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A generalization of the Lieb-Thirring inequalities in low dimensions

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Abstract. We give an estimate for the moments of the negative eigenvalues of elliptic operators on $\mathbb{R}^n$ in low dimensions. The estimate is a generalization of the Lieb-Thirring inequalities in one or two dimensions. We use the φ-transform decomposition of Frazier and Jawerth.

Key words: elliptic operator, eigenvalues, φ-transform, A_p-weights.

1. Introduction

For a real-valued measurable function V on $\mathbb{R}^n$ we set

$$V_+(x) = \max(V(x), 0) \quad \text{and} \quad V_-(x) = \max(-V(x), 0).$$

The Lieb-Thirring inequalities state

$$\sum_i |\lambda_i|^\gamma \le c_{n,\gamma} \int_{\mathbb{R}^n} V_-^{n/2+\gamma} dx \tag{1}$$

for suitable $\gamma \ge 0$, where $\lambda_1 \le \lambda_2 \le \cdots$ are the negative eigenvalues of the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^n)$. The inequality (1) holds if and only if

$$\begin{aligned} \gamma &\ge \frac{1}{2} && \text{for} \quad n = 1, \\ \gamma &> 0 && \text{for} \quad n = 2, \\ \gamma &\ge 0 && \text{for} \quad n \ge 3. \end{aligned}$$

The case $\gamma > 1/2$, $n = 1$, $\gamma > 0$, $n \ge 2$ was proved by Lieb and Thirring ([8]). They applied the inequality (1) to the problem of the stability of matter. The case $\gamma = 1/2$, $n = 1$ was proved by Weidl ([18]). The case $\gamma = 0$, $n \ge 3$ was established by Cwikel ([1]), Lieb ([7]) and Rozenbljum ([12], [13]). Some generalizations and variations of the Lieb-Thirring inequalities

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are known ([2], [6], [9], [14], [15]). In particular Egorov and Kondrat'ev ([2]) studied the estimate for $L_0 + V$ where L_0 is an elliptic operator of order $2m$.

In the present paper we give a generalization of a result by Egorov and Kondrat'ev's for certain degenerate elliptic partial differential operator in low dimension, for which the rate of degeneracy is regulated by the weight $w \in A_2$. A generalization of the higher dimensional cases is given in [17]. In the proof of our main theorem we use the φ-transform of Frazier-Jawerth ([3]).

First we recall the definition of A_p-weights. By a cube in $\mathbb{R}^n$ we mean a cube of which sides are parallel to coordinate axes. A locally integrable and nonnegative function w on $\mathbb{R}^n$ is an A_p-weight for some $p \in (1, \infty)$ if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(x)\,dx\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \le C$$

for all cubes $Q \subset \mathbb{R}^n$. The infimum of the constant C is called the A_p-constant of w.

We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(y)\,dy \le Cw(x) \quad a.e.\ x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. We write A_p for the class of A_p-weights. It turns out that $A_1 \subset A_p$ for $p > 1$.

For a non-negative and locally integrable function w on $\mathbb{R}^n$ we define

$$L^p(w) = \left\{f : f \text{ is measurable on } \mathbb{R}^n,\ \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx < \infty\right\}.$$

Next we consider an elliptic partial differential operator of order $2m$. For $m \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n)$ let

$$L_0 f(x) = \sum_{|\alpha|=|\beta|=m} (-1)^m D^\alpha\left(a_{\alpha\beta}(x)D^\beta f(x)\right),$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n,$$

$$a_{\alpha\beta} \in H^m_{\mathrm{loc}}(\mathbb{R}^n), \quad \text{and} \quad a_{\alpha\beta} = \overline{a_{\beta\alpha}}.$$

In the above definition the space $H^m_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the set of all $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that $D^\alpha f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for all $|\alpha| \le m$.

Let

$$a(f,g) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta f(x) \overline{D^\alpha g(x)}\, dx$$

for $f, g \in C_0^\infty(\mathbb{R}^n)$ and $\|\cdot\|$ be the norm of $L^2(\mathbb{R}^n)$.

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ the cube Q defined by

$$Q = \{(x_1, \ldots, x_n) : k_i \le 2^\nu x_i < k_i + 1,\ i = 1, \ldots, n\}$$

is called a dyadic cube in $\mathbb{R}^n$. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$. For any $Q \in \mathcal{Q}$ there exists a unique $Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and the side-length of Q' is double of that of Q. We call Q' the parent of Q.

