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# On the univalence of an integral on a subclass of meromorphic convex univalent functions

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**Abstract.** A nonlinear integral operator is studied on the class of convex meromorphic functions in the exterior of the unit disk. In this paper, we improve a sufficient condition for univalency of the operator obtained earlier by the first author.

*Key words*: univalent, meromorphic, convex functions and integral operators.

## 1. Introduction and main results

Let $\mathcal{S}$ be the class of normalized functions $f(z) = z + a_2 z^2 \ldots$, analytic and univalent in the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. In [5] an integral operator $P_\lambda[f]$ defined by

$$P_\lambda[f](z) = \int_0^z (f'(t))^\lambda \, dt$$

was shown to map $\mathcal{S}$ into itself, provided that $|\lambda| \leq (\sqrt{5}-2)/3 = 0.078\ldots$. Becker [3] established the univalence of $P_\lambda[f]$ for $|\lambda| \leq 1/6$ whereas Royster [10] gave an example implying that, unless $\lambda = 1$, for any $\lambda$ outside of the disk $|\lambda| \leq 1/3$ a function $f_0 \in \mathcal{S}$ can be found such that $P_\lambda[f_0] \notin \mathcal{S}$. Pfaltzgraff [8] improved the range of $\lambda$ to $|\lambda| \leq 1/4$. The question, whether the operator $P_\lambda$ preserves univalency for $1/4 < |\lambda| \leq 1/3$ still remains open.

A similar problem is completely solved for the subclass $\mathcal{K} \subsetneq \mathcal{S}$ of univalent convex functions. Namely, the inclusion $P_\lambda[\mathcal{K}] \subset \mathcal{S}$ holds if and only if either $|\lambda| \leq 1/2$ or $\lambda$ is real with $1/2 \leq \lambda \leq 3/2$ (see [2, 8]). More results of the similar type for other subclasses of $\mathcal{S}$ are obtained in [6, 9].

Counterparts of these problems for the case of meromorphic functions were studied by a number of authors (see [1, 2, 11]), however, the relevant constants are smaller than in the regular case. Denote by $\Sigma$ be the class of function

$$f(\zeta) = \zeta + \sum_{k=0}^{\infty} \alpha_k \zeta^{-k},$$

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regular and univalent in $E^- = \{\zeta : 1 < |\zeta| < \infty\}$ and having a simple pole at $\zeta = \infty$. Let $\Sigma_{\mathcal{K}}$ be its subclass consisting of convex univalent functions. Define the following integral operator

$$P_\lambda[f](\zeta) = \int_{\zeta_0}^{\zeta} (f'(t))^\lambda \, dt, \tag{1.1}$$

with $\lambda \in \mathbb{C}$, $\zeta_0 \in E^-$.

Recently the first author [7] applied a condition for univalency by J. Becker [4] to show that $P_\lambda[\Sigma] \subset \Sigma$ for all $|\lambda| \leq 1/4$. The following result regarding the set

$$\Lambda(\Sigma_{\mathcal{K}}, \Sigma) = \{\lambda \in \mathbb{C} : P_\lambda[\Sigma_{\mathcal{K}}] \subset \Sigma\}$$

was established in [7].

**Theorem 1.2** *We have*

$$\{\lambda = \mu e^{i\nu} : |\lambda| \leq \tilde{\mu}_0(\nu)\} \subset \Lambda(\Sigma_{\mathcal{K}}, \Sigma),$$

*with*

$$\tilde{\mu}_0(\nu) = \begin{cases} (3 + \cos\nu)/4 & \text{for } \nu_0 \leq |\nu| \leq \pi, \\ (1 + |\sin\nu|)^{-1} & \text{for } 0 \leq |\nu| \leq \nu_0. \end{cases}$$

*Here* $\nu_0 = \arccos t_0 = 0.9633\ldots$, *where* $t_0 = 0.5707\ldots$ *is the unique positive real root of the equation*

$$47 - 52t - 46t^2 - 12t^3 - t^4 = 0$$

*in the interval* $(0, 1)$.

The function $\tilde{\mu}_0(\nu)$ appears to have a jump at $\nu = \nu_0$ which is only due to the method of proof. In the present paper we improve the result by removing the discontinuity. However, the second expression for $\tilde{\mu}_0(\nu)$ is unsuperseded albeit in a small neighborhood of $\nu = 0$. Our main result is as follows.

