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Certain sufficient conditions for univalence

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Abstract. In this work some integral operators are studied and the author determines conditions for the univalence of these integral operators.

Key words: integral operator, univalence.

1. Introduction

Let $U = \{z : |z| < 1\}$ be the unit disk in the complex plane and let A be the class of functions which are analytic in the unit disk normalized with $f(0) = f'(0) - 1 = 0$.

Let S the class of the functions $f \in A$ which are univalent in U.

2. Preliminary results

In order to prove our main results we will use the theorems presented in this section.

Theorem A [2] *Assume that $f \in A$ satisfies condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, \quad z \in U, \tag{1}$$

then f is univalent in U.

Theorem B [3] *Let α be a complex number,* $\operatorname{Re}\alpha > 0$ *and $f(z) = z + a_2 z^2 + \cdots$ is a regular function in U. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2}$$

for all $z \in U$, then for any complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ *the function*

$$F_\beta(z) = \left[\beta \int_o^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} = z + \cdots \tag{3}$$

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is regular and univalent in U.

The Schwarz Lemma [1] *Let the analytic function $f(z)$ be regular in the unit circle $|z|<1$ and let $f(0)=0$. If, in $|z|<1$, $|f(z)|\leq 1$ then*

$$|f(z)|\leq |z|, \qquad |z|<1 \tag{4}$$

where equality can hold only if $f(z)=Kz$ and $|K|=1$.

3. Main results

Theorem 1 *Let $g\in A$ and γ be a complex number such that $\operatorname{Re}\gamma\geq 1$. If*

$$\left|\, zg'(z)\,\right|\leq 1, \qquad z\in U \tag{5}$$

and

$$|\gamma|\leq \frac{3\sqrt{3}}{2}, \tag{6}$$

then the function

$$T_\gamma(z)=\left[\gamma\int_0^z u^{\gamma-1}\left(e^{g(u)}\right)^\gamma du\right]^{\frac{1}{\gamma}} \tag{7}$$

is in the class S.

Proof. Let us consider the function

$$f(z)=\int_0^z \left(e^{g(u)}\right)^\gamma du \tag{8}$$

which is regular in U.

The function

$$p(z)=\frac{1}{|\gamma|}\frac{z\,f''(z)}{f'(z)} \tag{9}$$

where the constant $|\gamma|$ satisfies the inequality (6), is regular in U.

From (9) and (8) it follows that

$$p(z)=\frac{\gamma}{|\gamma|}z\,g'(z) \tag{10}$$

Using (10) and (5) we have

$$|p(z)|<1 \tag{11}$$

for all $z \in U$. From (10) we obtain $p(0) = 0$ and applying Schwarz-Lemma we obtain

$$\frac{1}{|\gamma|}\left|\frac{z f''(z)}{f'(z)}\right| \leq |z| \tag{12}$$

for all $z \in U$, and hence, we obtain

$$\left(1-|z|^2\right)\left|\frac{z f''(z)}{f'(z)}\right| \leq |\gamma||z|\left(1-|z|^2\right). \tag{13}$$

Let us consider the function $Q : [0,1] \to \mathrm{Re}, Q(x) = x\left(1-x^2\right)$, $x = |z|$. We have

$$Q(x) \leq \frac{2}{3\sqrt{3}} \tag{14}$$

for all $x \in [0,1]$. From (14), (13) and (6) we obtain

$$\left(1-|z|^2\right)\left|\frac{z f''(z)}{f'(z)}\right| \leq 1 \tag{15}$$

for all $z \in U$. From (8) we obtain $f'(z) = \left(e^{g(z)}\right)^{\gamma}$. Then, from (15) and Theorem B for $\mathrm{Re}\,\alpha = 1$ it follows that the function T_γ is in the class S.

□

Theorem 2 *Let $g \in A$, satisfy* (1), *γ be a complex number with* $\mathrm{Re}\,\gamma \geq 1$ *and* $|\gamma - 1| \leq \frac{54}{35+13\sqrt{13}}$. *If*

$$|g(z)| < 1, \quad z \in U, \tag{16}$$

then the function

$$H_\gamma(z) = \left[\gamma \int_0^z u^{2\gamma-2}\left(e^{g(u)}\right)^{\gamma-1} du\right]^{\frac{1}{\gamma}} \tag{17}$$

is in the class S.

Proof. We observe that

$$H_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1}\left(ue^{g(u)}\right)^{\gamma-1} du\right]^{\frac{1}{\gamma}}. \tag{18}$$

Let us consider the function

$$p(z) = \int_0^z \left(ue^{g(u)}\right)^{\gamma-1} du. \tag{19}$$

The function p is regular in U.

From (19) we obtain

$$\frac{p''(z)}{p'(z)} = (\gamma - 1)\,\frac{zg'(z)+1}{z} \tag{20}$$

and hence, we have

$$\left(1-|z|^2\right)\left|\frac{zp''(z)}{p'(z)}\right| = |\gamma - 1|\left(1-|z|^2\right)\left|zg'(z)+1\right| \tag{21}$$

for all $z \in U$. From (21) we get

$$\left(1-|z|^2\right)\left|\frac{zp''(z)}{p'(z)}\right| \le |\gamma - 1|\left(1-|z|^2\right)\left(\left|\frac{z^2\,g'(z)}{g^2(z)}\right|\frac{|g^2(z)|}{|z|}+1\right) \tag{22}$$

for all $z \in U$.

By the Schwartz Lemma also $|g(z)| \le |z|$, $z \in U$ and using (22) we obtain

$$\left(1-|z|^2\right)\left|\frac{z\,p''(z)}{p'(z)}\right| \le |\gamma - 1|\left(1-|z|^2\right)\left(\left|\frac{z^2\,g'(z)}{g^2(z)}-1\right||z|+|z|+1\right) \tag{23}$$

for all $z \in U$.

Since g satisfies the condition (1) then from (23) we have

$$\left(1-|z|^2\right)\left|\frac{z\,p''(z)}{p'(z)}\right| \le |\gamma - 1|\left(1-|z|^2\right)(2\,|z|+1) \tag{24}$$

for all $z \in U$.

Let us consider the function $G : [0,1] \to \Re$, $G(x) = \left(1-x^2\right)(2x+1)$, $x = |z|$.

We have

$$G(x) \le \frac{35+13\sqrt{13}}{54} \tag{25}$$

for all $x \in [0,1]$

Since $|\gamma - 1| \le \frac{54}{35+13\sqrt{13}}$, from (25) and (24) we conclude that

$$\left(1-|z|^2\right)\left|\frac{z\,p''(z)}{p'(z)}\right| \le 1 \tag{26}$$

for all $z \in U$.

Now (26) and Theorem B for $\operatorname{Re} \alpha = 1$ imply that the function H_γ is in the class S. □

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