We have the following theorem.

Theorem 1.1 *Let $n \le 2m$, $q \ge n/(2m)$, $\gamma > 0$ and $q+\gamma > 1$. We assume that there exists a $w \in A_2$ such that*

$$(L_0 f, f) \ge \int_{\mathbb{R}^n} w(x) \sum_{|\alpha|=m} |D^\alpha f(x)|^2\, dx \tag{2}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and

$$\int_{Q'} w\, dx \le 2^{2m} \int_Q w\, dx \tag{3}$$

for all $Q \in \mathcal{Q}$ and its parent Q'.

For a $u \in A_{q+\gamma}$ we suppose that

$$|Q|^{2m/n+1} \le c_1 \int_Q w\, dx \left(\int_Q u\, dx \right)^{1/q} \tag{4}$$

for all cubes $Q \subset \mathbb{R}^n$, where c_1 is a positive constant not depending on Q. For a real valued function V on $\mathbb{R}^n$ we assume that $V_+ \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} V_-^{q+\gamma} u\, dx < \infty. \tag{5}$$

Let $\mathcal{H}$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}} = \left\{ a(f,f) + \int_{\mathbb{R}^n} V_+ |f|^2 \, dx + \|f\|^2 \right\}^{1/2}.$$

Then we have the following.

(i) *There exists a unique self-adjoint operator L in $L^2(\mathbb{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that*

$$(Lf, g) = a(f,g) + \int_{\mathbb{R}^n} V f \overline{g} \, dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$.

(ii) *The negative spectrum of L is discrete.*

(iii) *There exists a positive constant c such that*

$$\sum_i |\lambda_i|^\gamma \le c \int_{\mathbb{R}^n} V_-^{q+\gamma} u \, dx, \tag{6}$$

where $\{\lambda_i\}$ is the set of all negative eigenvalues of L counting multiplicity and c depends only on n, m, q, γ, c_1, A_2-constant of w, and $A_{q+\gamma}$-constant of u.

The inequality (6) is a generalization of the Lieb-Thirring inequality for the case $\gamma > 1/2$, $n = 1$ and $\gamma > 0$, $n = 2$. Our result does not include the case $\gamma = 1/2$, $n = 1$. The case $w \equiv 1$ and $u(x) = |x - x_0|^{2mq-n}$ is proved by Egorov and Kondrat'ev ([2]). In [9] Netrusov and Weidl proved (6) for $w \equiv u \equiv 1$, $q = n/(2m) < 1$, $\gamma = 1 - n/(2m)$. Our result does not include their result.

We remark that the condition (4) is trivial by Hölder's inequality when $q = n/(2m)$ and $u = w^{-n/(2m)}$. An example of L_0 which satisfies the conditions in Theorem 1.1 is given in Section 4. We also prove in Section 4 that when $n = 2m$ the condition (3) means that w is "essentially" constant. We give a bound for the constant c in (6) in Section 4.

2. Preliminaries

First we recall some properties of A_p-weights which will be used in the following sections. Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes Q which contain x.

Proposition 2.1

(i) *Let $1 < p < \infty$ and w be a non-negative and locally integrable function on $\mathbb{R}^n$. Then there is a positive constant c such that*

$$\int_{\mathbb{R}^n} M(f)^p w\, dx \leq c \int_{\mathbb{R}^n} |f|^p w\, dx$$

for all $f \in L^p(w)$ if and only if $w \in A_p$.

(ii) *Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.*

(iii) *Let $0 < \tau < 1$ and f be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e. Then $(M(f))^\tau \in A_1$.*

(iv) *Let $1 \leq p < \infty$ and $w \in A_p$. Then there exists a positive constant c such that*

$$\int_{2Q} w\, dx \leq c \int_Q w\, dx$$

for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of Q.

The proofs of these facts are in [4, Chapter IV] or [16, Chapter V]. Property (iv) is called the doubling property of A_p-weights.

Let φ be a function which satisfies the following conditions.

(A1) $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(A2) $\operatorname{supp} \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$

(A3) $|\hat{\varphi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$.

(A4) $\sum_{\nu \in \mathbb{Z}} |\hat{\varphi}(2^\nu \xi)|^2 = 1$ for all $\xi \neq 0$.

