface S, let A be a small analytic disc, $C^{1,\gamma}$ up to the boundary, contained in $\bar{\Omega}$ and tangent to M at z^o. Then A must be contained in S in a suitable neighborhood of z^o.*

Proof. Since M is non-minimal, then one-sided extension does not hold. Our Theorem 2.1, in the restatement of the subsequent remark, implies that $A \cap B \subset M$ for a suitable ball B with center z^o. It remains to check that we have indeed $A \cap B \subset S$. For this we denote by p_1 and p_2 the projections to $T^{\mathbb{C}}_{z^o}M\,(= T_{z^o}S)$ from M and S respectively, and define $A_1 := p_2^{-1}p_1(A \cap B)$. Clearly A_1 is holomorphic (because p_2^{-1} is holomorphic), and A_1 coincides with $A \cap B$ (because they have the same image under p_1). (One way of proving this uniqueness principle is to refer to the uniqueness of the solution to Bishop's implicit function theorem in the classes $C^{1,\gamma}$ (cf. [1]).) □

We conclude our discussion by considering the case of a pseudoconvex domain Ω, and a small disc A, attached to $\bar{\Omega}$ that is verifying $A(\partial\Delta) \subset \bar{\Omega}$. First we have to notice that A must be contained in $\bar{\Omega}$, otherwise pseudoconvexity is clearly violated. Also, A must be transversal to M at any point of ∂A unless $A \subset M$. Indeed, otherwise Theorem 2.1 would again produce a contradition to pseudoconvexity. This statement is classical (cf. e.g. [1]) but the above proof is not.

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