**Theorem 1.3** *Given any real* $\nu$ *such that* $|\nu| \leq \nu_0$, *where* $\nu_0$ *is defined in the previous theorem, we have* $\{\lambda = \mu e^{i\nu} : |\lambda| \leq \mu_0(\nu)\} \subset \Lambda(\Sigma_{\mathcal{K}}, \Sigma)$, *with*

$$\mu_0(\nu) = \frac{2\cos\theta_\nu - 1 - \cos\nu}{2(1 - \cos(\nu - \theta_\nu))}, \tag{1.4}$$

*$\theta_\nu$ being the unique positive real root of the equation*

$$\begin{aligned}F(\theta) :=\ & 7 - 5\cos^2\nu + (17 - 6\cos\nu - 11\cos^2\nu)\cos\theta \\ & - (4 + 10\cos\nu - 8\cos^2\nu)\cos^2\theta - (12 - 16\cos^2\nu)\cos^3\theta \\ & - (8 + 14\cos\theta)\sin\nu\sin\theta - (1 - 4\cos\theta - 8\cos^2\theta)\sin 2\nu\sin\theta \\ & = 0 \qquad (1.5)\end{aligned}$$

*in the interval* $0 < \theta < \theta'_\nu = 2\arctan[\sin\nu/(3 + \cos\nu)]$.

*Proof.* By reasoning as in [7] we can conclude that the inclusion $P_\lambda[\Sigma_K] \subset \Sigma$ holds for a given $\lambda = \mu e^{i\nu}$ if there exists some $c$, $|c| \leq 1$, such that

$$\begin{aligned}H(\rho, \lambda, c) &= \frac{2\mu|\rho - c|}{|1 - c|(\rho + 1)} + \frac{1}{\rho}\left|\frac{2\lambda(\rho - c)}{(1 - c)(\rho + 1)} + c\right| \\ &\leq 1 \quad \text{for all } \rho > 1. \qquad (1.6)\end{aligned}$$

In particular, we have

$$H(1, \lambda, c) = \mu + |\lambda + c| \leq 1 \qquad (1.7)$$

and

$$H(\infty, \lambda, c) = 2\mu/|1 - c| \leq 1. \qquad (1.8)$$

Assume that the two bounds are attained and we try to determine the largest value of $\mu$ for which (1.6) holds. Set $c = 1 - 2\mu e^{i\theta}$, whence $|c| \leq 1$ is equivalent to the inequality $\mu \leq \cos\theta$. On the other hand, if the estimate in (1.7) is attained, then we have

$$\mu = \frac{2\cos\theta - 1 - \cos\nu}{2(1 - \cos(\nu - \theta))}. \qquad (1.9)$$

Due to the previous studies it suffices to consider the case when $0 < \nu < \nu_0$ and $|\theta| < \pi/2$. Since it is of interest only to find $\mu > 1/2$, we get from (1.9) that

$$0 < \theta < 2\arctan(\sin\nu/(2 + \cos\nu)).$$

Note that from (1.9) it also follows that $|c| \leq 1$.

By using a standard method of computation, we find that $\mu$ is an increasing function of $\theta$ on $[0, \theta'_\nu]$, besides, there is also some numerical evidence that the global maximum for $\mu(\nu)$ is to be found here than on the remaining part of the previous interval. So, in what follows we shall confine ourselves to considering of (1.9) on the interval $[0, \theta'_\nu]$.

It can be easily seen that (1.6) is equivalent to the following condition

$$\left|(\rho - c)e^{i(\nu-\theta)} + c(\rho+1)\right| \leq \rho(\rho + 1 - |\rho - c|), \quad \rho \geq 1, \tag{1.10}$$

where the right-hand side is positive if $|c| \leq 1$. Therefore, we can square both sides of the inequality (1.10) without violating it. Thus,

$$\rho^2|\rho - c| \leq \rho^3 - \rho^2 \operatorname{Re} c + A_1\rho + A_0, \quad \rho \geq 1, \tag{1.11}$$

where

$$A_1 = \operatorname{Re}(c - \bar{c}e^{i(\nu-\theta)}), \quad A_0 = -|c|^2(1 - \cos(\nu - \theta)). \tag{1.12}$$