For a dyadic cube Q such that

$$Q = \{(x_1, \ldots, x_n) : k_i \leq 2^\nu x_i < k_i + 1,\ i = 1, \ldots, n\}.$$

for $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we set

$$\varphi_Q(x) = 2^{\nu n/2} \varphi(2^\nu x - k).$$

3. Proof of Theorem 1.1

By (ii) of Proposition 2.1 there exists a constant s such that $1 < s < q+\gamma$ and $u \in A_{(q+\gamma)/s}$. It turns out that $V_- \in L^s_{\mathrm{loc}}(\mathbb{R}^n)$ (cf. [17, Section 3]).

Let $v(x) = (M(V_-^s)(x))^{1/s}$. We may assume that $v(x) > 0$ for all $x \in \mathbb{R}^n$. By the properties of the maximal operator we have $V_-(x) \le v(x)$ *a.e.* By (i) of Proposition 2.1 we get

$$\int_{\mathbb{R}^n} v^{q+\gamma} u\, dx = \int_{\mathbb{R}^n} M(V_-^s)^{(q+\gamma)/s} u\, dx \le c_1 \int_{\mathbb{R}^n} V_-^{q+\gamma} u\, dx < \infty.$$

Furthermore v is an A_1-weight by (iii) of Proposition 2.1.

We have the following lemmas.

Lemma 3.1 *There exists a positive constant α such that*

$$\alpha \sum_{Q\in\mathcal{Q}} |Q|^{-2m/n} |(f,\varphi_Q)|^2 \frac{1}{|Q|}\int_Q w\, dx \le \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

Lemma 3.2 *There exist positive constants β and β' such that*

$$\beta' \sum_{Q\in\mathcal{Q}} |(f,\varphi_Q)|^2 \frac{1}{|Q|}\int_Q v\, dx \le \int_{\mathbb{R}^n} |f|^2 v\, dx$$
$$\le \beta \sum_{Q\in\mathcal{Q}} |(f,\varphi_Q)|^2 \frac{1}{|Q|}\int_Q v\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

The proof of Lemma 3.1 is in [17, Proposition 2.2 and Lemma 3.2]. Lemma 3.2 is proved in [3].

Now we set

$$\mathcal{I} = \left\{ Q \in \mathcal{Q} : \beta \int_Q v(x)\, dx > \alpha |Q|^{-2m/n} \int_Q w(x)\, dx \right\},$$

where α and β are constants in Lemmas 3.1 and 3.2. We remark that $\mathcal{I}$ is not empty. In fact, if $\mathcal{I}$ is empty, then we have

$$\beta \int_Q v(x)\, dx \le \alpha |Q|^{-2m/n} \int_Q w(x)\, dx$$

for all $Q \in \mathcal{Q}$. Let $Q_0 \in \mathcal{Q}$ and $Q_0 \subset Q_1 \subset Q_2 \subset \cdots$ be the infinite sequence of dyadic cubes such that Q_{i+1} is the parent of Q_i for all $i = 1, 2, \ldots$. By (3) we have

$$|Q_{i+1}|^{-2m/n} \int_{Q_{i+1}} w(x)\,dx \le |Q_i|^{-2m/n} \int_{Q_i} w(x)\,dx$$

for all i. Hence we have

$$\beta \int_{Q_i} v(x)\,dx \le \alpha |Q_0|^{-2m/n} \int_{Q_0} w(x)\,dx$$

for all i. On the other hand, since $v \in A_1$, there exists a constant $d > 1$ such that

$$d \int_{Q_i} v\,dx \le \int_{Q_{i+1}} v\,dx$$

for all i (cf. [4, p. 141]). Hence we have

$$d^i \int_{Q_0} v\,dx \le \int_{Q_i} v\,dx$$

and

$$\lim_{i\to\infty} \int_{Q_i} v\,dx = \infty.$$

This is a contradiction. Therefore $\mathcal{I}$ is not empty.

Let $Q \in \mathcal{I}$ and Q' be the parent of Q. Then we have

$$\begin{aligned}\alpha|Q'|^{-2m/n} \int_{Q'} w(x)\,dx \le \alpha|Q|^{-2m/n} \int_{Q} w(x)\,dx < \beta \int_Q v(x)\,dx \\ \le \beta \int_{Q'} v(x)\,dx.\end{aligned}$$

Hence we have $Q' \in \mathcal{I}$. This fact means that $\mathcal{I}$ is an infinite set.

