Hokkaido Mathematical Journal Vol. 32 (2003) p. 193–208

Remark on application of distribution function inequality for Toeplitz and Hankel operators

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(Received January 15, 2002; Revised March 22, 2002)

Abstract. In this paper we characterize the compact product of analytic Toeplitz operator and Hankel operator, and the compact commutator of two Hankel operators, by using some distribution function inequalities.

Key words: Toeplitz and Hankel operators, distribution function inequality.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\partial\mathbb{D}$ be the unit circle. Let dA denote the normalized Lebesgue measure on $\mathbb{D}$ and $d\sigma$ denote the normalized Lebesgue measure on $\partial\mathbb{D}$. The Lebesgue space L^2 is the space of square integrable functions on $\partial\mathbb{D}$ and the Hardy space H^2 is the closed subspace of L^2 which is spanned by analytic polynomials. For f in L^{∞}, the space of essentially bounded functions on the unit circle, Toeplitz operator T_f and Hankel operator H_f on Hardy space H^2 is defined by $T_f g = P(fg)$ and $H_f g = J(I-P)(fg)$, where P is the orthogonal projection from L^2 onto H^2 and J is the unitary operator on L^2 defined by $Jg(w) = \overline{w}g(\overline{w})$. It is easily seen that $J^2 = I$, $J(I-P) = PJ$. This definition of Hankel operator may not be standard because many authors call next operator $\mathcal{H}_f$ Hankel operator: $\mathcal{H}_f g = (I-P)(fg)$. Clearly $\mathcal{H}_f$ is bounded transformation of H^2 to $(H^2)^{\perp}$ and $H_f = J\mathcal{H}_f$. H_f and $\mathcal{H}_f$ have many similar properties. For example matrix representatioins of H_f and $\mathcal{H}_f$ with respect to standard basis of H^2 and $(H^2)^{\perp}$ are both characterized that the entries on each skew-diagonal direction are the same constant. In this paper we are mainly interested in Hankel operator H_f.

Many authors have studied Toeplitz and Hankel operators with respect to the compact operators, and I think one of the most beautiful results of these operators are Axler-Chang-Sarason-Volberg theorem ([1], [13]). In 1970's they characterized the condition for the compactness of semi-

2000 *Mathematics Subject Classification* : 47B35.

comutator of Toeplitz operators by using function theory of H^∞ (Corona theorem, see [7], and Chang-Marshall theorem [4], [11] etc.) and a distribution function inequality. We remark that the idea to use the distribution function inequality to study Toeplitz and Hankel operators were first appeared in [1]. Also an elementary condition was obtained in [16] for the compactness of semi-commutator of Toeplitz operator in 1990's. These results are stated as follow:

Theorem 1.1 ([1], [13], [16]) *Let C be the set of continuous complex-valued functions on $\partial\mathbb{D}$. Then for $f, g \in L^\infty$, the following assertions are equivalent.*
(1) $([T_f, T_g) :=) T_f T_g - T_{fg}$ *is a compact operator on* H^2.
(2) $\lim_{z \to \partial\mathbb{D}} \|\mathcal{H}_{\overline{f}} k_z\| \|\mathcal{H}_g k_z\| = 0$.
(3) *For all m in $M(H^\infty + C)$,* $\overline{f}|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$ *or* $g|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$
(4) $H^\infty[\overline{f}] \cap H^\infty[g] \subseteq H^\infty + C$.

Here k_z is the normalized reproducing kernel in H^2 and $H^\infty[f]$ is the closed algebra generated by H^∞ and $f \in L^\infty$, and $\mathrm{supp}\, m$ is the closed support of representing measure of m (see Section 2 for precise definition). By refining the techniques of Theorem 1.1, Gorkin and Zheng ([8]) characterized the condition for the compactness of commutator of two Toeplitz operators.

Theorem 1.2 ([8]) *For f, g in L^∞, the following assertions are equivalent.*
(1) $([T_f, T_g] =) T_f T_g - T_g T_f$ *is a compact operator on* H^2.
(2) $\lim_{z \to \partial\mathbb{D}} \|(\mathcal{H}_{\overline{f}} k_z) \otimes (\mathcal{H}_g k_z) - (\mathcal{H}_{\overline{g}} k_z) \otimes (\mathcal{H}_f k_z)\| = 0$.
(3) *For all $m \in M(H^\infty + C)$, one of the followings holds.*
  (a) $f|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$ *and* $g|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$.
  (b) $\overline{f}|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$ *and* $\overline{g}|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$.
  (c) *There exist complex numbers a, b such that they are not zero at the same time and that $(af + bg)|_{\mathrm{supp}\, m}$ is a constant.*
(4) $H^\infty[f, g] \cap H^\infty[\overline{f}, \overline{g}] \cap \bigcap_{(a,b) \neq (0,0)} H^\infty[af + bg, \overline{af + bg}] \subseteq H^\infty + C$.

As special cases as above, we state here the results given by Brown and Halmos [3] that $T_f T_g - T_{fg} = 0$ if and only if $\overline{f} \in H^\infty$ or $g \in H^\infty$ and that $T_f T_g - T_g T_f = 0$ if and only if $f, g \in H^\infty$ or $\overline{f}, \overline{g} \in H^\infty$ or there exist complex numbers a, b such that they are not zero at the same time and that $af + bg$ is a constant.

In this paper we remark that we can characterize similarly conditions for the compactness of the product of analytic Toeplitz operator and Hankel operator (Theorem 4.3), and for the compactness of the commutator of two Hankel operators (Theorem 4.8), by using some distribution function inequalities stated in Section 3.