Clearly, both sides in (1.11) are positive for all $\rho \geq 1$, so we may square it again. After some elementary transformations, we write the resulting inequality as

$$a_4\rho^4 + a_3\rho^3 + a_2\rho^2 + a_1\rho + a_0 \geq 0, \quad \rho \geq 1, \tag{1.13}$$

where the coefficients are defined by the relations

$$a_4 = 2A_1 - (\operatorname{Im} c)^2, \quad a_3 = -2(A_1 \operatorname{Re} c - A_0), \tag{1.14}$$

$$a_2 = -2A_0 \operatorname{Re} c + A_1^2, \quad a_1 = 2A_0A_1, \ a_0 = A_0^2. \tag{1.15}$$

We recall that in [7] it was possible to prove that $a_4 \geq 0$ for $\theta = \theta'_\nu$, provided that $\nu_0 \leq |\nu| \leq \pi$. However, when $0 \leq |\nu| < \nu_0$, $\theta = \theta'_\nu$, we get $a_4 < 0$, so that the inequality (1.13) fails for sufficiently large $\rho$. Now, let us choose $\theta$ so that $a_4 = 0$, and prove that (1.13) is still valid.

Note that for $\rho = 1$ both sides of the inequality (1.10) are equal to $2\mu$, therefore $\rho = 1$ is a root of the polynomial in (1.13). Hence, it suffices to prove that

$$a_3\rho^2 + (a_3 + a_2)\rho - a_0 \geq 0, \quad \rho \geq 1. \tag{1.16}$$

This is implied by the following inequalities

$$a_3 \geq 0 \quad \text{and} \quad 2a_3 + a_2 - a_0 \geq 0. \tag{1.17}$$

Now we proceed to check (1.17). By substituting (1.9) into $c = 1 - 2\mu e^{i\theta}$ and then into (1.12) we obtain

$$\operatorname{Re} c = \frac{(1-\cos\theta)(1+2\cos\theta) - \sin\nu\sin\theta}{1-\cos(\nu-\theta)}, \tag{1.18}$$

$$\begin{aligned} A_1 = \frac{1-\cos\theta}{1-\cos(\nu-\theta)} \big[ & 3+\cos\theta + 2(1-\cos\theta - 2\cos^2\theta)\cos\nu \\ & - 2(1+2\cos\theta)\sin\nu\sin\theta\big], \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} A_0 = -\frac{1-\cos\theta}{1-\cos(\nu-\theta)} \big[ & 2+3\cos\theta + (1-2\cos\theta - 4\cos^2\theta)\cos\nu \\ & - 2(1+2\cos\theta)\sin\nu\sin\theta\big]. \end{aligned} \tag{1.20}$$

After lengthy computations the equation $a_4 = 0$ can be written in an equivalent form as (1.5). Clearly,

$$F(0) = 8(1-\cos\nu)^2 \geq 0.$$

On the other hand, we note that

$$F(\theta'_\nu) = \frac{(1-\cos\nu)^2(47 - 52\cos\nu - 46\cos^2\nu - 12\cos^3\nu - \cos^4\nu)}{(5+3\cos\nu)^3} < 0,$$

whenever $0 < |\nu| < \nu_0$. So, (1.5) has at least one root $\theta_\nu$ in the interval $(0, \theta'_\nu)$.

To prove the uniqueness of the root, let us verify that $F(\theta)$ is a convex function. We obtain

$$\begin{aligned} F''(\theta) = & -8 - 20\cos\nu + 16\cos^2\nu - (89 - 6\cos\nu - 107\cos^2\nu)\cos\theta \\ & + (16 + 40\cos\nu - 32\cos^2\nu)\cos^2\theta + (108 - 144\cos^2\nu)\cos^3\theta \\ & + (8\sin\nu + 17\sin 2\nu)\sin\theta + (56\sin\nu - 16\sin 2\nu)\sin\theta\cos\theta \\ & - 72\sin 2\nu\sin\theta\cos^2\theta. \end{aligned} \tag{1.21}$$