Lemma 3.3 *There exists a $c > 0$ such that*

$$\sum_{Q\in\mathcal{I}} \left(\frac{1}{|Q|} \int_Q v\,dx \right)^{\gamma} \le c \int_{\mathbb{R}^n} v^{q+\gamma} u\,dx$$

The proof of this lemma will be given later.

For $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int |f|^2 V_- \, dx \le \int |f|^2 v \, dx \le \beta \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx,$$

where we used Lemma 3.2. The last quantity is bounded by

$$\begin{aligned} &\beta \sum_{Q \in \mathcal{I}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx + \beta \sum_{Q \notin \mathcal{I}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx \\ &\le \beta K \sum_{Q \in \mathcal{I}} |(f, \varphi_Q)|^2 + \alpha \sum_{Q \notin \mathcal{I}} |(f, \varphi_Q)|^2 |Q|^{-2m/n} \frac{1}{|Q|} \int_Q w \, dx \\ &\le cK \|f\|_2^2 + \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha| = m} |D^\alpha f|^2 \right\} w \, dx \end{aligned}$$

where

$$K = \max_{Q \in \mathcal{I}} \frac{1}{|Q|} \int_Q v \, dx$$

and we used Lemma 3.1. We remark that K is finite by Lemma 3.3.

By the condition (2) we have

$$\int_{\mathbb{R}^n} |f|^2 V_- \, dx \le \int_{\mathbb{R}^n} |f|^2 v \, dx \le cK \|f\|_2^2 + (L_0 f, f). \tag{7}$$

Hence we have

$$a(f, f) + \int_{\mathbb{R}^n} V |f|^2 \, dx \ge -cK \|f\|_2^2$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. Therefore

$$b(f, g) = a(f, g) + \int_{\mathbb{R}^n} V f \overline{g} \, dx$$

is a lower semi-bounded quadratic form on $\mathcal{H}$.

By the assumption of the coefficients of L_0 and $V_+ \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ we can show that $b(f, g)$ is a closed form on $\mathcal{H}$ (cf. [17]). Since $b(f, g)$ is a closed and lower semi-bounded quadratic form on $\mathcal{H}$, there exists a unique self-adjoint operator L in $L^2(\mathbb{R}^n)$ with domain $\mathcal{D} \subset \mathcal{H}$ such that

$$(Lf, g) = a(f, g) + \int_{\mathbb{R}^n} V f \overline{g} \, dx$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$ ([10, Theorem VIII.15]).

We set

$$\lambda_1 = \inf_{f \in \mathcal{D},\, \|f\|=1} (Lf, f)$$

and

$$\lambda_k = \sup_{\phi_1,\ldots,\phi_{k-1} \in L^2} \inf_{\substack{f \in \mathcal{D},\, \|f\|=1,\\ (f,\phi_j)=0,\, j=1,\ldots,k-1}} (Lf, f)$$

for $k \geq 2$.

For each fixed $k \in \mathbb{N}$ either:

(i) there are k eigenvalues counting multiplicity below the infimum of the essential spectrum of L, and λ_k is the kth eigenvalue of L;

or

(ii) λ_k is the infimum of the essential spectrum of L and $\lambda_k = \lambda_{k+1} = \lambda_{k+2} = \cdots$ and there are at most $k-1$ eigenvalues counting multiplicity below λ_k.

The proof of this fact is in [11, Theorem XIII.1].

We have the following lemma.

Lemma 3.4 *Let $A > 0$ and*

$$\mathcal{I}_A = \left\{ Q \in \mathcal{I} : \alpha|Q|^{-1-2m/n} \int_Q w\, dx - \beta|Q|^{-1} \int_Q v\, dx \leq -A \right\}.$$

Then $\mathcal{I}_A$ is a finite set.

Proof. Let $Q \in \mathcal{I}_A$. Then we have

$$A \leq \frac{\beta}{|Q|} \int_Q v\, dx.$$

By Lemma 3.3 we conclude that $\mathcal{I}_A$ is a finite set. □

Let $\{\mu_k\}_{k=1}^{\infty}$ be the non-decreasing rearrangement of

$$\left\{ \alpha|Q|^{-1-2m/n} \int_Q w\, dx - \beta|Q|^{-1} \int_Q v\, dx \right\}_{Q \in \mathcal{I}}.$$

Then

$$\mu_1 \leq \mu_2 \leq \cdots$$

and

$$\lim_{k\to\infty} \mu_k = 0$$

by Lemma 3.4.