2. Preliminary

For f in L^∞ we define the another operator S_f on $(H^2)^\perp$ called dual-Toeplitz operator by $S_f g = (I-P)(fg)$. Next are elementary properties of these operators.

Lemma 2.1 *For f and g in L^∞, the followings hold.*
(1) $H_f^* = H_{f^*}$, *where* $f^*(w) = \overline{f(\overline{w})}$.
(2) $\mathcal{H}_f^* \mathcal{H}_g = H_f^* H_g = T_{\overline{f}g} - T_{\overline{f}} T_g$.
(3) $T_f^* H_g = H_g T_{f^*}$ *and* $S_f \mathcal{H}_g = \mathcal{H}_g T_f$ *if* $f \in H^\infty$.

Proof. They are computed easily by the definitions. □

For f and g in L^2, $f \otimes g$ is the operator of rank one on L^2 defined by $(f \otimes g)h = \langle h, g \rangle f$. For $z \in \mathbb{D}$ and $w \in \partial\mathbb{D}$, let $k_z(w)$ be the normalized reproducing kernel in H^2 defined by $(1-|z|^2)^{1/2}/(1-\overline{z}w)$, and let φ_z be

$$\varphi_z(w) = \frac{z-w}{1-\overline{z}w}.$$

Then it is easily seen that φ_z is in H^∞, $\varphi_z^{-1} = \varphi_z$, $|\varphi_z'(w)| = |k_z(w)|^2$ and $I - T_{\varphi_z} T_{\overline{\varphi_z}}$ is the rank one projection $k_z \otimes k_z$.

Let B be a Douglas algebra (i.e. closed algebra between H^∞ and L^∞) and denote $M(B)$ be the maximal ideal space (space of nonzero complex homomorphisms) of B. It is known that $H^\infty + C$ is the smallest Douglas algebra and its maximal ideal space $M(H^\infty + C)$ is identified to $M(H^\infty) \setminus \mathbb{D}$. By the Corona theorem (see [7]) $\mathbb{D}$ is dense in $M(H^\infty)$. For m in $M(H^\infty)$ there is a unique representing measure μ_m on $M(L^\infty)$. The closed supprot of this measure μ_m is called support set of m and denoted by $\operatorname{supp} m$. The supprot set is a weak peak set ([9] page 207) and $H^\infty|_{\operatorname{supp} m}$ is a uniformly closed subalgebra of $C(\operatorname{supp} m)$ ([6] page 57). Next lemmas are useful for our arguments.

Lemma 2.2 ([8]) *For f in L^∞ and m in $M(H^\infty + C)$, the following assertions are equivalent.*

(1) $f|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$.

(2) $\liminf_{z \to m} \|H_f k_z\|_2 = 0$ (*if and only if* $\lim_{z \to m} \|H_f k_z\|_2 = 0$).

(3) *There is a net* $\{z_\alpha\}$ *which converges to* m *such that* $\liminf_{z_\alpha \to m} \|H_f k_{z_\alpha}\|_2 = 0$.

Lemma 2.3 ([14]) *For f in L^∞ and m in $M(H^\infty + C)$, the following assertions are equivalent.*

(1) $\overline{f}|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ *and* $f(m) = 0$.

(2) $\liminf_{z \to m} \|T_f k_z\|_2 = 0$ (*if and only if* $\lim_{z \to m} \|T_f k_z\|_2 = 0$).

(3) *There is a net* $\{z_\alpha\}$ *which converges to* m *such that* $\liminf_{z_\alpha \to m} \|T_f k_{z_\alpha}\| = 0$.

3. Distribution function inequality

For f in L^1, denote the Hardy-Littlewood maximal function of f by $M(f)$ and, for $h \in L^2$ and $1 < r < 2$, let $\Lambda_r h(w) = [M(|h|^r)(w)]^{1/r}$. For w in $\partial\mathbb{D}$, let Γ_w denote the angle with vertex w and opening $\pi/2$ which is bisected by the radius to w. The set of points z in Γ_w saisfying $|z - w| < \gamma$ is denoted by $\Gamma_{w,\gamma}$ for $0 < \gamma < 1$. For h in L^1 and $0 < \varepsilon < 1$, we define the truncated Lusin integral of h to be

$$[A_\varepsilon(h)](w) = \left[\int_{\Gamma_{w,\varepsilon}} |\nabla h(z)|^2 dA(z) \right]^{\frac{1}{2}}$$

where $h(z)$ is the classical harmonic extension of h i.e.

$$h(z) = \int_{\partial\mathbb{D}} h(w) P_z(w) d\sigma(w),$$

where $P_z(w)$ is Poisson kernel. This integral is finite for almost all w in $\partial\mathbb{D}$ ([10]). I_z is the closed subarc of $\partial\mathbb{D}$ with center $z/|z|$ and the measure $\delta(z) = 1 - |z|$. The Lebesgue measure of subset E of $\partial\mathbb{D}$ is denoted by $|E|$. For f is in L^2, we put $f_+ = Pf$ and $f_- = (I - P)f$.