By substituting $t = \tan(\theta/2)$ into this expression we can show that $F''(\theta)$ has the same sign as the polynomial

$$\begin{aligned} P(t) = \sum\nolimits_{j=0}^{6} b_j t^j \equiv\ & 27 + 26\cos\nu - 53\cos^2\nu \\ & + (128\sin\nu - 142\sin 2\nu)t - (453 + 94\cos\nu - 619\cos^2\nu)t^2 \\ & + (32\sin\nu + 356\sin 2\nu)t^3 + (373 - 106\cos\nu - 459\cos^2\nu)t^4 \\ & - (96\sin\nu + 78\sin 2\nu)t^5 - (11 - 14\cos\nu + 21\cos^2\nu)t^6, \end{aligned} \tag{1.22}$$

where $0 < t < \tau = \sin\nu/(3+\cos\nu)$. Obviously,

$$b_0 = (1-\cos\nu)(27+53\cos\nu) > 0.$$

Therefore,

$$b_0 + b_1 t \geq t(b_0 + b_1\tau)/\tau,$$

and we see that

$$b_0 + b_1\tau = (1-\cos\nu)(209+30\cos\nu - 231\cos^2\nu)/(3+\cos\nu)$$

is positive. Furthermore, it follows that

$$b_0 + b_1 t + b_2 t^2 \geq t^2(b_0 + b_1\tau + b_2\tau^2)\tau^{-2}.$$

At the same time, we get

$$b_0 + b_1\tau + b_2\tau^2 = (1-\cos\nu)(174 - 248\cos\nu - 138\cos^2\nu \\ + 388\cos^3\nu)(3+\cos\nu)^{-2}$$

where the latter expression is positive for $0 < |\nu| < \nu_0 = 0.9633\ldots$.

The rest of the proof is done by the same token, just taking into account the following equalities

$$b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 \\ = (1-\cos\nu)(554 + 174\cos\nu + 18\cos^2\nu + 282\cos^3\nu \\ - 324\cos^4\nu)(3+\cos\nu)^{-3},$$

$$b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 \\ = (1-\cos\nu)(2035 + 1343\cos\nu - 710\cos^2\nu + 138\cos^3\nu \\ - 125\cos^4\nu + 135\cos^5\nu)(3+\cos\nu)^{-4},$$

$$b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 + b_5\tau^5 \\ = (1-\cos\nu)(6009 + 5812\cos\nu - 751\cos^2\nu + 208\cos^3\nu \\ - 21\cos^4\nu + 28\cos^5\nu - 21\cos^6\nu)(3+\cos\nu)^{-5},$$

$$b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 + b_5\tau^5 + b_6\tau^6 \\ = (18016 + 23448\cos\nu + 3616\cos^2\nu - 112\cos^3\nu + 64\cos^4\nu \\ + 24\cos^5\nu)(1-\cos\nu)(3+\cos\nu)^{-6}.$$

So, for any $\nu$, $0 < |\nu| < \nu_0$, there exists a unique $\theta_\nu$, $0 < \theta_\nu < \theta'_\nu$, such that $F(\theta_\nu) = 0$. Equivalently, there exists a unique $d_\nu$, $3 < d_\nu < \infty$, such that

$$\theta_\nu = 2\arctan[\sin\nu/(d_\nu + \cos\nu)].$$

However, to prove the required inequalities we need to further specify the location of $d_\nu$. To this end, let us substitute $\theta = 2\arctan(\sin\nu/(d + \cos\nu))$ in (1.5) and write this equation in the equivalent form

$$\begin{aligned} p(d, \cos\nu) &= -4(d-2)^2\cos^4\nu - (12d^3 - 40d^2 + 20d + 24)\cos^3\nu \\ &\quad - (9d^4 - 12d^3 - 50d^2 + 76d + 1)\cos^2\nu \\ &\quad + (2d^5 - 30d^4 + 88d^3 - 68d^2 - 10d + 2)\cos\nu \\ &\quad + (4d^6 - 22d^5 + 39d^4 - 16d^3 - 14d^2 + 6d - 1) \\ &= 0, \end{aligned} \tag{1.23}$$

where $p(d, x)$ is a polynomial with respect to both $d$ and $x$.