When

$$\mu_k = \alpha|Q|^{-1-2m/n} \int_Q w\,dx - \beta|Q|^{-1} \int_Q v\,dx,$$

we define $\psi_k = \varphi_Q$.

By (7) and the density argument we have $\int_{\mathbb{R}^n} |f|^2 v\,dx < \infty$ for all $f \in \mathcal{D}$ and the inequalities in Lemmas 3.1 and 3.2 hold for $f \in \mathcal{D}$. Hence we have

$$\begin{aligned}
(Lf, f) &= a(f, f) + \int_{\mathbb{R}^n} V|f|^2\,dx \\
&\geq \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w\,dx - \int_{\mathbb{R}^n} V_-|f|^2\,dx \\
&\geq \int_{\mathbb{R}^n} \left\{ \sum_{|\alpha|=m} |D^\alpha f|^2 \right\} w\,dx - \int_{\mathbb{R}^n} |f|^2 v\,dx \\
&\geq \sum_{Q\in\mathcal{Q}} |(f, \varphi_Q)|^2 \left\{ \alpha|Q|^{-2m/n-1} \int_Q w\,dx - \beta|Q|^{-1} \int_Q v\,dx \right\}
\end{aligned}$$

for all $f \in \mathcal{D}$. Therefore we have

$$\begin{aligned}
\lambda_k &\geq \inf_{\substack{f\in\mathcal{D},\, \|f\|=1, \\ (f,\psi_i)=0,\, i=1,\ldots,k-1}} (Lf, f) \\
&\geq \inf_{\substack{f\in\mathcal{D},\, \|f\|=1, \\ (f,\psi_i)=0,\, i=1,\ldots,k-1}} \sum_{j=1}^{\infty} |(f, \psi_j)|^2 \mu_j \\
&\geq \mu_k \sup_{\substack{f\in\mathcal{D},\, \|f\|=1, \\ (f,\psi_i)=0,\, i=1,\ldots,k-1}} \sum_{j=k}^{\infty} |(f, \psi_j)|^2 \geq c\mu_k,
\end{aligned}$$

where we used the fact $\mu_k < 0$ and $\sum_j |(f, \psi_j)|^2 \leq c\|f\|^2$.

Since $\lim_{k\to\infty} \mu_k = 0$, the negative spectrum of L is discrete. Furthermore we have

$$\begin{aligned}\sum_{k,\lambda_k<0} |\lambda_k|^\gamma &\le c\sum_{k=1}^{\infty} |\mu_k|^\gamma \\ &= c\sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1}\int_Q v\,dx - \alpha|Q|^{-1-2m/n}\int_Q w\,dx\right)^\gamma \\ &\le c\sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1}\int_Q v\,dx\right)^\gamma \le c\int_{\mathbb{R}^n} v^{q+\gamma}u\,dx \le c\int_{\mathbb{R}^n} V_-^{q+\gamma}u\,dx,\end{aligned}$$

where we used Lemma 3.3. This ends the proof of Theorem 1.1.

Proof of Lemma 3.3. For $Q\in\mathcal{I}$ we have

$$\begin{aligned}\alpha|Q|^{-2m/n}\int_Q w(x)\,dx &< \beta\int_Q v(x)\,dx \\ &\le \beta\left(\int_Q v^{q+\gamma}u\,dx\right)^{1/(q+\gamma)}\left\{\int_Q u^{-1/(q+\gamma-1)}\,dx\right\}^{(q+\gamma-1)/(q+\gamma)}.\end{aligned}$$

Since $u\in A_{q+\gamma}$ means

$$\frac{1}{|Q|}\int_Q u\,dx\left\{\frac{1}{|Q|}\int_Q u^{-1/(q+\gamma-1)}\,dx\right\}^{q+\gamma-1} \le c,$$