Lemma 3.1 (D.F.I. for $\mathcal{H}_f$ [16]) *Fix $l > 2$. Then, for all $p, r \in (1,2)$ with $1/l + 1/r = 1/p$*[1)] *and sufficiently large $a > 0$, there exists $C_a > 0$ such*

[1)] There exist l, p, r like this. For $0 < \varepsilon < (l-2)/2$, we may put $p = l/(l - \varepsilon)$ and $r = pl/(l - p)$.

that

$$\left|\left\{w \in I_z : A_{2\delta(z)}(\mathcal{H}_f u)(w) < a[|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w\in I_z} \Lambda_r(u)(w)\right\}\right|$$
$$\geq C_a|I_z|$$

for all $f \in L^\infty$, $u \in H^\infty$ *and* $1/2 < |z| < 1$. *Moreover, the constant* C_a *can be chosen to satisfy* $\lim_{a\to\infty} C_a = 1$.

We will next prove the distributioin function inequality for operator S_f.

Lemma 3.2 (D.F.I. for S_f) *Fix* $l > 2$. *Then for all* $p, r \in (1,2)$ *with* $1/l + 1/r = 1/p$ *and* $a > 0$ *sufficiently large, there exists* $C_a > 0$ *such that*

$$\left|\left\{w \in I_z : A_{2\delta(z)}(S_f u)(w) < a[|f|^l(z)]^{1/l} \inf_{w\in I_z} \Lambda_r(u)(w)\right\}\right| \geq C_a|I_z|$$

for all $f \in L^\infty$, $u \in (H^2)^\perp$ *and* $1/2 < |z| < 1$. *Moreover, the constant* C_a *can be chosen to satisfy* $\lim_{a\to\infty} C_a = 1$.

Proof. For $u \in (H^2)^\perp$ and $1/2 < |z| < 1$, we write $S_f u$ as $S_f u = (I - P)(fu) = (I-P)u_1 + (I-P)u_2$ where $u_1 = fu\chi_{2I_z}$ and $u_2 = fu\chi_{\partial\mathbb{D}\setminus 2I_z}$. Remainder of the proof is similar to the proof of [16] Theorem 6. □

Remark 3.1 We can prove the similar distribution function inequalities for operators T_f, $\mathcal{H}_f^*$ and H_f. But they are not used in this paper, so we will not mention here.

By using Lemma 3.1 and 3.2, we can prove the next distribution function inequality, which will be used after in order to characterize the condition for compactness of product of analytic Toeplitz operator and Hankel operator.

Proposition 3.3 *Fix* $l > 2$. *Then, for all* $p, r \in (1,2)$ *with* $1/l + 1/r = 1/p$ *and sufficiently large* $a > 0$, *there exists* $C_a > 0$ *such that*

$$\begin{aligned}\Big|\Big\{w \in I_z : A_{2\delta(z)}(S_f u)(w) A_{2\delta(z)}(\mathcal{H}_g v)(w) < a^2 \left[|f|^l(z)\right]^{1/l} \\ \times \left[|g_- - g_-(z)|^l(z)\right]^{1/l} \inf_{w\in I_z} \Lambda_r(u)(w) \inf_{w\in I_z} \Lambda_r(v)(w)\Big\}\Big| \\ \geq C_a|I_z| \qquad (3.1)\end{aligned}$$

for all f *and* g *in* L^∞, u *in* $(H^2)^\perp$, v *in* H^∞ *and* $1/2 < |z| < 1$. *Moreover, the constant* C_a *can be chosen to satisfy* $\lim_{a\to\infty} C_a = 1$.

For f, g in L^∞ and u, v in H^2, we define $B_\gamma(u,v)(w)$ by

$$B_\gamma(u,v)(w) = \int_{\Gamma_{w,\gamma}} |\nabla(\mathcal{H}_f u)\cdot\nabla(\mathcal{H}_{g^*}v) - \nabla(\mathcal{H}_g u)\cdot\nabla(\mathcal{H}_{f^*}v)|\, dA(z).$$

For $z \in \mathbb{D}$ we define F_z to be

$$F_z = (H_f k_z)\otimes\left(H_g^* k_z\right) - (H_g k_z)\otimes\left(H_f^* k_z\right).$$

Lemma 3.4 *Asuume that for some $\lambda \in \mathbb{C}$, $H_f k_z \perp H_{g-\lambda f} k_z$ or $H_f^* k_z \perp H_{g-\lambda f}^* k_z$. Then*

$$\begin{aligned}\|F_z\| &\le \left[\|H_f k_z\|_2^2 \left\|H_{g-\lambda f}^* k_z\right\|_2^2 + \left\|H_{g-\lambda f} k_z\right\|_2^2 \left\|H_f^* k_z\right\|_2^2\right]^{\frac{1}{2}} \\ &\le \sqrt{2}\|F_z\|\end{aligned}$$

for all $z \in \mathbb{D}$.

Proof. The proof is similar to [8] Lemma 2.8. □

Proposition 3.5 *Fix $f, g \in L^\infty$. For all $l \in (2,3)$ and sufficiently large $a > 0$, there exist $K_a > 0$ and $r \in (1,2)$,*

$$\begin{aligned}&\Big|\Big\{w \in I_z : B_{2\delta(z)}(u,v)(w) \\ &\qquad < a^2 N_l \|F_z\|^{\frac{l-1}{l}} \inf_{w\in I_z} \Lambda_r(u)(w) \inf_{w\in I_z} \Lambda_r(v)(w)\Big\}\Big| \ge K_a |I_z|\end{aligned}$$

for all $1/2 < |z| < 1$ and $u, v \in H^\infty$, where $N_l > 0$ depends only on l. Moreover constant K_a can be choosen to satisfy $\lim_{a\to\infty} K_a = 1$.