Now, it is possible to verify the required estimates. From (1.14)–(1.20) we get

$$\begin{aligned} &[1 - \cos(\nu - \theta)]^2(1 - \cos\theta)^{-1}a_3 \\ &= -18 - 6\cos\nu + 8\cos^2\nu + (-28 + 12\cos\nu + 20\cos^2\nu)\cos\theta \\ &\quad + (10 + 26\cos\nu - 12\cos^2\nu)\cos^2\theta + (28 - 24\cos^2\nu)\cos^3\theta \\ &\quad - 16\cos\nu\cos^4\theta + 6(3 + \cos\nu)\sin\nu\sin\theta \\ &\quad + 4(7 - 3\cos\nu)\sin\nu\sin\theta\cos\theta - 24\sin\nu\cos\nu\sin\theta\cos^2\theta \\ &\quad - 16\sin\nu\sin\theta\cos^3\theta \end{aligned}$$

and

$$\begin{aligned} &[1 - \cos(\nu - \theta)]^2(1 - \cos\theta)^{-1}(2a_3 + a_2 - a_0) \\ &= -31 - 16\cos\nu + 9\cos^2\nu + (-29 + 38\cos\nu + 39\cos^2\nu)\cos\theta \\ &\quad + (8 + 26\cos\nu - 16\cos^2\nu)\cos^2\theta \\ &\quad + (36 - 16\cos\nu - 48\cos^2\nu)\cos^3\theta + 6(5 + 2\cos\nu)\sin\nu\sin\theta \\ &\quad + 2(19 - 8\cos\nu)\sin\nu\sin\theta\cos\theta - 16(1 + 3\cos\nu)\sin\nu\sin\theta\cos^2\theta. \end{aligned}$$

After substituting $\theta = 2\arctan(\sin\nu/(d + \cos\nu))$ into the last two expressions, we can see that it suffices to prove that the following two expressions are positive:

$$\begin{aligned}\sum_{k=0}^{8} q_k d^k \equiv &-2d^8 + (15 - \cos\nu)d^7 - (46 - 29\cos\nu - 19\cos^2\nu)d^6 \\ &+ (65 - 131\cos\nu - 46\cos^2\nu + 38\cos^3\nu)d^5 \\ &- (32 - 215\cos\nu + 71\cos^2\nu + 150\cos^3\nu - 28\cos^4\nu)d^4 \\ &- (11 + 123\cos\nu - 264\cos^2\nu - 128\cos^3\nu + 128\cos^4\nu - 8\cos^5\nu)d^3 \\ &+ (10 - 9\cos\nu - 183\cos^2\nu + 104\cos^3\nu + 172\cos^4\nu - 40\cos^5\nu)d^2 \\ &+ (3 + 23\cos\nu + 6\cos^2\nu - 134\cos^3\nu - 40\cos^4\nu + 64\cos^5\nu)d \\ &- 2 - 3\cos\nu + 11\cos^2\nu + 14\cos^3\nu - 32\cos^4\nu - 32\cos^5\nu, \quad (1.24)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{6} r_k d^k \equiv &-8d^6 + (52 + 4\cos\nu)d^5 - (119 - 28\cos\nu - 27\cos^2\nu)d^4 \\ &+ (92 - 172\cos\nu - 76\cos^2\nu + 28\cos^3\nu)d^3 \\ &+ (18 + 232\cos\nu - 14\cos^2\nu - 96\cos^3\nu + 12\cos^4\nu)d^2 \\ &- (24 + 32\cos\nu - 148\cos^2\nu - 60\cos^3\nu + 48\cos^4\nu)d \\ &- 15 - 60\cos\nu - 61\cos^2\nu + 40\cos^3\nu + 48\cos^4\nu. \quad (1.25)\end{aligned}$$

In order to do this, let us partition the ranges for both $d$ and $\nu$ into several parts and find the relevant estimates separately. Assuming that $\nu_1 \le \nu \le \nu_2$, $d_1 \le d \le d_2$, we can easily notice that

$$q_8 d^8 + q_7 d^7 \ge (15 - 2d_2 - \cos\nu)d^7 \ge (15 - 2d_2 - \cos\nu)d_1 d^6,$$

the first factor being positive. Similarly, it follows that

$$\begin{aligned}&q_8 d^8 + q_7 d^7 + q_6 d^6 \\ &\qquad \ge \left[15d_1 - 2d_1 d_2 - 46 + (29 - d_1)\cos\nu + 19\cos^2\nu\right] d_1 d^5.\end{aligned}$$

However, in this case, in order to prove that the factor is positive we need to make the necessary evaluations for each of the following cases separately.