the last term is bounded by

$$\begin{aligned}&c\left(\int_Q v^{q+\gamma}u\,dx\right)^{1/(q+\gamma)}|Q|\left(\int_Q u\,dx\right)^{-1/(q+\gamma)} \\ &\le c\left(\int_Q v^{q+\gamma}u\,dx\right)^{1/(q+\gamma)}|Q|\left(\int_Q u\,dx\right)^{\gamma/\{q(q+\gamma)\}-1/q} \\ &\le c\left(\int_Q v^{q+\gamma}u\,dx\right)^{1/(q+\gamma)}|Q|\left(\int_Q u\,dx\right)^{\gamma/\{q(q+\gamma)\}}|Q|^{-2m/n-1}\int_Q w\,dx, \qquad (8)\end{aligned}$$

where we used (4). Therefore we have

$$0<c\le\int_Q v^{q+\gamma}u\,dx\left(\int_Q u\,dx\right)^{\gamma/q}.$$

By this inequality we conclude that if $Q_1\supset Q_2\supset\cdots$ are cubes in $\mathcal{I}$, then this sequence must have a minimal element with respect to the inclusion relation. Let $\mathcal{M}$ be the set of all such minimal cubes in $\mathcal{I}$.

Lemma 3.5 *Let $Q \in \mathcal{Q}$ and $Q_1, Q_2, \ldots, Q_{2^n}$ be the half-size dyadic subcubes of Q. Then we have*

$$\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma \leq 2^{-n\min\{1,\gamma\}}\sum_{i=1}^{2^n}\left(\frac{1}{|Q_i|}\int_{Q_i} v\,dx\right)^\gamma. \tag{9}$$

Proof. We have

$$\frac{1}{|Q|}\int_Q v\,dx = 2^{-n}\sum_{i=1}^{2^n}\frac{1}{|Q_i|}\int_{Q_i} v\,dx.$$

If $0 < \gamma < 1$, then we can get (9) easily. If $\gamma > 1$, then (9) is a consequence of the convexity of the function $y = x^\gamma$, $x > 0$. □

Let $\mathcal{N}$ be the set of all $Q \in \mathcal{Q}$ such that $Q \notin \mathcal{I}$ and its parent $Q' \in \mathcal{I} \setminus \mathcal{M}$. Then using Lemma 3.5 repeatedly we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{I}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma \\
&\quad\leq \sum_{Q\in\mathcal{M}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma\left\{\sum_{k=0}^{\infty}2^{-kn\min\{1,\gamma\}}\right\} \\
&\qquad + \sum_{Q\in\mathcal{N}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma\left\{\sum_{k=1}^{\infty}2^{-kn\min\{1,\gamma\}}\right\} \\
&\quad\leq c\sum_{Q\in\mathcal{M}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma + c\sum_{Q\in\mathcal{N}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma.
\end{aligned}$$

Let $Q \in \mathcal{I}$. Then by (4) and (8) we get

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma &\leq c\left(\int_Q v^{q+\gamma}u\,dx\right)^{\gamma/(q+\gamma)}\left(\int_Q u\,dx\right)^{-\gamma/(q+\gamma)} \\
&\leq c\left(\int_Q v^{q+\gamma}u\,dx\right)^{\gamma/(q+\gamma)}\left(|Q|^{-2m/n-1}\int_Q w\,dx\right)^{q\gamma/(q+\gamma)} \\
&\leq c\left(\int_Q v^{q+\gamma}u\,dx\right)^{\gamma/(q+\gamma)}\left(|Q|^{-1}\int_Q v\,dx\right)^{q\gamma/(q+\gamma)}.
\end{aligned}$$

Therefore we have

$$\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma \le c\int_Q v^{q+\gamma}u\,dx.$$

Similarly we have this inequality for $Q \in \mathcal{N}$ because the parent Q' of Q belongs to $\mathcal{I}$ and the inequality

$$|Q|^{-2m/n-1}\int_Q w\,dx \le c'|Q|^{-1}\int_Q v\,dx$$

holds by the doubling property of v.

Therefore we conclude

$$\begin{aligned}\sum_{Q\in\mathcal{I}}\left(\frac{1}{|Q|}\int_Q v\,dx\right)^\gamma &\le c\sum_{Q\in\mathcal{M}}\int_Q v^{q+\gamma}u\,dx + c\sum_{Q\in\mathcal{N}}\int_Q v^{q+\gamma}u\,dx\\ &\le c\int_{\mathbb{R}^n} v^{q+\gamma}u\,dx,\end{aligned}$$

where we used the fact that the cubes in $\mathcal{M}\cup\mathcal{N}$ are mutually disjoint. Hence Lemma 3.3 is proved. □