Proof. By using Lemma 3.4 the proof is similar to [8] page $105 \sim 108$. □

4. Main result

Lemma 4.1 *Asume K is a compact operator on H^2. Then*

$$\lim_{z\to\partial\mathbb{D}} \left\|K - T_{\varphi_z}^* K T_{\varphi_z}\right\| = 0 \quad \textit{and} \quad \lim_{z\to\partial\mathbb{D}} \|K - T_{\varphi_z} K T_{\overline{\varphi_z}}\| = 0.$$

Proof. First part of this lemma is [16] Lemma 2. We will prove the second part. K can be approximated by finite sum of the form $f \otimes g$ where f and

g is in H^2, so we may put $K = f \otimes g$. Because of

$$\overline{z} - \varphi_{\overline{z}}(w) = \frac{1-|z|^2}{1-zw} w \to 0 \ \text{ a.e. } \ (z \to \partial\mathbb{D}),$$

by Lebesgue dominated theorem $\|\overline{z}F - \varphi_{\overline{z}}F\|_2 \to 0$ and $\|zF - \varphi_z F\|_2 \to 0$ when $z \to \partial\mathbb{D}$ for all $F \in L^2$. Therefore, for all ξ in $\partial\mathbb{D}$, $\left\|\overline{\xi}F - \varphi_{\overline{z}}F\right\|_2 \to 0$ and $\|\xi F - \varphi_z F\|_2 \to 0$ when $z \to \xi$ and

$$\begin{aligned}
&\|f \otimes g - T_{\varphi_z}(f \otimes g)T_{\varphi_{\overline{z}}}\| \\
&\leq \|(\xi f - T_{\varphi_z} f) \otimes (\xi g)\| + \|(T_{\varphi_z} f) \otimes (\xi g - T_{\overline{\varphi_{\overline{z}}}} g)\| \\
&\leq \|\xi f - \varphi_z f\| \|g\| + \|f\| \|\xi g - \overline{\varphi_{\overline{z}}} g\| \\
&\to 0 \ \ (z \to \xi)
\end{aligned}$$

This proves the second part of this Lemma. □

Remark 4.1 As we show later, in the case where $K = H_f H_g - H_g H_f$, K is compact operator on H^2 if and only if $\lim_{z\to\partial\mathbb{D}} \left\|K - T^*_{\varphi_z} K T_{\varphi_z}\right\| = 0$ (Theorem 4.8 (2) ⇒ (1)) and in the case where $K = T_f H_g$ ($f \in H^\infty$, $g \in L^\infty$), K is compact operator on H^2 if and only if $\lim_{z\to\partial\mathbb{D}} \|K - T_{\varphi_z} K T_{\varphi_{\overline{z}}}\| = 0$ (Theorem 4.3 (2) ⇒ (1)).

Proposition 4.2 *If $T_f H_g = 0$ where $f \in H^\infty$ and $g \in L^\infty$, then $f = 0$ or $g \in H^\infty$.*

Proof. By using Brown-Halmos techniques in [3], we can prove easily. □

Next theorem is a characterization for the compactness of the product of analytic Toeplitz operator and Hankel operator, concerning Proposition 4.2.

Theorem 4.3 *For $f \in H^\infty$ and $g \in L^\infty$, the following assertions are equivalent.*

(1) *$T_f H_g$ is a compact operator on H^2.*

(2) $\lim_{z\to\partial\mathbb{D}} \|T_f k_z\| \left\|H_g^* k_z\right\| = 0$.

(3) *For all $m \in M(H^\infty + C)$, $f|_{\operatorname{supp} m} = 0$ or $g^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$.*

(4) *$H^\infty[\overline{f}, fg^*] \cap H^\infty[g^*] \subseteq H^\infty + C$.*

Proof. (1) ⇒ (2): By Lemma 2.1 and 4.1

$$\begin{aligned}
&\|T_f H_g - T_{\varphi_z} T_f H_g T_{\varphi_{\overline{z}}}\| = \left\|T_f H_g - T_f T_{\varphi_z} T^*_{\varphi_z} H_g\right\| \\
&\quad = \|T_f(k_z \otimes k_z) H_g\| = \|T_f k_z\| \left\|H^*_g k_z\right\| \\
&\quad \to 0 \quad (z \to \partial\mathbb{D})
\end{aligned}$$

(2) $\Rightarrow$ (1): For all u, v in H^∞, by Littlewood-Paley formula (cf. [17] page 167),

$$\begin{aligned}
\langle T_f H_g v, u\rangle &= \left\langle H_g v, T_{\overline{f}} u\right\rangle = \left\langle \mathcal{H}_g v, JP(\overline{f}u)\right\rangle \\
&= \left\langle \mathcal{H}_g v, (I-P)J(\overline{f}u)\right\rangle = \langle \mathcal{H}_g v, (I-P)f^*(Ju)\rangle \\
&= \langle \mathcal{H}_g v, S_{f^*} Ju\rangle \\
&= \int_{\mathbb{D}} \nabla(\mathcal{H}_g v) \cdot \nabla(S_{f^*} Ju) \log\frac{1}{|z|} dA(z) \\
&= \int_{|z|>R} + \int_{|z|\le R} = I_R + II_R
\end{aligned}$$

where R is in $(1/2, 1)$. Here ∇ means gradient and $\cdot$ means inner product of $\mathbb{C}^2$. We can easily check that there is a compact operator K_R such that $II_R = \langle K_R v, u\rangle$.

Claim We define $\Xi_l(z) = \left[|f^*|^l(z)\right]^{1/l} \left[|g_- - g_-(z)|^l(z)\right]^{1/l}$ for $z \in \mathbb{D}$ and $l > 2$. Then there exists a constant $C > 0$ such that

$$|I_R| \le C \sup_{|z|>R} \Xi_l(z) \|u\|_2 \|v\|_2$$

for all $R \in (1/2, 1)$ and $l > 2$.