As a result, we can write

$$q_8 d^8 + q_7 d^7 + q_6 d^6 + q_5 d^5 \ge \begin{cases} M_1 d_1^5 & \text{if } M_1 \ge 0, \\ M_1 d_2^5 & \text{if } M_1 < 0, \end{cases} \tag{1.26}$$

where

$$\begin{aligned}M_1 = 15d_1^2 - 2d_1^2 d_2 - 46d_1 + (29d_1 - d_1^2 - 131)\cos\nu_1 \\ + (19d_1 - 46)\cos^2\nu_2 + 38\cos^3\nu_2.\end{aligned} \tag{1.27}$$

In a similar way,

$$q_4 d^4 \geq Q'_2 d_1^4, \quad \text{if } M_2 \geq 0, \tag{1.28}$$

where

$$M_2 = -32 + 215\cos\nu_2 - 71\cos^2\nu_1 - 150\cos^3\nu_1 + 28\cos^4\nu_2, \tag{1.29}$$

otherwise we shall use rather a crude estimate

$$\begin{aligned} q_4 d^4 \geq (215\cos\nu_2 + 28\cos^4\nu_2)d_1^4 \\ - (32 + 71\cos^2\nu_1 + 150\cos^3\nu_1)d_2^4. \end{aligned} \tag{1.30}$$

Since the polynomial

$$g(x) = -11 - 123x + 264x^2 + 128x^3 - 128x^4 + 8x^5$$

is positive and increases with respect to $x$ on the whole segment $0.5707\cdots \leq x \leq 1$, we get

$$\begin{aligned} q_3 d^3 \geq (-11 - 123\cos\nu_2 + 264\cos^2\nu_2 + 128\cos^3\nu_2 \\ - 128\cos^4\nu_2 + 8\cos^5\nu_2)d_1^3. \end{aligned} \tag{1.31}$$

Next, we note that

$$q_2 d^2 \geq M_3 d_1^2, \quad \text{if } M_3 \geq 0, \tag{1.32}$$

where

$$\begin{aligned} M_3 = 10 - 9\cos\nu_1 - 183\cos^2\nu_1 + 104\cos^3\nu_2 \\ + 172\cos^4\nu_2 - 40\cos^5\nu_1, \end{aligned} \tag{1.33}$$

otherwise take

$$\begin{aligned} q_2 d^2 \geq (10 + 104\cos^3\nu_2 + 172\cos^4\nu_2)d_1^2 \\ - (9\cos\nu_1 + 183\cos^2\nu_1 + 40\cos^5\nu_1)d_2^2. \end{aligned} \tag{1.34}$$

Since the polynomial

$$h(x) = 3 + 23x + 6x^2 - 134x^3 - 40x^4 + 64x^5$$

is negative and decreases with respect to $x$ for all $0.5707\dots \le x \le 1$, we get

$$q_1 d \ge (3 + 23\cos\nu_1 + 6\cos^2\nu_1 - 134\cos^3\nu_1 - 40\cos^4\nu_1 + 64\cos^5\nu_1)d_2. \quad (1.35)$$

Finally, we have

$$q_0 \ge -2 - 3\cos\nu_1 + 11\cos^2\nu_1 + 14\cos^3\nu_1 - 32\cos^4\nu_1 - 32\cos^5\nu_1. \quad (1.36)$$

The polynomial expression in (1.25) is estimated quite analogously. Thus, we have

$$r_6 d^6 + r_5 d^5 + r_4 d^4 \ge \begin{cases} M_4 d_1^4 & \text{if } M_4 \ge 0, \\ M_4 d_2^4 & \text{if } M_4 < 0, \end{cases} \quad (1.37)$$

where

$$M_4 = 52d_1 - 8d_1 d_2 - 119 + (28 + 4d_1)\cos\nu_2 + 27\cos^2\nu_2. \quad (1.38)$$