4. Some remarks

Let m and n be natural numbers such that $n/2 < m < 3n/2$. Let α be a number such that $m - n/2 < \alpha < \min\{2m-n, n\}$. Then

$$L_0 f(x) = (-1)^m \sum_{|\beta|=m} D^\beta\left(|x|^\alpha D^\beta f(x)\right)$$

satisfies the conditions in Theorem 1.1 with $w(x) = |x|^\alpha$ and $u = w^{-n/(2m)}$. We shall show this. First we have $|x|^\alpha \in H^m_{\mathrm{loc}}(\mathbb{R}^n)$. It is known that for $p \in (1,\infty)$ and $\delta \in \mathbb{R}$ we have $|x|^\delta \in A_p$ on $\mathbb{R}^n$ if and only if $-n < \delta < n(p-1)$ (cf. [4, p. 407]). Hence we can easily show that $w = |x|^\alpha \in A_2$ and $u = w^{-n/(2m)} = |x|^{-n\alpha/(2m)} \in A_{n/(2m)+\gamma}$ for any $\gamma > 0$ such that $n/(2m) + \gamma > 1$. We shall prove that w satisfies the condition (3). Let $Q' \in \mathcal{Q}$ such that

$$Q' = \{(x_1,\dots,x_n) : k_i \le 2^\nu x_i < k_i + 1,\ i = 1,\dots,n\}$$

for $k = (k_1,\dots,k_n) \in \mathbb{Z}^n$ and $\nu \in \mathbb{Z}$. We shall show that the inequality

$$\int_{Q'} |x|^\alpha \, dx \le 2^{2m} \int_Q |x|^\alpha \, dx \tag{10}$$

holds for all half-size dyadic sub-cubes Q of Q'. Since $w(x) = |x|^\alpha$ is a radial function, it is enough to consider the case $k_i \ge 0$ for all $i = 1, \dots, n$. Since $|x|^\alpha \le |x'|^\alpha$ for all $x, x' \in \mathbb{R}^n$ such that $0 \le x_i \le x_i'$, $i = 1, \dots, n$, it is enough to prove (10) for

$$Q = \{(x_1, \dots, x_n) : k_i \le 2^\nu x_i < k_i + 1/2, \ i = 1, \dots, n\}.$$

If we set $2^\nu x = y$ in (10), then the inequality we shall show is

$$\int_{R'} |y|^\alpha \, dy \le 2^{2m} \int_R |y|^\alpha \, dy, \tag{11}$$

where

$$R' = \{(y_1, \dots, y_n) : k_i \le y_i < k_i + 1, \ i = 1, \dots, n\}$$

and

$$R = \{(y_1, \dots, y_n) : k_i \le y_i < k_i + 1/2, \ i = 1, \dots, n\}.$$

Setting $z = 2y - k$, we get

$$\int_R |y|^\alpha \, dz = 2^{-n-\alpha} \int_{R'} |z + k|^\alpha \, dz \ge 2^{-n-\alpha} \int_{R'} |z|^\alpha \, dz.$$

Hence we proved (11) since $n + \alpha < 2m$.

Next we consider the condition (3) when $n = 2m$. Let $I_0 = (-\infty, 0)$ and $I_1 = [0, \infty)$. For $r_1, \dots, r_n \in \{0, 1\}$, we set

$$K_{r_1, \dots, r_n} = \{(x_1, \dots, x_n) : x_i \in I_{r_i}, \ i = 1, \dots, n\}.$$

We have the following proposition.

Proposition 4.1 *Let w be a real-valued locally integrable function on $\mathbb{R}^n$. We assume that*

$$\int_{Q'} w(x) \, dx \le 2^n \int_Q w(x) \, dx \tag{12}$$

for all $Q \in \mathcal{Q}$ and its parent Q'. Then $w(x)$ is constant almost everywhere on each $K_{r_1, \dots, r_n}$.