Proof of the claim. By Proposition 3.3, there exist $r \in (1, 2)$ and $a, K_a > 0$ such that

$$\begin{aligned}
&\Big|\Big\{w \in I_z : A_{2\delta(z)}(S_{f^*} Ju)(w) A_{2\delta(z)}(\mathcal{H}_g v)(w) \\
&\qquad < a^2 \Xi_l(z) \inf_{w\in I_z} \Lambda_r(Ju)(w) \inf_{w\in I_z} \Lambda_r(v)(w)\Big\}\Big| \ge K_a |I_z|
\end{aligned}$$

for all $u, v \in H^\infty$, $1/2 < |z| < 1$. Fix $R \in (1/2, 1)$. For w in $\partial\mathbb{D}$, define

$$\begin{aligned}
\rho(w) = \max\Big\{\gamma : &A_\gamma(S_{f^*} Ju)(w) A_\gamma(\mathcal{H}_g v)(w) \\
&\le a^2 \sup_{|z|>R} \Xi_l(z) \Lambda_r(Ju)(w) \Lambda_r(v)(w)\Big\}.
\end{aligned}$$

Then,

$$\begin{aligned}\int_{\partial\mathbb{D}} & A_{\rho(w)}(S_{f^*}Ju)(w)A_{\rho(w)}(\mathcal{H}_g v)(w)d\sigma(w) \\ &\leq a^2 \sup_{|z|>R} \Xi_l(z) \int_{\partial\mathbb{D}} \Lambda_r(Ju)\Lambda_r(v)d\sigma \\ &\leq a^2 \sup_{|z|>R} \Xi_l(z)\|\Lambda_r(Ju)\|_2\|\Lambda_r v\|_2.\end{aligned}$$

Because Hardy-Littlewood maximal function is L^p ($p \in (1,\infty)$) bounded (cf. [7] page 24) and $\frac{2}{r} \in (1,2)$, there exists $A_r > 0$ such that

$$\begin{aligned}\|\Lambda_r(Ju)\|_2 &= \|M(|u^*|^r)^{1/r}\|_2 = \|M(|u^*|^r)\|_{2/r}^{1/r} \\ &\leq A_r(\||u^*|^r\|_{2/r})^{1/r} = A_r\|u^*\|_2 = A_r\|u\|_2.\end{aligned}$$

Moreover there exist $A_r' > 0$ such that

$$\begin{aligned}\int_{\partial\mathbb{D}} & A_{\rho(w)}(S_{f^*}Ju)(w)A_{\rho(w)}(\mathcal{H}_g v)(w)d\sigma(w) \\ &\leq a^2 A_r A_r' \sup_{|z|>R} \Xi_l(z)\|u\|_2\|v\|_2. \qquad (4.1)\end{aligned}$$

On the other hand, let χ_w denote the characteristic function of $\Gamma_{w,\rho(w)}$. Then

$$\begin{aligned}\int_{\partial\mathbb{D}} & A_{\rho(w)}(S_{f^*}Ju)(w)A_{\rho(w)}(\mathcal{H}_g v)(w)d\sigma(w) \\ &= \int_{\partial\mathbb{D}}\left(\int_{\Gamma_{w,\rho(w)}}|\nabla(S_{f^*}Ju)|^2dA(z)\right)^{\frac{1}{2}}\left(\int_{\Gamma_{w,\rho(w)}}|\nabla(\mathcal{H}_g v)|^2dA(z)\right)^{\frac{1}{2}}d\sigma(w) \\ &\geq \int_{\partial\mathbb{D}}\int_{|z|>R}\chi_w(z)|\nabla(S_{f^*}Ju)||\nabla(\mathcal{H}_g v)|dA(z)d\sigma(w).\end{aligned}$$

If we define $E_z = \{w \in I_z : \rho(w) \geq 2(1-|z|)\}$, then by Proposition 3.3

$$|E_z| \geq K_a|I_z| = K_a(1-|z|)$$

for all $|z| > R$. Because z is in $\Gamma_{w,\rho(w)}$ for w in E_z,

$$\begin{aligned}\int_{\partial\mathbb{D}} & A_{\rho(w)}(S_{f^*}Ju)(w)A_{\rho(w)}(\mathcal{H}_g v)(w)d\sigma(w) \\ &\geq \int_{|z|>R}|E_z||\nabla(S_{f^*}Ju)||\nabla(\mathcal{H}_g v)|dA(z)\end{aligned}$$

$$\geq K_a \int_{|z|>R} |\nabla(S_{f^*}Ju)||\nabla(\mathcal{H}_g v)|(1-|z|)dA(z)$$
$$\geq K_a \int_{|z|>R} |\nabla(S_{f^*}Ju)||\nabla(\mathcal{H}_g v)|\log\frac{1}{|z|}dA(z)$$
$$\geq K_a\,|I_R|\,. \tag{4.2}$$