Furthermore,

$$r_3 d^3 + r_2 d^2 \ge \begin{cases} M_5 d_1^2 & \text{if } M_5 \ge 0, \\ M_5 d_2^2 & \text{if } M_5 < 0, \end{cases} \quad (1.39)$$

where

$$M_5 = 92d_2 + 18 + (232 - 172d_2)\cos\nu_1 - (76d_2 + 14)\cos^2\nu_1 + 24(d_2 - 4)\cos^3\nu_1 + 4d_2\cos^3\nu_2 + 12\cos^4\nu_2. \quad (1.40)$$

The polynomial

$$\phi(x) = -24 - 32x + 148x^2 + 60x^3 - 48x^4$$

is positive and increases on the segment $0.5707\dots \le x \le 1$ and hence, we have

$$r_1 d \ge (-24 - 32\cos\nu_2 + 148\cos^2\nu_2 + 60\cos^3\nu_2 - 48\cos^4\nu_2)d_1. \quad (1.41)$$

As the relevant expression is not monotonic with respect to $\nu$ on the whole interval, let us take

$$r_0 \ge -15 - 60\cos\nu_1 - 61\cos^2\nu_1 + 40\cos^3\nu_2 + 48\cos^4\nu_2. \quad (1.42)$$

Now, we show how the requested estimates are verified. In the first case, assume that $\nu_1 = 0$ and $\nu_2 = 0.3$. For $d = 3.91$ solve the algebraic equation $p(d, x) = 0$, the function $p(d, x)$ being defined in (1.23). Since it has only one positive root $x = 1.0034\ldots$, we deduce that $p(3.91, \cos\nu) > 0$ for all $\nu$.

On the other hand, by setting in this equation $d = 3.81$ we find that its positive root $x = 0.9529\ldots$ is less than $\cos 0.3 = 0.9553\ldots$, so that $p(3.81, \cos\nu) < 0$ for all $\nu \in [0, 0.3]$.

Therefore, if $0 \le \nu \le 0.3$, then the solution of (1.5) corresponds to some $d_\nu$ such that $3.81 \le d_\nu \le 3.91$.

In view of estimates (1.26)–(1.42), we conclude that $Q_+ \ge 8704.9$, and $R_+ \ge 360.0$, where $Q_+$, $R_+$ are lower bounds for the expressions in (1.24) and (1.25), respectively.

Table 1 includes estimates obtained in each of the remaining cases. We see that both required expressions are positive, and therefore, the proof of the Theorem 1.3 is complete. □

Note that the increment of $\nu$ should be kept steadily decreasing because $\cos\nu$ changes more rapidly for larger values of $\nu$ and so does the difference between the bounds of $d$.

Table 1.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\nu_1$ | 0.3 | 0.43 | 0.52 | 0.59 | 0.65 | 0.7 | 0.74 |
| $\nu_2$ | 0.43 | 0.52 | 0.59 | 0.65 | 0.7 | 0.74 | 0.78 |
| $d_1$ | 3.72 | 3.63 | 3.56 | 3.49 | 3.42 | 3.37 | 3.31 |
| $d_2$ | 3.82 | 3.73 | 3.64 | 3.57 | 3.5 | 3.43 | 3.375 |
| $Q_+$ | 4418.1 | 2507.4 | 5128.4 | 3414.6 | 2278.8 | 2070.6 | 1073.7 |
| $R_+$ | 103.6 | 24.5 | 114.5 | 38.1 | 5.8 | 103.1 | 15.0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\nu_1$ | 0.78 | 0.81 | 0.84 | 0.87 | 0.89 | 0.91 | 0.93 | 0.95 |
| $\nu_2$ | 0.81 | 0.84 | 0.87 | 0.89 | 0.91 | 0.93 | 0.95 | 0.964 |
| $d_1$ | 3.26 | 3.219 | 3.169 | 3.13 | 3.09 | 3.06 | 3.026 | 3 |
| $d_2$ | 3.32 | 3.27 | 3.221 | 3.171 | 3.14 | 3.105 | 3.063 | 3.03 |
| $Q_+$ | 996.9 | 667.9 | 243.2 | 501.6 | 166.9 | 62.6 | 6.0 | 159.6 |
| $R_+$ | 53.8 | 48.1 | 9.7 | 73.6 | 19.1 | 5.7 | 3.1 | 33.1 |