Proof. Let $Q_0 \in \mathcal{Q}$ and $z \in Q_0$. Then there exists an infinite decreasing sequence of dyadic cubes $Q_0 \supset Q_1 \supset Q_2 \supset \cdots$ such that $z \in Q_i$ and Q_i is the parent of Q_{i+1} for all $i = 0, 1, \ldots$. By (12) we have

$$\frac{1}{|Q_i|} \int_{Q_i} w(x)\, dx \le \frac{1}{|Q_{i+1}|} \int_{Q_{i+1}} w(x)\, dx$$

for all $i = 0, 1, \ldots$. Since w is locally integrable and $\lim_{i\to\infty} |Q_i| = 0$, we have

$$\lim_{i\to\infty} \frac{1}{|Q_i|} \int_{Q_i} w(x)\, dx = w(z)$$

for almost every $z \in Q_0$. Hence we have

$$\frac{1}{|Q_0|} \int_{Q_0} w(x)\, dx \le \operatorname*{ess\,inf}_{z\in Q_0} w(z).$$

Therefore we conclude that w is constant almost everywhere on Q_0. Since we can choose any dyadic cube as Q_0, we get the conclusion of Proposition 4.1.

□

Finally we give a bound of the constant c in (6). The result is a little complicated. We need explicit calculation of the constants in propositions and lemmas in this paper and weighted norm inequalities in [4]. Let w and u be weights satisfying the conditions in Theorem 1.1. Let A and B be the A_2-constant of w and the $A_{q+\gamma}$-constant of u respectively. We set

$$\delta_1 = \frac{A - 2^{-2}}{A}, \quad \varepsilon_1 = \frac{1}{2} \min\left\{1, -\frac{\log \delta_1}{(n+1)\log 2}\right\},$$

$$\delta_2 = \frac{B^{-1/(q+\gamma-1)} - 2^{-(q+\gamma)}}{B^{-1/(q+\gamma-1)}}, \quad \varepsilon_2 = \frac{1}{2} \min\left\{1, -\frac{\log \delta_2}{(n+1)\log 2}\right\},$$

$$s = \frac{(q+\gamma)(1+\varepsilon_2)}{q+\gamma+\varepsilon_2}, \quad D = B\left(\frac{2^{n\varepsilon_2}}{2^{-\varepsilon_2} - 2^{n\varepsilon_2}\delta_2}\right)^{(q+\gamma-1)/(1+\varepsilon_2)},$$

$$\delta_3 = \frac{D^{-s/(q+\gamma-s)} - 2^{-(q+\gamma)/s}}{D^{-s/(q+\gamma-s)}}, \quad \varepsilon_3 = \frac{1}{2} \min\left\{1, -\frac{\log \delta_3}{(n+1)\log 2}\right\},$$

$$\kappa = \frac{q+\gamma+s\varepsilon_3}{s(1+\varepsilon_3)}, \quad E = 3^{n/s}\left(1 + \frac{3^n 4^n}{s-1}\right),$$

$$\delta_4 = \frac{E - 2^{-2}}{E}, \quad \varepsilon_4 = \frac{1}{2} \min\left\{1, -\frac{\log \delta_4}{(n+1)\log 2}\right\}.$$

Then the constant c in (6) is bounded by

$$A^{4q}B^2c_1^qE^{4\gamma+5q}12^{n\kappa}2^{2\kappa-(q+\gamma)/s}\frac{q+\gamma}{q+\gamma-\kappa s}c_n\tilde{c}_n^{q+\gamma}c_{m,n}^q\left(\sum_{k=0}^{\infty}2^{-kn\min\{1,\gamma\}}\right)$$
$$\times 2^{2q(2n+2)(1+\varepsilon_1)/\varepsilon_1}\left(\frac{2^{n\varepsilon_1}}{2^{-\varepsilon_1}-2^{n\varepsilon_1}\delta_1}\right)^{2q/\varepsilon_1}\left(\frac{2^{n\varepsilon_2}}{2^{-\varepsilon_2}-2^{n\varepsilon_2}\delta_2}\right)^{(q+\gamma-1)/(1+\varepsilon_2)}$$
$$\times\left(\frac{2^{n\varepsilon_3}}{2^{-\varepsilon_3}-2^{n\varepsilon_3}\delta_3}\right)^{(q+\gamma-s)/(s(1+\varepsilon_3))}$$
$$\times 2^{2(q+\gamma)(2n+3)(1+\varepsilon_4)/\varepsilon_4}\left(\frac{2^{n\varepsilon_4}}{2^{-\varepsilon_4}-2^{n\varepsilon_4}\delta_4}\right)^{2(q+\gamma)/\varepsilon_4},$$

where c_1 is the constant in (4) and c_n, $\tilde{c}_n$, $c_{m,n}$ are constants depending only on m and n. We omit the proof.

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