Therefore by combining (4.1) and (4.2), we have

$$|I_R| \leq C \sup_{|z|>R} \Xi_l(z)\|u\|_2\|v\|_2.$$

For all $u, v \in H^\infty$, $|\langle(T_f H_g - K_R)v, u\rangle| \leq C\sup_{|z|>R}\Xi_l(z)\|u\|_2\|v\|_2$ and

$$\|T_f H_g - K_R\| \leq C \sup_{|z|>R} \Xi_l(z). \tag{4.3}$$

On the other hand, if we fix $2 < l < 3$, then

$$\begin{aligned}
\Xi_l(z) &= \left[|f^*|^l(z)\right]^{1/l}\left[|g_- - g_-(z)|^l(z)\right]^{1/l} \\
&= \left[|f|^l(\overline{z})\right]^{1/l}\left[|(g_-)^* - (g_-)^*(\overline{z})|^l(\overline{z})\right]^{1/l} \\
&\leq \left[|f|^2(\overline{z})\right]^{(l-1)/2l}\left[|f|^{\frac{2}{3-l}}(\overline{z})\right]^{(3-l)/2l} \\
&\quad \times \left[|(g_-)^* - (g_-)^*(\overline{z})|^2(\overline{z})\right]^{(l-1)/2l} \\
&\qquad\qquad \left[|(g_-)^* - (g_-)^*(\overline{z})|^{2/(3-l)}(\overline{z})\right]^{(3-l)/2l} \\
&\leq C\|T_f k_{\overline{z}}\|^{(l-1)/l}\|\mathcal{H}_{g^*}k_{\overline{z}}\|^{(l-1)/l}\|(I-P)(g^*\circ\varphi_{\overline{z}})\|^{\frac{1}{2}}_{\frac{2}{3-l}} \\
&\leq C_l\|T_f k_{\overline{z}}\|^{(l-1)/l}\|\mathcal{H}_{g^*}k_{\overline{z}}\|^{(l-1)/l} \qquad (\sharp) \\
&\to 0 \quad (z \to \partial\mathbb{D})
\end{aligned}$$

We use the boundedness of $(I - P)$ on L^p where $p \in (1, \infty)$ for $(\sharp)$ and the assumption for the last limit operation. Therefore, by (4.3), $T_f H_g$ is a compact operator.

(2) $\Rightarrow$ (3): Fix m in $M(H^\infty + C)$. By the Corona theorem there is a net $\{z\}$ in $\mathbb{D}$ which converges to m. By (2) we have $\liminf_{z\to m}\|T_f k_z\| = 0$ or $\liminf_{z\to m}\|H_{g^*}k_z\| = 0$. From Lemma 2.3, $\liminf_{z\to m}\|T_f k_z\| = 0$ if and only if $\overline{f}|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ and $f(m) = 0$ but because of $f \in H^\infty$, we easily have $f|_{\operatorname{supp} m} = 0$. On the other hand if $\liminf_{z\to m}\|H_{g^*}k_z\| = 0$, then we have $g^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ by Lemma 2.2.

(3) $\Rightarrow$ (2): If (2) is false, then there exists a net $\{z\}$ in $\mathbb{D}$ such that it

converges to $m \in M(H^\infty + C)$ and that $\limsup_{z \to m} \|T_f k_z\| \|H_g^* k_z\| > \delta > 0$. Then we can lead easily to the contradiction from each condition of (3).

(3) $\Rightarrow$ (4): For all $m \in M(H^\infty + C)$, by (3) it is easily seen that m is multiplicative on $H^\infty[\overline{f}, fg^*]$ or $H^\infty[g^*]$. So we have $m \in M(H^\infty[\overline{f}, fg^*] \cap H^\infty[g^*])$. Therefore $M(H^\infty + C) \subseteq M(H^\infty[\overline{f}, fg^*] \cap H^\infty[g^*])$ and by Chang-Marshall theorem we have (4).

(4) $\Rightarrow$ (3): By Sarason's result (see [8] Lemma 1.3.) we have $M(H^\infty + C) \subseteq M(H^\infty[\overline{f}, fg^*]) \cup M(H^\infty[g^*])$. Fix m in $M(H^\infty + C)$. If $m \in M(H^\infty[\overline{f}, fg^*])$ and $m \notin M(H^\infty[g^*])$, then by [8] Lemma 1.5. $\overline{f}|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$ and $(fg^*)|_{\mathrm{supp}\, m} = (f|_{\mathrm{supp}\, m})(g^*|_{\mathrm{supp}\, m}) \in H^\infty|_{\mathrm{supp}\, m}$ and $g^*|_{\mathrm{supp}\, m} \notin H^\infty|_{\mathrm{supp}\, m}$. Therefore we have $f|_{suppm} = 0$. On the other hand, if m is in $M(H^\infty[g^*])$, then we have $g^*|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$. $\square$

Remark 4.2 If we put $f(z) = z - 1$ and $g(z) = \overline{b(z)}$ where b is the Blaschke product whose zero points approach 1 along the real axis, then this is the example that $T_f H_g$ ($f \in H^\infty$) is a compact operator although $f \neq 0$ and $g \notin H^\infty + C$.

Remark 4.3 Because of $T_f \mathcal{H}_g^* J = T_f H_g^*$, we have directly the characterization for the compactness of $T_f \mathcal{H}_g^*$ from this theorem.

As a special case of Theorem 4.3 we have the next corollary.

Corollary 4.4 *For f in L^∞, the following assertions are equivalent.*
(1) *H_f is a compact operator on H^2.*
(2) $\lim_{z \to \partial\mathbb{D}} \|H_f k_z\| = 0$.
(3) *For all $m \in M(H^\infty + C)$, $f|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$.*
(4) $f \in H^\infty + C$.

Remark 4.4 The equivalence of (1) and (4) is the well-known result by Hartman, and the equivalence of (1) and (2) is proved in [2]. Also by using equivalence of (1), (3) and (4), we get Sarason's result:

Corollary 4.5 ([12])
(1) $H^\infty + C = \{f \in L^\infty; f|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$ *for all* $m \in M(H^\infty + C)\}$.
(2) $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$
$= \{f \in L^\infty; f|_{\mathrm{supp}\, m}$ *is a constant for all* $m \in M(H^\infty + C)\}$.

Proof. We will show (2): f is in QC if and only if H_f and $H_{\overline{f}}$ are compact operators on H^2 if and only if $f|_{\mathrm{supp}\, m} \in H^\infty|_{\mathrm{supp}\, m}$ and $\overline{f}|_{\mathrm{supp}\, m} \in$

$H^\infty|_{\operatorname{supp} m}$ for all m in $M(H^\infty + C)$. Therefore $f|_{\operatorname{supp} m}$ is a constant for all m in $M(H^\infty + C)$, because $\operatorname{supp} m$ is the anti-symmetric set. $\square$

Before proving characterization of essentially commuting Hankel operators, we state the next results as special cases. The proofs are not difficult by using Brown-Halmos techniques in [3].

Proposition 4.6 ([15]) *Let f and g be in L^∞. Then $H_fH_g - H_gH_f = 0$ if and only if there are complex numbers a, b such that they are not zero at the same time and that $af + bg$ is in H^∞.*

Corollary 4.7 ([15]) *Let f be in L^∞. Then H_f is the normal operator if and only if there is a complex number α such that its absolute value is 0 or 1 and that $f + \alpha f^*$ in H^∞.*

Concerning Proposition 4.6, we have the following:

Theorem 4.8 *Fix f and g in L^∞. Then the following assertions are equivalent.*

(1) *$([H_f, H_g] =) H_fH_g - H_gH_f$ is a compact operator on H^2.*

(2) $\|F_z\| = \|(H_fk_z) \otimes (H_g^*k_z) - (H_gk_z) \otimes (H_f^*k_z)\| \to 0 \ \ (z \to \partial\mathbb{D})$.

(3) *For all $m \in M(H^\infty + C)$, one of the following conditions holds.*

   (a) *$f|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ and $g|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$.*

   (b) *$f^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ and $g^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$.*

   (c) *There exist $a, b \in \mathbb{C}$, such that they are not zero at the same time and that $(af + bg)|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ and $(af + bg)^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$.*

Moreover if $f, g \in \overline{H^\infty}$, the following is also equivalent.

(4) $H^\infty[f, g] \cap H^\infty[f^*, g^*] \cap \bigcap_{(a,b)\neq(0,0)} H^\infty[af + bg, (af + bg)^*] \subseteq H^\infty + C$.

Proof. (1) $\Rightarrow$ (2): By Lemma 4.1

$$\begin{aligned}
&\left\|H_fH_g - H_gH_f - T_{\varphi_z}^* \left(H_fH_g - H_gH_f\right) T_{\varphi_z}\right\| \\
&\quad = \left\|H_f \left(I - T_{\varphi_{\overline{z}}}T_{\varphi_{\overline{z}}}^*\right) H_g - H_g \left(I - T_{\varphi_{\overline{z}}}T_{\varphi_{\overline{z}}}^*\right) H_f\right\| \\
&\quad = \left\|H_f \left(k_{\overline{z}} \otimes k_{\overline{z}}\right) H_g - H_g \left(k_{\overline{z}} \otimes k_{\overline{z}}\right) H_f\right\| \\
&\quad = \left\|(H_fk_{\overline{z}}) \otimes \left(H_g^*k_{\overline{z}}\right) - (H_gk_{\overline{z}}) \otimes \left(H_f^*k_{\overline{z}}\right)\right\| \\
&\quad \to 0 \ \ (z \to \partial\mathbb{D})
\end{aligned}$$

$(2) \Rightarrow (1)$: For all $u, v \in H^\infty$, by Littlewood-Paley formula

$$\begin{aligned}
&\langle (H_f H_g - H_g H_f)u, v\rangle \\
&\quad = \langle \mathcal{H}_g u, \mathcal{H}_{f^*} v\rangle - \langle \mathcal{H}_f u, \mathcal{H}_{g^*} v\rangle \\
&\quad = \int_{\mathbb{D}} [\nabla(\mathcal{H}_g u)\cdot\nabla(\mathcal{H}_{f^*} v) - \nabla(\mathcal{H}_f u)\cdot\nabla(\mathcal{H}_{g^*} v)] \log\frac{1}{|z|} dA(z) \\
&\quad = \int_{|z|>R} + \int_{|z|\le R} = I_R + II_R
\end{aligned}$$

where $R \in \left(\frac{1}{2}, 1\right)$. It is easily seen that there is a compact operator K_R on H^2 such that $II_R = \langle K_R u, v\rangle$ and by Proposition 3.5 and by the similar argument of [8] page $102 \sim 104$ we have $|I_R| \le C \sup_{|z|>R} \|F_z\|^{(l-1)/l} \|u\|_2 \|v\|_2$ for some constant C, and we have (1).

$(2) \Leftrightarrow (3)$: $(\Rightarrow)$ Fix m in $M(H^\infty + C)$. By the Corona theorem there is a net z which converges to m. If $\liminf_{z\to m} \|H_f k_z\|_2 = 0$ and $\liminf_{z\to m} \|H_f^* k_z\|_2 = 0$ then $f|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ and $f^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ by Lemma 2.2 and this is the case of $(a,b) = (1,0)$ in (c) of (3). We first asume that $\liminf_{z\to m} \|H_f k_z\|_2 \ge c > 0$. If we put $\lambda_z = \langle H_g k_z, H_f k_z\rangle / \|H_f k_z\|^2$, then $|\lambda_z| \le \|g\|_\infty / c$. Therefore we may think there is a complex number a such that $\lambda_z \to a$ when $z \to m$. By using Lemma 3.4 there is a constant C independent of z such that,

$$\begin{aligned}
\|H_f k_z\|^2 \|H_{g-af} k_z\|^2 + \|H_{g-af} k_z\|^2 \|H_f^* k_z\|^2 &\le C\{\|F_z\|^2 + |\lambda_z - a|^2\} \\
&\to 0 \quad (z \to m)
\end{aligned}$$

Therefore $\|H_{g-af}^* k_z\| \to 0$ and $\|H_{g-af} k_z\| \|H_f^* k_z\| \to 0$ when $z \to m$. By using Lemma 2.2, we have

$$(g - af)^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m} \text{ and } (g - af)|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m} \tag{4.4}$$

or

$$(g - af)^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m} \text{ and } f^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m} \tag{4.5}$$

If (4.4) is true, then this is the case of (c). On the other hand (4.5) is equivalent to $f^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ and $g^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$, and this is the case of (b). When $\liminf_{z\to m} \|H_f^* k_z\|_2 \ge c > 0$ we also have (3) similarly.

($\Leftarrow$) If (2) is false, then there exist $\delta > 0$ and net $\{z\}$ such that it converges to m and that $\limsup_{z\to m}\|F_z\| \geq \delta$. On the other hand

$$\|F_z\| \leq \|H_f k_z\| \left\|H^*_{g+af} k_z\right\| + \|H_{g+af} k_z\| \left\|H^*_{\bar{f}} k_z\right\|$$

for all a in $\mathbb{C}$. Therefore each of the condition (3) leads to $\|F_z\| \to 0$ when $z \to m$, and this is the contradiction.
(3) $\Leftrightarrow$ (4) : The proof is similar to [8] Lemma 1.1. □

Remark 4.5 If $f|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ is equivalent to $f^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$, then the condition of Theorem 4.8 (3) become more simple. But $f|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ is not equivalent to $f^*|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ generally.

For $\lambda \in \partial\mathbb{D}$ let $M_\lambda = \{m \in M(H^\infty) : m(\chi_1) = \lambda\}$ where $\chi_1(w) = w$ for w in $\partial\mathbb{D}$ and we call M_λ the fiber over λ. Put $f(z) = \exp(\frac{z+i}{z-i})$ for $z \in \mathbb{D}$. Then f is the singular inner function. Because $f(z)$ tends to 0 when z approaches i along the imaginaly axis, there exists m in the fiber M_i such that $m(f) = 0$ by [9] page 161. If $\overline{f}|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$, then $f|_{\operatorname{supp} m}$ is a constant, and $f|_{\operatorname{supp} m} = 0$. But we have a contradiction because $|f| = 1$ on $M(L^\infty)$ (Silov boundary of H^∞) and $\operatorname{supp} m \subset M(L^\infty) \cap M_i$ ([5] page 156). Therefore $\overline{f}|_{\operatorname{supp} m} \notin H^\infty|_{\operatorname{supp} m}$. On the other hand we have $\overline{f}^*|_{\operatorname{supp} m} = 1 \in H^\infty|_{\operatorname{supp} m}$ because $f = 1$ on M_{-i} by the two theorems in [9] page 161.

As corollary we can characterize the essentially normal Hankel operators concerning Corollary 4.7. The operator A is said to be the essentially normal if $A^*A - AA^*$ is a compact operator.

Corollary 4.9 *For $f \in L^\infty$, the following assertions are equivalent.*
(1) *H_f is an essentially normal operator.*
(2) $\|H_f k_z \otimes H_f k_z - H^*_{\bar{f}} k_z \otimes H^*_{\bar{f}} k_z\| \to 0 \quad (z \to \partial\mathbb{D})$.
(3) *There exists $a \in \mathbb{C}$ whose absolute value is 1 or 0 such that $(f + af^*)|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$ and $(\overline{a}f + f^*)|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}$.*

Moreover if $f, g \in \overline{H^\infty}$ the following is also equivalent.
(4) $\bigcap_{|a|=1,0} H^\infty[f + af^*, \overline{a}f + f^*] \subseteq H^\infty + C$.

Proof. (1) $\Rightarrow$ (3) : If H_f is an essentially normal, the condition (a), (b) of Theorem 4.8 (3) are both $f|_{\operatorname{supp} m}$ *and* $f^*|_{\operatorname{supp} m}$ in $H^\infty|_{\operatorname{supp} m}$, and this is the case of $a = 0$ for (3). Next the condition (c) of Theorem 4.8 (3) is that there exist $a, b \in \mathbb{C}$ such that they are not zero at the same time and that

$$(af + bf^*)|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m} \quad \text{and} \quad (\overline{b}f + \overline{a}f^*)|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}.$$

If $|a| \neq |b|$, then $f|_{\mathrm{supp}\, m}$ *and* $f^*|_{\mathrm{supp}\, m}$ are in $H^\infty|_{\mathrm{supp}\, m}$ and we have the case $a = 0$ of (3). If $|a| = |b|$ we have (3) clearly. Remainder of the proof is easy by Theorem 4.8. □

Acknowledgement The author expresses special gratitude to Professors Takashi Yoshino and Keiji Izuchi for their valuable comments. He also thanks to Professor Takashi Yoshino for having lead me to the attractive theory of Toeplitz and Hankel operators.

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